Math 108 Combinatorics Fall 2005
Homework 1 Solutions

(Problem 1: Derangements) Given n letters and n addressed envelopes, in how many ways can the letters be placed in the envelopes so that no letter is in the correct envelope? Solve this problem for the special cases $n = 3$, $n = 4$, and $n = 5$. Calculate the ratio of this number to $n!$ in each case.

Solution. We will refer to the letters and envelopes by numbers and write the letter numbers in the order they are placed in envelopes. For example, when $n = 3$, $(1, 3, 2)$ means that letter 1 is placed in envelope 1, letter 3 is placed in envelope 2, and letter 2 is placed in envelope 3.

(n = 3) There are $3! = 6$ total ways in which the letters can be placed in envelopes, namely $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, and $(3, 2, 1)$. Of these, only $(2, 3, 1)$ and $(3, 1, 2)$ are derangements, so there are 2 derangements of three letters. The proportion of placements that are derangements is $2/3! = 1/3$.

(n = 4) There are $4! = 24$ total ways in which the four letters can be placed in four envelopes. The derangements are $(2, 1, 4, 3)$, $(2, 3, 4, 1)$, $(2, 4, 1, 3)$, $(3, 1, 4, 2)$, $(3, 4, 1, 2)$, $(3, 4, 2, 1)$, $(4, 1, 2, 3)$, $(4, 3, 1, 2)$, and $(4, 3, 2, 1)$. So there are 9 derangements of four letters. The proportion of placements that are derangements is $9/4! = 3/8$.

(n = 5) For $n = 5$, let's adopt a more clever technique. Let $d(n)$ denote the number of derangements of n letters. To construct a derangement of n letters, first choose k with $2 \leq k \leq n$, and place letter k in envelope 1. Then letter 1 is either placed in envelope k or not. If letter 1 is placed in envelope k, then we complete our derangement by choosing any derangement of $\{2, 3, \ldots, k-1, k+1, \ldots, n\}$, which can be done in $d(n-2)$ ways. If letter 1 is not placed in envelope k, then we can choose a derangement of $\{2, 3, \ldots, n\}$, and replace k with 1. This can be done in $d(n-1)$ ways. So $d(n) = (n-1)(d(n-1) + d(n-2))$. For $n = 5$, we find that $d(5) = 4(d(4) + d(3)) = 4(9 + 2) = 44$. The proportion of placements that are derangements is $44/5! = 11/30$.

(General Case) From the above discussion, we know that $d(n)$ satisfies the recursion $d(n) = (n-1)(d(n-1) + d(n-2))$ with base cases of $d(1) = 0$ and $d(2) = 1$. Using induction, we can show that

$$d(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

To prove this, note that the formula is correct for the base cases $d(1) = 0$ and $d(2) = 1$.

Assume that it is correct for $d(n-1)$ and $d(n-2)$. Then, using the inductive hypothesis,

$$\begin{aligned} d(n) &= (n-1)(d(n-1)+d(n-2)) \\ &= (n-1)\left[(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}+(n-2)!\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}\right] \\ &= (n-1)((n-1)!+(n-2)!)\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}+(n-1)(n-1)!\frac{(-1)^{n-1}}{(n-1)!} \\ &= n!\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}+n!\frac{(-1)^{n-1}}{(n-1)!}-(n-1)!\frac{(-1)^n}{(n-1)!} \\ &= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}, \end{aligned}$$

proving the result. To obtain the formula without guessing, it is easiest to use (exponential) generating functions, which are explained in Chapter 14. From this formula for $d(n)$, it is clear that

$$\lim_{n\to\infty}\frac{d(n)}{n!}=\frac{1}{e}.$$

□

(Problem 2: Kirkman's Schoolgirls) Fifteen schoolgirls walk each day in five groups of three. Arrange the girls' walks for a week so that, in that time, each pair of girls walks together in a group exactly once. Solve this problem for nine schoolgirls walking for four days.

Solution. The following arrangement gives one solution for nine schoolgirls:

Day 1: $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$
Day 2: $\{1,4,7\}$, $\{2,5,8\}$, $\{3,6,9\}$
Day 3: $\{1,5,9\}$, $\{2,6,7\}$, $\{3,4,8\}$
Day 4: $\{1,6,8\}$, $\{2,4,9\}$, $\{3,5,7\}$

This is the unique solution up to permutations of the girls. For fifteen schoolgirls, this arrangement works:

Day 1: $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{10,11,12\}$, $\{13,14,15\}$
Day 2: $\{1,4,8\}$, $\{2,5,11\}$, $\{3,9,15\}$, $\{6,12,14\}$, $\{7,10,13\}$
Day 3: $\{1,5,13\}$, $\{2,8,14\}$, $\{3,7,11\}$, $\{4,9,12\}$, $\{6,10,15\}$
Day 4: $\{1,6,9\}$, $\{2,12,15\}$, $\{3,8,10\}$, $\{4,11,13\}$, $\{5,8,15\}$
Day 5: $\{1,7,12\}$, $\{2,4,10\}$, $\{3,6,13\}$, $\{5,8,15\}$, $\{9,11,14\}$
Day 6: $\{1,10,14\}$, $\{2,9,13\}$, $\{3,5,12\}$, $\{4,7,15\}$, $\{6,8,11\}$
Day 7: $\{1,11,15\}$, $\{2,6,7\}$, $\{3,4,14\}$, $\{5,9,10\}$, $\{8,12,13\}$

There are seven distinct solutions for fifteen schoolgirls up to permutations of the girls.

