Homework 1

Due: April 6, 2005

1. §1.1: 9, 12, 15
2. §1.2: 6, 15
3. §1.3: 1, 6, 10, 15
4. This is §1.4 problem 11, written out. The *Heisenberg group over F* is

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in F \right\}.$$

   Let

$$X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}$$

   be elements of $H(F)$.

   (a) Compute the matrix product XY and deduce that $H(F)$ is closed under matrix multiplication. Exhibit explicit matrices such that $XY \neq YX$ (so that $H(F)$ is always non-abelian).

   (b) Find an explicit formula for X^{-1} and deduce that $H(F)$ is closed under taking inverses.

5. An *involution* in a group G is an element $g \in G$ that satisfies $g^2 = 1$.

   (a) Show that if every element of a group G is an involution, then G is abelian.

   (b) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the multiplication rules:

$$ij = k = -ij, \quad jk = i = -ki, \quad ki = j = -ik, \quad i^2 = j^2 = k^2 = -1,$$

   and 1 and -1 multiply as expected. Show that Q_8 is a group containing exactly one involution.