

Math 108 Combinatorics Spring 2007

Homework 2 Solutions

1. Given n letters, of which m are identical and the rest are all distinct, find a formula for the number of words which can be made.

Solution. If we use i of the $n - m$ distinct letters and j of the m identical letters, we can make $(i + j)!/j!$ words. There are $\binom{n-m}{i}$ ways to choose the i distinct letters to use. So in total, we can make

$$\sum_{i=0}^{n-m} \sum_{j=0}^m \binom{n-m}{i} \cdot \frac{(i+j)!}{j!}$$

different words. □

2. Give a recursion and a direct formula for the numbers in the sequence $\{a_0, a_1, a_2, \dots\}$ given by

$$a_n = |\{\text{sequences of 0's and 1's of length } n \text{ with no two consecutive 0's}\}|.$$

For example, there are three such sequences of length 2, given by 01, 11, and 10.

Solution. For each integer $n \geq 0$, let

$$A_n = \{\text{sequences of 0's and 1's of length } n \text{ with no two consecutive 0's}\}$$

We claim that there is a bijection $\theta : A_{n-1} \cup A_{n-2} \rightarrow A_n$ for $n \geq 2$. For a sequence s in A_{n-1} , let $\theta(s)$ be the sequence in A_n that results from appending a 1 to s . For a sequence s in A_{n-2} , let $\theta(s)$ be the sequence in A_n that results from appending 10 to s .

The map θ is injective because the image of a sequence from A_{n-1} ends with a 1 but the image of a sequence from A_{n-2} ends with a 0. The map θ is also surjective, because any sequence s in A_n either ends with a 1 or a 10, and so is the image of the sequence obtained by removing either 1 or 10 from the end of s . Therefore, θ is a bijection and $|A_n| = |A_{n-1}| + |A_{n-2}|$. That is, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. The base cases are $a_0 = 1$ and $a_1 = 1$. This is the same recursion that defines the Fibonacci sequence. Solving the recurrence shows that

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].$$

□

3. Find an interpretation for the coefficients a_n in the generating function

$$\prod_{k \geq 1} (1 + x^k) = \sum_{n \geq 0} a_n x^n.$$

Solution. Almost by definition, the left-hand side is equal to

$$\prod_{k \geq 1} (1 + x^k) = \sum_{\text{all finite } S \subset \mathbb{P}} x^{\text{sum}(S)},$$

where $\text{sum}(S)$ indicates the sum of all the elements of S and \mathbb{P} is the set of positive integers. Thus a_n is just the number of finite subsets of \mathbb{P} whose sum is n . This is the number of partitions of n into distinct parts. \square

4. The *Bernoulli numbers* b_n are given by $b_0 = 1$ and

$$\sum_{k=0}^n \binom{n+1}{k} b_k = 0.$$

(a) Show that the exponential generating function

$$B(x) = \sum_{n \geq 0} \frac{b_n x^n}{n!} = \frac{x}{e^x - 1}.$$

Proof. For $n \geq 1$,

$$\begin{aligned} (n+1)b_n &= - \sum_{k=0}^{n-1} \binom{n+1}{k} b_k \\ b_n &= - \sum_{k=0}^{n-1} \frac{n!}{k!(n+1-k)!} b_k \end{aligned}$$

Therefore,

$$\begin{aligned}
B(x) &= \sum_{n=0}^{\infty} \frac{b_n x^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} (-1) \sum_{k=0}^{n-1} \frac{n!}{k!(n+1-k)!} \cdot \frac{b_k x^n}{n!} \\
&= 1 - \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{b_k x^n}{k!(n+1-k)!} \\
&= 1 - \sum_{k=0}^{\infty} \frac{b_k}{k!} \cdot \sum_{n=k+1}^{\infty} \frac{x^n}{(n+1-k)!} \\
&\quad \text{(Substitute } m = n + 1 - k) \\
&= 1 - \sum_{k=0}^{\infty} \frac{b_k}{k!} \cdot \sum_{m=2}^{\infty} \frac{x^{m+k-1}}{m!} \\
&= 1 - \sum_{k=0}^{\infty} \frac{b_k x^{k-1}}{k!} \cdot (e^x - x - 1) \\
&= 1 - \frac{e^x - x - 1}{x} \cdot B(x) \\
\frac{e^x - 1}{x} \cdot B(x) &= 1 \\
B(x) &= \frac{x}{e^x - 1}.
\end{aligned}$$

□

- (b) Show that $B(x) + \frac{1}{2}x$ is an even function in x , and deduce that $b_n = 0$ for all odd $n \geq 3$.

Proof.

$$\begin{aligned}
B(-x) - \frac{1}{2}x &= \frac{-x}{e^{-x} - 1} - \frac{1}{2}x \\
&= \frac{-xe^x}{1 - e^x} - \frac{1}{2}x \\
&= \frac{-2xe^x}{2(1 - e^x)} - \frac{x(1 - e^x)}{2(1 - e^x)} \\
&= \frac{-x - xe^x}{2(1 - e^x)} \\
&= \frac{x + xe^x}{2(e^x - 1)} \\
&= \frac{x}{e^x - 1} + \frac{1}{2}x
\end{aligned}$$

Therefore $B(x) + \frac{1}{2}x$ is even. It follows that the coefficient of x^n in the Taylor series expansion of $B(x)$ is 0 for all odd $n \geq 3$. Hence $b_n = 0$ for all odd $n \geq 3$. □

5. Choose three items from the first 10 pages of the Catalan addendum and use bijective maps to show that they are counted by Catalan numbers.

