

Math 108 Combinatorics Spring 2007

Homework 3 Solutions

1. How many positive integers less than 1000 have no factor strictly between 1 and 10?

Solution. A positive integer does not have a factor between 1 and 10 if and only if it does not have a factor of 2, 3, 5, or 7. Let $X = \{1, 2, \dots, 999\}$ and for each positive integer i , let $A_i \subset X$ be the multiples of i in X . If p_1, p_2, \dots, p_k are distinct primes, then

$$A_{p_1} \cap A_{p_2} \cap \dots \cap A_{p_k} = A_{p_1 p_2 \dots p_k}.$$

The size of A_i is $\lfloor 999/i \rfloor$. By the Inclusion-Exclusion Principle,

$$\begin{aligned} & |X \setminus (A_2 \cup A_3 \cup A_5 \cup A_7)| \\ &= |X| - |A_2| - |A_3| - |A_5| - |A_7| + |A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| \\ &\quad + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7| - |A_2 \cap A_3 \cap A_5| - |A_2 \cap A_3 \cap A_7| \\ &\quad - |A_2 \cap A_5 \cap A_7| - |A_3 \cap A_5 \cap A_7| + |A_2 \cap A_3 \cap A_5 \cap A_7| \\ &= |X| - |A_2| - |A_3| - |A_5| - |A_7| + |A_6| + |A_{10}| + |A_{14}| + |A_{15}| \\ &\quad + |A_{21}| + |A_{35}| - |A_{30}| - |A_{42}| - |A_{70}| - |A_{105}| + |A_{210}| \\ &= 999 - 449 - 333 - 249 - 142 + 166 + 99 + 71 + 66 \\ &\quad + 47 + 28 - 33 - 23 - 14 - 9 + 4 \\ &= 228 \end{aligned}$$

So there are 228 positive integers less than 1000 with no factor strictly between 1 and 10. □

2. Prove

$$\sum_{n \geq 0} S(n, n-2)x^n = \frac{x^3(1+2x)}{(1-x)^5}.$$

Solution. Let $P(x) = \sum_{n \geq 0} S(n, n-2)x^n$. Recall that $S(n, k) = kS(n-1, k) + S(n-1, k-1)$ for all n and k . Note that a set partition of $\{1, 2, \dots, n\}$ into $n-1$ parts must have exactly one part of size 2 and $n-2$ parts of size 1. There are $\binom{n}{2}$ ways to choose the part of size 2, so $S(n, n-1) = \binom{n}{2}$.

$$\begin{aligned} P(x) &= \sum_{n \geq 0} S(n, n-2)x^n \\ &= \sum_{n \geq 3} S(n, n-2)x^n \\ &= \sum_{n \geq 3} ((n-2)S(n-1, n-2) + S(n-1, n-3))x^n \\ &= \sum_{n \geq 3} \frac{(n-1)(n-2)^2}{2} \cdot x^n + \sum_{n \geq 3} S(n-1, n-3)x^n \\ &= \sum_{n \geq 3} \frac{(n-1)(n-2)^2}{2} \cdot x^n + xP(x). \end{aligned}$$

To evaluate the first sum, we have

$$\begin{aligned}
 \frac{x^2}{1-x} &= \sum_{n \geq 0} x^{n+2} \\
 \left(\frac{x^2}{1-x} \right)'' &= \sum_{n \geq 0} (n+2)(n+1)x^n \\
 \left(x \cdot \left(\frac{x^2}{1-x} \right)'' \right)' &= \sum_{n \geq 0} (n+2)(n+1)^2 x^n \\
 x^3 \cdot \left(x \cdot \left(\frac{x^2}{1-x} \right)'' \right)' &= \sum_{n \geq 0} (n+2)(n+1)^2 x^{n+3} \\
 &= \sum_{n \geq 3} (n-1)(n-2)^2 x^n \\
 \frac{x^3(2+4x)}{(1-x)^4} &= \sum_{n \geq 3} (n-1)(n-2)^2 x^n
 \end{aligned}$$

Thus

$$\begin{aligned}
 P(x) &= \frac{x^3(1+2x)}{(1-x)^4} + xP(x) \\
 P(x) &= \frac{x^3(1+2x)}{(1-x)^5}.
 \end{aligned}$$

□

3. Suppose $\{b_n\}_{n \geq 1}$ is the sequence of Bernoulli numbers. Show that

$$b_n = \sum_{k=1}^n \frac{(-1)^k k! S(n, k)}{k+1}.$$

Solution. Let $c_n = \sum_{k=1}^n \frac{(-1)^k k! S(n, k)}{k+1}$. We must show that $b_n = c_n$ for all n .

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{c_n x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=1}^n \frac{(-1)^k k! S(n, k) x^n}{(k+1)n!} \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^k k!}{k+1} \cdot \sum_{n=k}^{\infty} \frac{S(n, k) x^n}{n!} \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^k k!}{k+1} \cdot \frac{(e^x - 1)^k}{k!} \quad (\text{using Proposition 5.4.1}) \\
 &= \sum_{k=1}^{\infty} \frac{(1 - e^x)^k}{k+1}.
 \end{aligned}$$

Let $f(x) = \sum_{k=1}^{\infty} \frac{(1-e^x)^k}{k+1}$. Then

$$\begin{aligned} [f(x)(1-e^x)]' &= \sum_{k=1}^{\infty} (1-e^x)^k (-e^x) \\ &= \frac{-e^x}{1-(1-e^x)} \\ &= -1 \\ f(x)(1-e^x) &= -x \\ f(x) &= \frac{x}{e^x-1}. \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} \frac{c_n x^n}{n!} = \frac{x}{e^x-1}.$$

From Homework 2, we know the same is true when c_n is replaced by b_n ; it follows that $b_n = c_n$ for all n . \square

4. The *girth* of a graph is the length of the smallest cycle in the graph. Let G be a graph such that the girth of G is at least 5 and every vertex in G has degree at least d . Prove that G has at least $d^2 + 1$ vertices. Can equality hold?

Solution. Let v be any vertex of G . There are at least d neighbors of v . Each of those neighbors has at least $d-1$ neighbors different from v . Note that no two neighbors of v can have another common neighbor; this would create a cycle of length 4. Also, no two neighbors of v can be adjacent. Therefore G contains v , at least d neighbors of v , and at least $d(d-1)$ vertices at distance two from v , and these vertices are all distinct. Thus there are at least $1 + d + d(d-1) = d^2 + 1$ vertices in G .

The only two examples for which equality holds are the cycle C_5 (for which $d = 2$) and the *Petersen graph* found in Figure 1.4 (for which $d = 3$). The Petersen graph is remarkable for how often it can be used to provide an example or a counterexample. \square

5. Prove that every finite simple graph (on at least two vertices) has at least two vertices with the same degree.

Solution. Let G be a finite simple graph with $n \geq 2$ vertices. If G has no isolated vertices, then $1 \leq \deg(v) \leq n-1$ for all $v \in V$. If G has at least one isolated vertex, then $0 \leq \deg(v) \leq n-2$ for all $v \in V$ (no vertex is adjacent to the isolated vertex, so a vertex can be connected to at most $n-2$ other vertices). In either case, there are $n-1$ possible vertex degrees and n vertices; by the Pigeonhole Principle, two vertices have the same degree. \square