Homework 3

Due: Friday, October 28

Note: these problems are taken from van Lint and Wilson.

1. Let $r \in \mathbb{Z}_{>0}$.
   
   (a) Prove that there exists a minimal number $N(r) \in \mathbb{Z}_{>0}$ such that if the set $\{1, 2, \ldots, n\}$ is colored by $r$ colors and $n \geq N(r)$, then there exists $x, y, z \in \{1, 2, \ldots, n\}$ (not necessarily distinct) such that $x, y$ and $z$ have the same color and $x + y = z$.
   
   (b) Find $N(2)$.
   
   (c) Show that $N(3) > 13$.

2. Suppose $G$ is a simple graph on $n$ vertices with $\lfloor n^2/4 \rfloor$ edges and no $K_3$ as a subgraph. Show that
   $$G = \begin{cases} K_{k,k}, & \text{if } n = 2k, \\ K_{k,k+1}, & \text{if } n = 2k + 1. \end{cases}$$

3. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in $a_{ij} \in \mathbb{R}_{\geq 0}$ such that the row sums and column sums are all equal to $\ell \in \mathbb{R}$. Show that $A$ is a linear combination of permutation matrices.

4. Let $A_i = \{i - 1, i, i + 1\} \cap \{1, 2, \ldots, n\}$. Find
   $$SDR(A_1, \ldots, A_n), \quad \text{and} \quad \lim_{n \to \infty} SDR(A_1, \ldots, A_n)^{1/n}.$$ 

5. Let $a_1, a_2, \ldots, a_{n^2+1}$ be a permutation of the integers $1, 2, \ldots, n^2+1$. Use Dilworth’s theorem to show that there is a subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ of $a_1, a_2, \ldots, a_{n^2+1}$ (of length $n + 1$) such that either
   $$a_{i_1} > a_{i_2} > \cdots > a_{i_{n+1}} \quad \text{or} \quad a_{i_1} < a_{i_2} < \cdots < a_{i_{n+1}}.$$