Sample Solution. We will construct bijections between one of the standard interpretations of the Catalan numbers,

(i): Dyck paths from $(0, 0)$ to $(2n, 0)$, i.e., lattice paths with steps $(1, 1)$ and $(1, -1)$, never falling below the x -axis,

and each of four other sets. Write each lattice path in (i) as the sequence of lattice points $(x_0, y_0), (x_1, y_1), \dots, (x_{2n}, y_{2n})$, where necessarily

$$\begin{aligned}(x_0, y_0) &= (0, 0), \\(x_1, y_1) &= (1, 1), \\(x_{2n-1}, y_{2n-1}) &= (2n - 1, 1), \text{ and} \\(x_{2n}, y_{2n}) &= (2n, 0).\end{aligned}$$

(l^4): Lattice paths from $(0, 0)$ to $(n - 1, n - 1)$ with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$, never going above the line $y = x$, such that the steps $(1, 1)$ only appear on the line $y = x$.

(i) \Leftrightarrow (l^4): First, there is a bijection from (i) to (h): lattice paths from $(0, 0)$ to (n, n) using steps $(1, 0)$ and $(0, 1)$ and not going above the line $y = x$, given by

- replacing every $(1, 1)$ step by a $(1, 0)$ step and
- replacing every $(1, -1)$ step by a $(0, 1)$ step.

Lattice paths in (i) are characterized by having n $(1, 1)$ steps and n $(1, -1)$ steps, and at any point along the path, there have been at least as many $(1, 1)$ steps as $(1, -1)$ steps. Lattice paths in (h) are characterized by having n $(1, 0)$ steps and n $(0, 1)$ steps, and at any point along the path, there have been at least as many $(1, 0)$ steps as $(0, 1)$ steps. Then the given map is clearly a bijection.

An equivalent description for (l^4) is: lattice paths from $(1, 0)$ to $(n, n - 1)$ with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$, never going above the line $y = x - 1$, such that the steps $(1, 1)$ only appear on the line $y = x - 1$. We need to construct a bijection from (h) to (l^4).

Given a lattice path P in (h), construct a lattice path P' from $(1, 0)$ to $(n, n - 1)$ by removing any lattice point (i, i) in P . That P' is in (l^4) follows from two observations: first, any point (i, j) in P' has $i < j$, so P' never goes above $y = x - 1$; and second, any step $(1, 1)$ occurs where a point (i, i) was removed, and hence is necessarily a step from $(i, i - 1)$ to $(i + 1, i)$ for some i .

This is a bijection, because given any path P' in (l^4), we can reconstruct P in (h): namely, append $(0, 0)$ and $(1, 1)$ to the beginning and end, and insert (i, i) whenever a step $(1, 1)$ occurs from $(i, i - 1)$ to $(i + 1, i)$.

(z^4) : Sequences a_1, \dots, a_n of nonnegative integers satisfying $a_1 + \dots + a_i \geq i$ and $\sum a_j = n$.

$(i) \Leftrightarrow (z^4)$: Given a lattice path P in (i) , let a_i be the number of $(1, 1)$ steps between the $(i-1)$ -st and i -th $(1, -1)$ steps for $i = 0, 1, \dots, n-1$. Then the sequence a_1, \dots, a_n must be in (z^4) : the total number of $(1, 1)$ steps is n , so $\sum a_j = n$, and after i $(1, -1)$ steps, we cannot be below the x -axis, so the number of $(1, 1)$ steps, $a_1 + \dots + a_i$, must be at least i .

This is a bijection since we can reconstruct P from any sequence in (z^4) : take a_1 $(1, 1)$ steps, then one $(1, -1)$ step, then a_2 $(1, 1)$ steps, etc. The conditions on sequences in (z^4) ensure the resulting path is in (i) .

(b^5) : Sequences a_1, \dots, a_{2n} of nonnegative integers with $a_1 = 1$, $a_{2n} = 0$ and $a_i - a_{i-1} = \pm 1$.

$(i) \Leftrightarrow (b^5)$: Given a lattice path P in (i) , let $a_i = a_{i-1} + 1$ if the i -th step was $(1, 1)$ and $a_i = a_{i-1} - 1$ if the i -th step was $(1, -1)$ (let $a_0 = 0$). Then $a_1 = 1$, $a_{2n} = 0$ (since there are equal numbers of $(1, 1)$ and $(1, -1)$ steps), $a_i - a_{i-1} = \pm 1$, and the a_i are all nonnegative, because at any point, there have been at least as many $(1, 1)$ steps as $(1, -1)$ steps. So we get a sequence in (b^5) .

This is a bijection since we can reconstruct P from any sequence in (b^5) , where the i -th step is $(1, 1)$ if $a_i - a_{i-1} = 1$ and $(1, -1)$ otherwise.

(c^5) : Sequences of $n-1$ 1's and any number of -1's such that every partial sum is nonnegative.

$(b^5) \Leftrightarrow (c^5)$: To each sequence in c^5 , append a 1 followed by enough -1 's to form a sequence with n 1's and n -1's. This defines a bijection between (c^5) and (C^5) : sequences of n 1's and n -1's in which every partial sum is nonnegative.

Obviously the partial sums of a sequence in (C^5) give a sequence in (b^5) and the successive differences of a sequence in (b^5) give a sequence in (C^5) . \square