Generalizations of this problem in design theory include Kirkman Triple Systems, Steiner Triple Systems, and Oberwolfach problems. In particular, suppose there are n schoolgirls who walk every day in $n/3$ groups of 3. For what values of n can you arrange the girls' walks so that over some number of days, each pair of girls walks together in a group exactly once? That is equivalent to asking for the existence of a certain Kirkman Triple System.

There are two obvious necessary conditions. First, n must be divisible by 3. Also, the number of days required is

$$\frac{\binom{n}{2}}{3 \cdot \frac{n}{3}} = \frac{n-1}{2}$$

(this is the number of pairs of schoolgirls divided by the number of pairs of schoolgirls with a group of size 3 and the number of groups each day). Therefore n must be odd. These two conditions on n say precisely that n must be congruent to 3 modulo 6. Ray-Chaudhuri and Wilson [2] showed that there is always a solution for n congruent to 3 modulo 6. □

(Problem 3: Knight's Tour Problem) Let K_{mn} be an $m \times n$ chessboard. Is there a way to move a knight around the board so that he lands on every square exactly once? Solve this problem for 3×3, 3×4, and 3×5 chessboards. Are there any more general statements you can make? Find at least one infinite family of chessboards for which no knight's tour exists.

Solution. For notational purposes, we will label the three rows of the chessboard by A, B, and C and the columns by $1, 2, \ldots$.

(3×3) There is no knight's tour, since the center square B2 is not reachable from any other square.

(3×4) One knight's tour is A1, B3, C1, A2, B4, C2, A3, C4, B2, A4, C3, B1.

(3×5) There is no knight's tour. To prove this, color the squares alternately black and white as on a regular chessboard, with A1 colored white. Every time the knight makes a move, it must switch from a black square to a white square or vice versa. Since there are eight white squares and seven black squares, the knight must start and end on white squares. On the other hand, the black squares B1 and B5 are each only reachable from A3 and C3. Since the knight cannot start or end at B1 or B5, there is no way to visit both squares. Thus there is no knight's tour.

It is obvious that no $1 \times n$ or $2 \times n$ chessboard has a knight's tour. It turns out that the only other chessboards with no knight's tour are 3×6 and 4×4. You can easily prove that these boards do not have a knight's tour by similar arguments as above. Showing that all the other boards have knight's tours involves several constructions.

If we require that the knight end in the same square as it started (with the starting square the only one visited twice), this is called a knight's circuit or a closed knight's path. There is a simple and beautiful argument showing that for m and n odd, the $m \times n$ chessboard has no knight's circuit. As in the 3×5 case, color the squares of the chessboard alternately black and white, as on a regular chessboard. When m and n are odd, there are an odd number of squares overall, so there are unequal numbers of white and black squares. But any circuit alternating black and white must visit an equal number of white and black squares. So there is no knight's circuit on such chessboards. The general problem remains open.

□

(Problem 4: A Ramsey Game) This two-player game requires a sheet of paper and pencils of two colors, say red and blue. Six points on the paper are chosen, with no three in a line. The players take one pencil each and take turns drawing a line connecting two of the chosen points. The first player to complete a triangle of his/her own color loses. (Only triangles with vertices at the chosen points count.) Can the game ever result in a draw? Test the assertion that the Ramsey game cannot end in a draw by playing it with a friend. Develop heuristic rules for successful play.

Proof. Let's show that it is not possible to color the lines connecting the points in such a way that no monochromatic triangle is formed. This will show that at some point during the game, one player will complete a triangle of his/her own color and will lose.

Color the lines red and blue in any way. Choose any one of the six points and call it A. There are five lines drawn from point A, so there must be (at least) three of one color, say red. So A is connected to (at least) three points, say B, C, and D, by red lines. If any line between two of B, C, and D is red, then those two points with A form a monochromatic (red) triangle. On the other hand, if all lines between B, C, and D are blue, then B, C, and D form a monochromatic (blue) triangle. So any coloring has a monochromatic triangle.

In the language of Ramsey theory, this shows that $R(2, 2, 3) \leq 6$. To show that $R(2, 2, 3) = 6$, it suffices to show a coloring of the lines connecting five points in which there are no monochromatic edges. Representing these lines as a five-pointed star inscribed in a pentagon, such a coloring is achieved by coloring the star red and the pentagon blue.

It turns out that the second player can always win this game. All known winning strategies for the second player are quite complicated. One good, though fallible, strategy taken from [1] is described here (assume that the second player uses red). Upon his/her turn, the second player should:

1. Find all uncolored edges which, when colored red, will not complete a triangle with two black edges.

2. Of the edges in the previous step, find those which, when colored red, create a minimum number of triangles that have two red edges and one uncolored edge.

3. Of the edges in the previous step, find those which, when colored red, complete as many triangles as possible with two red edges and one black edge.

4. Of the edges in the previous step, find those which, when colored red, create as many triangles as possible with one red edge, one black edge, and one uncolored edge.

5. Of the edges in the previous step, color one of them red.

A complicated variant of this strategy gives the second player a winning strategy. $\square$

References

[1] E. Mead, A. Rosa, and C. Huang, The Game of Sim: A Winning Strategy for the Second Player, *Math. Mag.* 47(5), 243–247, 1974.

[2] D. K. Ray-Chaudhuri and R. M. Wilson, Solution of Kirkman's schoolgirl problem. Combinatorics (Proc. Sympos. Pure Math., Vol. XIX), 187–203, 1971.