MATH 42 Final

Autumn 2006

Buyukboduk, Helleloid, Paquin, Thiem, Weiner

Name: ____________________

Student ID: ____________________

Name of your TA: ____________________

Instructions. Print your name, student ID number and TA name. Sign below to indicate that you accept **the honor code**. There are **seventeen** pages including this one, and **nine** questions. Before you begin the exam, please make sure that you have all the pages. Read each question carefully and, unless specified otherwise, show all your work and explain your answers. Calculators are not permitted. You have **3 hours** to complete the exam.

Signature: ____________________

Page	Score	Maximum
3		5+5
4		5+5
5		3+3+3+3
6		4+4+4
7		8
8		7
9		4+4+4
10		4+4+4
11		5+5
12		5
13		8+4
14		5
15		3+5
16		4+6
17		6
Total		139

Overflow I

1. Evaluate the following integrals.

(a) $\int \ln(x^2)\ dx$

Answer. Integrate by parts $u = \ln(x^2)$ and $dv = dx$, so

$$\int \ln(x^2)\ dx = x\ln(x^2) - \int x\frac{2x}{x^2}dx = x\ln(x^2) - 2x + C,$$

for some C in $\mathbb{R}$.

(b) $\int_0^2 \frac{x+1}{x^2-4x+3}\ dx$

Answer. Note that for $0 \leq x \leq 1$,

$$0 \leq \frac{1}{-3(x-1)} \leq \frac{x+1}{(x-3)(x-1)}.$$

Since

$$-\frac{1}{3}\int_0^1 \frac{1}{x-1}dx$$

diverges, by the comparison test, so does

$$\int_0^1 \frac{x+1}{x^2-4x+3}\ dx$$

(and therefore the entire integral).

(c) $\displaystyle\int \frac{\sqrt{x^2-9}}{x^4}\,dx$

Answer. Substitute $3\sec(\theta) = x$, so that $dx = 3\sec(\theta)\tan(\theta)d\theta$ and $\sqrt{x^2-9} = 3\tan(\theta)$. Thus,

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \int \frac{9\tan^2(\theta)\sec(\theta)}{3^4\sec^4(\theta)}d\theta = \frac{1}{9}\int \sin^2(\theta)\cos(\theta)d\theta$$

Substitute $u = \sin(\theta)$ to obtain

$$\frac{1}{9}\int \sin^2(\theta)\cos(\theta)d\theta = \frac{1}{9}\int u^2\,du = \frac{u^3}{27} + C = \frac{\sin^3(\theta)}{27} + C$$

Since $\sin(\theta) = \frac{\sqrt{x^2-9}}{x}$, we have

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \frac{(x^2-9)^{3/2}}{27x^3} + C,$$

for some C in $\mathbb{R}$.

(d) $\displaystyle\int_0^1 \frac{\sin(x)}{x}\,dx$

Answer. Since the integral is improper and

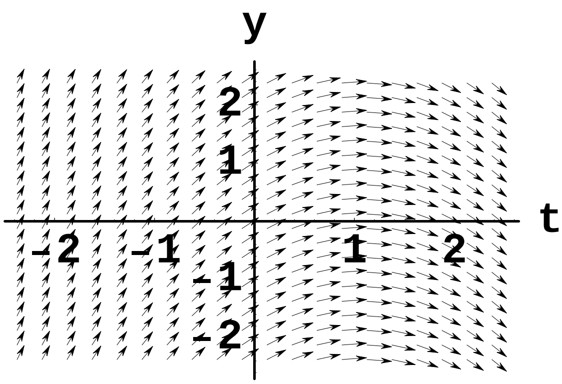
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

we have

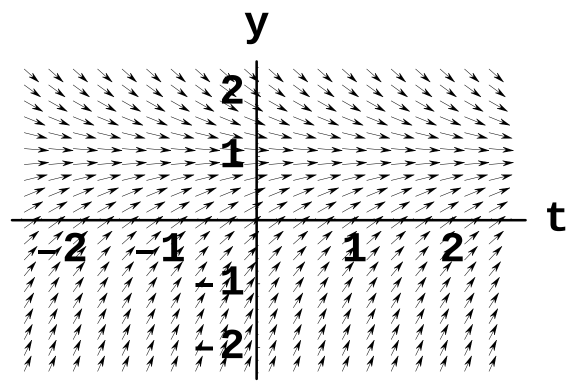
$$\begin{aligned}\int_0^1 \frac{\sin(x)}{x}\,dx &= \lim_{t\to 0}\int_t^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}dx \\ &= \lim_{t\to 0}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\right|_t^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}.\end{aligned}$$

