Math 108 Combinatorics Spring 2007
Homework 4 Solutions

1. Let (G, c) be a network with distinguished vertices s and t. A *minimum cut* in G is a pair of sets (X, Y) such that $V = X \cup Y$, $s \in X$, $t \in Y$, $X \cap Y = \emptyset$, and $c(X, Y)$ is minimal. Suppose (X, Y) and (A, B) are minimal cuts. Show that $(X \cup A, Y \cap B)$ is also a minimal cut.

   Solution. Note that $(X \cup A, Y \cap B)$ is, in fact, a partition of V with $s \in X \cup A$ and $t \in Y \cap B$. Let f be a maximum flow on G and let M be the value of f. The capacity of (X, Y) is M, so for every edge e from X to Y, $f(e) = c(e)$, and for every edge e from Y to X, $f(e) = 0$. The same is true for edges between A and B.

   Every edge e from $X \cup A$ to $Y \cap B$ is either an edge from X to Y or from A to B. Thus $f(e) = c(e)$. Every edge e from $Y \cap B$ to $X \cup A$ is either an edge from Y to X or from B to A, so $f(e) = 0$. Thus the capacity of the cut $(X \cup A, Y \cap B)$ equals the flow across the cut, which is M. So $(X \cup A, Y \cap B)$ is a minimal cut. □

2. Find a maximum flow in the given network.

   Solution. The edges $\{Bt, Et, GH\}$ form a cut of capacity 20. So the maximum value of a flow is at most 20. On the other hand, we can exhibit a flow f with value 20:

$$\begin{aligned}
f(sA) &= 6\\
f(sD) &= 6\\
f(sG) &= 8\\
f(AB) &= 5\\
f(AC) &= 1\\
f(DC) &= 2\\
f(DF) &= 4\\
f(GF) &= 4\\
f(GH) &= 4\\
f(CB) &= 3\\
f(FE) &= 8\\
f(Bt) &= 8\\
f(Et) &= 8\\
f(Ht) &= 4
\end{aligned}$$

   So f is a maximum flow. □

3. Let G be an undirected graph and let s and t be non-adjacent vertices in G. Show that the maximum number of pairwise vertex-disjoint paths from s to t equals the minimum number of vertices whose removal separates s from t.

Solution. Construct a weighted directed graph H as follows. The vertices are s, t, and a pair of vertices v_1 and v_2 for each vertex x of G other than s and t. For each edge (s, v) of G, put arcs (s, v_1) and (v_2, s) of infinite capacity into H. For each edge (t, v) of G, put arcs (t, v_1) and (v_2, t) of infinite capacity into H. For each edge (v, w) of G where v and w are not s or t, put arcs (v_2, w_1) and (w_2, v_1) of infinite capacity into H. Finally, for each vertex v of G other than s and t, put an arc (v_1, v_2) of capacity 1 into H.

Any path $s, v, w, \ldots, t$ in G has a corresponding path $s, v_1, v_2, w_1, w_2, \ldots, t$ in H. On the other hand, any path from s to t in H has the form $s, v_1, v_2, w_1, w_2, \ldots, t$, and corresponds to a path $s, v, w, \ldots, t$ in G.

Choose a max flow in H of value k. Every edge of H enters a vertex for which the only outgoing edge has capacity 1, or leaves a vertex for which the only incoming edge has capacity 1, or is an edge (v_1, v_2) of capacity 1. Thus the flow along every edge is either 0 or 1. Starting at s, we can choose an edge leading out of s with flow 1, and by conservation of flow, we can follow that flow of 1 until we reach t. This gives us an $s-t$ path in H. There are k such edges leading out of s, so we get k such paths. These paths must be pairwise vertex-disjoint, since each vertex v_1 only has outgoing capacity 1, and each vertex v_2 only has incoming capacity 1, and hence can only belong to one path. Under the correspondence between paths in H and paths in G, we have found k pairwise vertex-disjoint $s - t$ paths in G. So the value of the max flow in H is at most the maximum number of pairwise vertex-disjoint paths from s to t in G.

Let C be a cut of minimum capacity in H. Then C of $|C|$ arcs (v_1, v_2), since all other arcs in H have infinite capacity. Let B be the set of vertices in G so that $v \in B$ if and only if $(v_1, v_2) \in C$. Since C is a cut, every $s - t$ path in H goes through one of the arcs in C. Due to the correspondence between $s - t$ paths in H and $s - t$ paths in G, every $s - t$ path in G goes through one of the vertices in B. Thus the removal of B separates s from t in G. This shows that the minimum number of vertices whose removal separates s from t is at most the size of a min cut in H.

Therefore, by the max-flow min-cut theorem, the minimum number of vertices whose removal separates s from t in G is at most the maximum number of pairwise vertex-disjoint paths from s to t in G. On the other hand, given k pairwise vertex-disjoint paths from s to t in G, separating s from t clearly requires the removal of at least k vertices. So the two numbers are equal. □

4. Let $A_i = \{i-1, i, i+1\} \cap \{1, 2, \ldots, n\}$ for $i = 1, 2, \ldots, n$. Let S_n denote the number of SDR's of the collection $\{A_1, A_2, \ldots, A_n\}$. Determine S_n and $\lim_{n\to\infty} S_n^{1/n}$.

   Solution. First, $S_1 = 1$ and $S_2 = 2$. Suppose $n \geq 3$. Since there are n subsets and each is contained in $\{1, 2, \ldots, n\}$, any SDR must use each of the numbers $\{1, 2, \ldots, n\}$ as a representative. In an SDR for $\{A_1, A_2, \ldots, A_n\}$, either n is the representative for A_n or it is the representative for A_{n-1}.

   If n represents A_n, then choosing representatives for $A_1, \ldots, A_{n-1}$ is the same as choosing representatives for $B_i = \{i-1, i, i+1\} \cap \{1, 2, \ldots, n-1\}$ with $i = 1, \ldots, n-1$, which can be done in S_{n-1} ways. If n represents A_{n-1}, then $n-1$ represents A_n, and choosing representatives for $A_1, \ldots, A_{n-2}$ is the same as choosing representatives for $C_i = \{i-1, i, i+1\} \cap \{1, 2, \ldots, n-2\}$, which can be done in S_{n-2} ways. Thus $S_n = S_{n-1} + S_{n-2}$.

Since this is the same base case and recurrence as for the Fibonacci numbers, it follows that

$$S_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{2}.$$

Since $((1+\sqrt{5})/2)^{(n+1)/n}$ goes to $(1+\sqrt{5})/2$ as $n \to \infty$, we find

$$\begin{aligned}
\lim_{n\to\infty} \frac{S_n}{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}} &= \frac{1}{\sqrt{5}} \\
\lim_{n\to\infty} \frac{S_n^{1/n}}{\left(\frac{1+\sqrt{5}}{2}\right)^{(n+1)/n}} &= 1 \\
\lim_{n\to\infty} S_n^{1/n} &= \frac{1+\sqrt{5}}{2}.
\end{aligned}$$

□