Homework 5

Due: Tuesday, June 5

1. Suppose $G = KA$, where $K \cap A = 1$ and A is an abelian normal subgroup of G. For any irreducible character $\gamma : A \to \mathbb{C}$ we have a subgroup

$$K_\gamma = \{k \in K \mid \gamma(kak^{-1}) = \gamma, a \in A\} \subseteq G.$$

   (a) Show that if $\chi : K_\gamma \to \mathbb{C}$ is irreducible, then

$$\mathrm{Ind}_{K_\gamma}^G(\chi)$$

   is irreducible.
   Hint: One approach is to use the corresponding modules. Let U be the K_γ-module corresponding to χ and let $V = \mathrm{Ind}_{K_\gamma}^G(U)$. As an A-module

$$V = \bigoplus_{\mu: A \to \mathbb{C}^\times} V_\mu, \qquad \text{where} \quad V_\mu = \{v \in V \mid va = \mu(a)v, a \in A\}.$$

   Determine which μ appear, show that $V_\gamma = U$, and use the irreducibility of U to show V is irreducible.

   (b) Find the irreducible characters of

$$U_3(\mathbb{F}_5) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F}_5 \right\}.$$

   Hint: Note that the subgroup generated by the last column is normal in U_3 and abelian.

2. (a) Find an invariant Hermetian inner product on each of the irreducible representations of D_8, and find an orthonormal basis for each irreducible representation with respect to this inner product.

   (b) Use (i) to give an explicit orthonormal basis for $\mathrm{Maps}(D_8, \mathbb{C})$ using matrix coefficients and Schur orthogonality.

   (c) Choose a nonzero function $f \in \mathrm{Maps}(D_8, \mathbb{C})$, and decompose f in terms of the basis from (ii).

3. Let $|G|$ be odd and suppose χ is a nontrivial irreducible character. Show that $\chi \neq \overline{\chi}$. (That is, χ is not a real-valued character).
   Hint: Use orthogonality to show that $\chi(1) = 2a$ for some algebraic integer a.