2. Match the following direction fields to their differential equations.

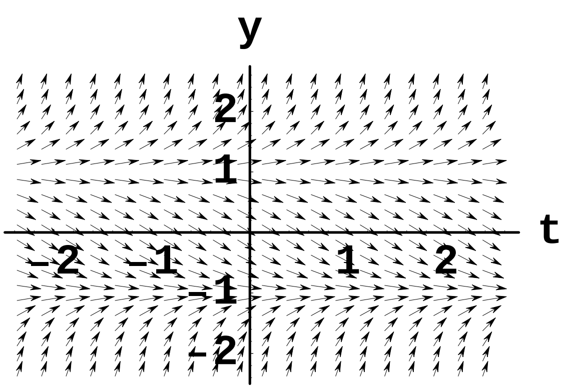
A. **4**



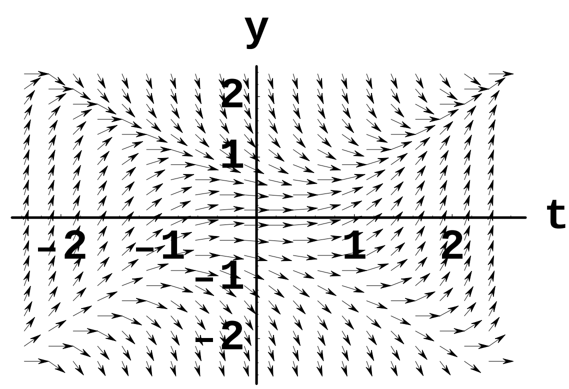
B. **5**



C. **8**



D. **6**



1. $\dfrac{dy}{dt} = t - 1$ 2. $\dfrac{dy}{dt} = 1 - y^2$ 3. $\dfrac{dy}{dt} = y^2 - t^2$ 4. $\dfrac{dy}{dt} = 1 - t$

5. $\dfrac{dy}{dt} = 1 - y$ 6. $\dfrac{dy}{dt} = t^2 - y^2$ 7. $\dfrac{dy}{dt} = 1 + y$ 8. $\dfrac{dy}{dt} = y^2 - 1$

3. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$

Answer. Note that $|\sin(1/n)| \leq 1$, we have

$$0 \leq \frac{\sin(1/n)}{n^2} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ converges as a p-series, by the comparison test, so does our series.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$

Answer. Note that by L'Hospital's rule,

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0,$$

so our series converges by the alternating sign test.

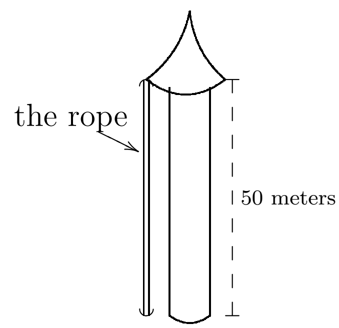
(c) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + 1}{4^n + 2}$

Answer. Use the limit comparison test with $(3/4)^n$,

$$\lim_{n\to\infty} \frac{\frac{3^n+1}{4^n+2}}{\frac{3^n}{4^n}} = \lim_{n\to\infty} \frac{1 + 1/3^n}{1 + 2/4^n} = 1 > 0.$$

Thus, since $\sum_{n=2}^{\infty} 3^n/4^n$ converges as a geometric series, so does our series.

4. Mickey Mouse has a 50 meter rope which he accidentally got wet. To dry the rope he hung it from the top of one of his castle towers (which happens to be 50 meters high).



After a couple of hours the rope has not yet dried completely, but since water has accumulated at the bottom we can describe the mass of the rope at a distance y above the ground as

$$e^{-y^2} \qquad \text{kilograms/meter} \qquad \longleftarrow \qquad \text{DIFFERENT FROM MIDTERM 2.}$$

(a) What is the total mass of the rope. Hint: you may leave your final answer as a series of numbers.

