Homework 2

Due: Thursday, April 19

1. Given n letters, of which m are identical and the rest are all distinct, find a formula for the number of words which can be made.

2. Give a recursion and a direct formula for the numbers in the sequence $\{a_0, a_1, a_2, \ldots\}$ given by

$$a_n = |\{\text{sequences of 0's and 1's of length } n \text{ with no two consecutive 0's}\}|.$$

   For example, there are three such sequences of length 2, given by 01, 11 and 10.

3. Find an interpretation for the coefficients a_n in the generating function

$$\prod_{k\geq 1}(1+x^k) = \sum_{n\geq 0} a_n x^n.$$

4. The *Bernoulli numbers* b_n are given by

$$\sum_{k=0}^{n} \binom{n+1}{k} b_k = 0, \qquad \text{where } b_0 = 1.$$

   (a) Show that the exponential generating function

$$B(x) = \sum_{n\geq 0} \frac{b_n x^n}{n!} = \frac{x}{e^x - 1}.$$

   (b) Show that $B(x) + \frac{1}{2}x$ is an even function in x, and deduce that $b_n = 0$ for all odd $n \geq 3$.

5. Download the "Catalan addendum" from

   `http://www-math.mit.edu/~rstan/ec/.`

   Choose 3 of the items from the first 10 pages (choose ones we did not cover in class), and use bijective maps to show that they are counted by Catalan numbers (that is, you need at least three bijections).