Homework 3

Due: Tuesday, May 8

1. If G is a nonabelian group of order 6, what are the possible dimensions of its irreducible modules? If the group has order 8? If the group is order 12? For each case, list all the possible combinations.

2. Let λ be a partition of n. Let

$$V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \text{ a standard tableau of shape } \lambda\}.$$

For $i \in \{1, 2, \ldots, n-1\}$, define

$$v_T s_i = \begin{cases} v_{(Ts_i)}, & \text{if } Ts_i \text{ is standard,} \\ v_T, & \text{otherwise} \end{cases}$$

Is V^λ an S_n-module under this action? If so, what is its decomposition into irreducibles?

3. The natural module of S_n is the vector space

$$V = \mathbb{C}\text{-span}\{v_1, v_2, \ldots, v_n\},$$

with the action

$$\begin{array}{ccc} V \times S_n & \longrightarrow & V \\ (v_i, w) & \mapsto & v_{(iw)} \end{array}$$

Decompose V into irreducible S_n-modules (you do not need to give an inner direct sum; an outer direct sum will suffice).

4. Suppose $V \subseteq \mathbb{C}G$ is a submodule.

   (a) Show that there exists and idempotent $e \in \mathbb{C}G$ such that

   $$V = e\mathbb{C}G.$$

   (b) Show that if $\theta \in \mathrm{Hom}_{\mathbb{C}G}(V, \mathbb{C}G)$, then

   $$(v\theta) = xv, \qquad \text{for some } x \in \mathbb{C}G.$$

   (c) Show that

   $$\mathrm{Hom}_{\mathbb{C}G}(V, V) \cong e\mathbb{C}Ge.$$

   Remark: The space $e\mathbb{C}Ge$ is an algebra (a Hecke algebra), and one can show that these two spaces are in fact anti-isomorphic.