Answer. The mass of a small Δy section of the rope at distance y above the ground has mass approximately

$$e^{-y^2}\Delta y.$$

Thus, if we add up all the sections and take the limit as the number of sections goes to infinity, we get

$$\text{Mass} \;=\; \int_0^{50} e^{-y^2}dy.$$

Since $e^y = \sum_{y=0}^{\infty} y^n/n!$, we can substitute $-y^2$ for y, and obtain,

$$\text{Mass} \;=\; \int_0^{50}\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{n!}dy = \left(\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{n!(2n+1)}\right|_0^{50} = \sum_{n=0}^{\infty}\frac{(-1)^n(50)^{2n+1}}{n!(2n+1)}.$$

(b) How much work would it take to lift the entire rope up to the top of the tower? (Recall, if force is constant, then Work = (Force)(Distance); in general, Force = (Mass)(Acceleration), and you can take the acceleration of gravity to be 10 meters/second^2).

Answer. The amount of work to lift a small section of width Δy at height y above the ground to the tower is approximately

$$\text{[Distance][Force]} = [(50 - y)][10e^{-y^2}\Delta y].$$

Thus, if we take the limit as the number of sections goes to infinity, we get

$$\text{Work} = \int_0^{50} 10(50-y)e^{-y^2}\,dy = 500\int_0^{50} e^{-y^2}\,dy - 10\int_0^{50} ye^{-y^2}\,dy$$

and by substituting $u = -y^2$ in the second integral

$$= 500\sum_{n=0}^{\infty}\frac{(-1)^n(50)^{2n+1}}{n!(2n+1)} + 5\int_0^{-50^2} e^u\,du$$

$$= 500\sum_{n=0}^{\infty}\frac{(-1)^n(50)^{2n+1}}{n!(2n+1)} + 5(e^{-50^2} - 1).$$

5. Give examples of the following, or explain why no such example exists. Be sure to justify why the example is appropriate.

   (a) A differential equation that is not separable.

Answer.

$$y' = x + y.$$

   (b) A bounded divergent sequence.

Answer. The sequence $\{(-1)^n\}$ is bounded by 1 above and -1 below, but diverges.

   (c) A convergent sequence that is not monotonic.

Answer. The sequence $\{(-1)^n/n\}_{n=1}^{\infty}$ is a sequence that is not monotonic, but it converges by the squeeze theorem using $\{-1/n\}_{n=1}^{\infty}$ and $\{1/n\}_{n=1}^{\infty}$.

(d) A differential equation with 3 equilibrium solutions.

Answer. The differential equation

$$y' = y(1-y)(y-2)$$

has equilibrium solutions at $y = 0$, $y = 1$, and $y = 2$.

(e) An alternating series that diverges.

Answer. The series

$$\sum_{n=1}^{\infty} (-1)^n n$$

diverges by the alternating series test, since $\lim_{n\to\infty} n = \infty$.

(f) A power series with radius of convergence $R = 2$.

Answer. The power series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n}$$

has a radius of convergence 2.

6. Consider the curve $f(x) = \sin(x)$ for $\pi \leq x \leq 2\pi$.

(a) Set up an integral that gives the length of the curve. You do not need to evaluate the integral.

Answer. Since $\sin(x)$ is periodic, we can work from 0 to π. Thus, the arclength is

$$\int_0^\pi \sqrt{1+\cos^2(x)}dx.$$

(b) Use Simpson's rule with 4 steps to estimate the length of the curve.

Answer. Using Simpson's rule, we have

$$\int_0^\pi \sqrt{1+\cos^2(x)}dx \approx \frac{\pi}{12}(\sqrt{2}+4\sqrt{3/2}+2+4\sqrt{3/2}+\sqrt{2}).$$

(c) Let R be the region above the curve and below the x-axis. Set up an integral that gives the volume obtained by rotating R around the y-axis. You do not need to evaluate the integral.

Answer. Using cylindrical shells, the volume is

$$\left| \int_{\pi}^{2\pi} 2\pi x \sin(x) dx \right|.$$

7. (a) Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n}(x-1)^n.$$

Answer. Use the ratio test to obtain

$$\lim_{n\to\infty} \frac{\frac{5^{n+1}|x-1|^{n+1}}{n+1}}{\frac{5^n|x-1|^n}{n}} = \lim_{n\to\infty} \frac{5|x-1|n}{n+1} = 5|x-1|.$$

Thus, the power series converges absolutely for $|x-1| < 1/5$. At $x = 1+1/5$, we have

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges as the alternating harmonic series, and when $x = 1 - 1/5$, we have

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n(-5)^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges as the harmonic series. Thus, the interval of convergence is $4/5 < x \leq 6/5$.

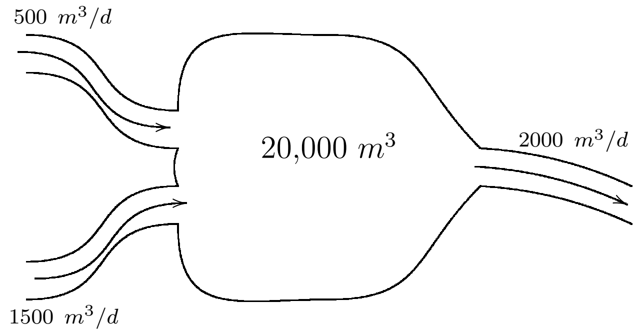
(b) Change the power series so that the new power series has the same center and the same radius of convergence as in (a), but a different interval of convergence.

Answer. The series

$$\sum_{n=1}^{\infty} (-5)^n (x-1)^n$$

has the same radius of convergence, but diverges at both endpoints.

8. A lake with volume 20,000 m^3 has



- One river flowing in at the rate of $500m^3/d$ with a $8kg/1000m^3$ concentration of fertilizer from nearby fields.
- Another river flowing in at the rate of $1500m^3/d$ with a $2kg/1000m^3$ concentration of fertilizer from nearby fields.
- A river flowing out at the rate of $2000m^3/d$.

(a) Give a differential equation describing the change in total amount of fertilizer in the the lake over time (in days).

Answer. Let $F(t)$ be the amount of fertilizer in the lake after t days. Then

$$F' = 500\frac{8}{1000} + 1500\frac{2}{1000} - 2000\frac{F(t)}{20000} = 7 - \frac{F(t)}{10}.$$

(b) Identify any equilibrium solutions of the system.

Answer. The system has an equilibrium point at $F = 70$.

(c) Find a function that describes the behavior of the system assuming the lake begins with $10kg$ of fertilizer.

Answer. This differential equation is separable, so we solve

$$\int \frac{1}{70 - F} dF = \int \frac{1}{10} dt$$

so

$$-\ln|70 - F| = \frac{t}{10} + C.$$

Since $F(0) = 10$, we have $C = -\ln(60)$. Thus,

$$70 - F = e^{-t/10 + \ln(60)},$$

or

$$F = 70 - 60e^{-t/10}.$$

9. Consider

$$(7.5)^{1/3}.$$

(a) Where would be a good idea to center your Taylor series of $x^{1/3}$ if you want to use it to estimate $(7.5)^{1/3}$? Justify your answer, and be sure to explain why centering at $a = 7.5$ would NOT be a good idea.

Answer. We would want to center the Taylor series at $a = 8$, since we know the value of $8^{1/3}$, and it is close to 7.5. The value 7.5 would be ill-informed since we would then have to use $(7.5)^{1/3}$ to approximate $(7.5)^{1/3}$.

(b) Use you answer to (a) to estimate $(7.5)^{1/3}$ using a degree three polynomial (ie. the first four terms of your Taylor series).

Answer. If $f(x) = x^{1/3}$, then

$$f'(x) = \frac{1}{3x^{2/3}}, \quad f''(x) = -\frac{2}{9x^{5/3}}, \quad f'''(x) = \frac{10}{27x^{8/3}}.$$

Thus, the first four terms of the Taylor series are

$$T_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{9 \cdot 2^4 \cdot 2!}(x-8)^2 + \frac{5}{27 \cdot 2^7 \cdot 3!}(x-8)^3.$$

Thus,

$$(7.5)^{1/3} \approx 2 - \frac{1}{24} - \frac{1}{9 \cdot 2^6} - \frac{5}{27 \cdot 2^{10} \cdot 3!}.$$

(c) What is the maximum error of your estimate?

Answer. Since

$$|f^{(4)}(x)| = \left|\frac{80}{81x^{11/3}}\right| \leq \frac{80}{81}$$

on the interval $7.5 \leq x \leq 8.5$, we have by Taylor's inequality that the error

$$|R_3(x)| \leq \frac{80}{81 \cdot 4!}|x-8|^4 \leq \frac{80}{81 \cdot 4! \cdot 2^4} = \frac{10}{81 \cdot 4!}.$$

Overflow II