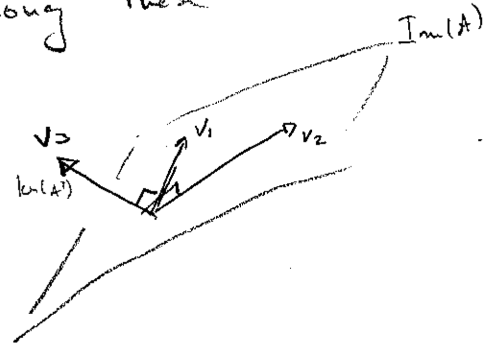
5.4

2. A basis for $\ker(A^T)$ is $\underbrace{\begin{bmatrix}1\\-2\\1\end{bmatrix}}_{v_3}$.

$\operatorname{Im}(A)$ is perpendicular to it.

$\operatorname{Im}(A) = \operatorname{span}\left\{ \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{v_2} \right\}$

Something along these lines



20. Do some math, find

$x^* = \begin{bmatrix}2\\2\end{bmatrix}$ $\quad b - Ax^* = \begin{bmatrix}-1\\1\\1\end{bmatrix}$

Image spanned by $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$.

Notice the dot product of these vectors with $b - Ax^* = 0$.

26. Apparently, the answer to this one is

$A^TAx = A^Tb \quad \begin{bmatrix}66 & 78 & 90\\78 & 93 & 108\\90 & 108 & 126\end{bmatrix} x = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$x = \begin{bmatrix}t-7/6\\1-2t\\t\end{bmatrix}$ for any $t \in \mathbb{R}$

28. $u_1, \dots, u_n$ orthonormal. So $u_i \cdot u_j = 0$ when $i \neq j$

and $u_i \cdot u_i = 1$. Let $A = \left(u_1 \,|\, \cdots \,|\, u_{n-1} \right)$

We solve

$$A^T A x = A^T u_n \quad . \quad \text{Notice} \quad A^T u_n = \begin{bmatrix} u_1 \cdot u_n \\ \vdots \\ u_{n-1} \cdot u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and
$$A^T A = \begin{pmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 \cdots & u_1 \cdot u_n \\ u_2 \cdot u_1 & & \\ \vdots & u_i \cdot u_j & \\ u_n \cdot u_1 & & \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} = I_{n-1}$$

since u_i's orthonormal.

So $\quad I_{n-1} x = 0 \quad \Rightarrow \quad x = 0.$

32. Want $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ of $f(t) := c_0 + c_1 t + c_2 t^2$

$$\begin{aligned} 27 &= c_0 + 0c_1 + 0c_2 = c_0 \\ 0 &= c_0 + 1 \cdot c_1 + 1^2 c_2 = c_0 + c_1 + c_2 \\ 0 &= c_0 + 2 \cdot c_1 + 2^2 \cdot c_2 = c_0 + 2c_1 + 4c_2 \\ 0 &= c_0 + 3 \cdot c_1 \cdot 3^2 \cdot c_2 = c_0 + 3c_1 + 9c_2 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{=A} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Check that $\ker(A) = 0$. So

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = (A^T A)^{-1} A^T \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25.65 \\ -28.35 \\ 6.75 \end{bmatrix}$$

$$f(t) = 25.65 - 28.35t + 6.75t^2$$

38. We want $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ such that

$$\begin{aligned} 110 &= c_0 + 2c_1 + c_2 \\ 180 &= c_0 + 12c_1 + 0c_2 \\ 120 &= c_0 + 5c_1 + c_2 \\ 160 &= c_0 + 11c_1 + c_2 \\ 160 &= c_0 + 6c_1 + 0c_2 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 12 & 0 \\ 1 & 5 & 1 \\ 1 & 11 & 1 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 110 \\ 180 \\ 120 \\ 160 \\ 160 \end{bmatrix}$$

Solving for $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ using Least-Squares gives

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 125 \\ 5 \\ -25 \end{bmatrix} \quad \text{so} \quad w = 125 + 5h - 25g$$

If $h=0$, $g=0$ c_0 gives weight of a 5' male, Obviously, he should weigh some positive #, so c_0 should be positive. c_1 should also be positive since taller people tend to weigh more. c_2 should be negative since women tend to be lighter than men of equal height.

6.1

4. $\det\begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix} = 0$

8. $\det\begin{bmatrix}1 & 2 & 3\\ 1 & 1 & 1\\ 3 & 2 & 1\end{bmatrix} = 1\cdot\det\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix} - 2\cdot\det\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix} + 3\cdot\det\begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}$

$= -1 - 2(-2) + 3(-1) = -1+4-3 = 0$

22. $\det\begin{bmatrix}\cos k & 1 & -\sin k\\ 0 & 2 & 0\\ \sin k & 0 & \cos k\end{bmatrix} = 2\cos^2 k + 2\sin^2 k$

$= 2(\cos^2 k + \sin^2 k) = 2$

matrix invertible $\Leftrightarrow$ det $\neq 0$, Since $2 \neq 0$ $\forall$ k,
matrix invertible for all $k \in \mathbb{R}$.

26. $\det\begin{bmatrix}4-\lambda & 2\\ 2 & 7-\lambda\end{bmatrix} = \lambda^2 - 11\lambda + 28 - 4 = \lambda^2 - 11\lambda + 24 = (\lambda-8)(\lambda-3)$

not invertible for $\lambda = 3, 8$.

28. determinant of an upper triangular matrix is equal to the product of its diagonal entries. So

$\det\begin{bmatrix}5-\lambda & 7 & 11\\ 0 & 3-\lambda & 13\\ 0 & 0 & 2-\lambda\end{bmatrix} = (5-\lambda)(3-\lambda)(2-\lambda)$. Matrix not invertible for $\lambda = 2, 3, 5$.

44/ Use induction, $\det(kA) = k^n \det(A)$ for A $n \times n$ matrix.

Base Case A is 1×1 is obvious.

Assume true for $(n-1) \times (n-1)$ matrices.

Let $A = \begin{pmatrix} a_{ij} \end{pmatrix}$ and let A_{1i} be the $n-1 \times n-1$ matrix obtained by deleting the ith column and 1^{st} row of A.

$\det(kA) =$ by expanding along top row

$$= ka_{11} \det(kA_{11}) - ka_{12} \det(kA_{12}) + \cdots + (-1)^{n-1} ka_{1n} \det(kA_{1n}).$$

By induction $\boxed{\det(kA_{1j}) = k^{n-1} \det(A_{1j})}$

$$= ka_{11} k^{n-1} \det(A_{11}) \mp \cdots + (-1)^{n-1} ka_{1n} k^{n-1} \det(A_{1n})$$

$$= k^n \left(a_{11} \det(A_{11}) \mp \cdots \pm (-1)^{n-1} a_{1n} \det(A_{1n}) \right)$$

$$= k^n \det(A).$$

OR/ note that $kA = (kI_n)A$.

using the fact that $\det(AB) = \det(A)\det(B)$, we see $\det(kA) = \det(kI_n)\det(A)$. But $kI_n = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix}$

so $\det(kI_n) = k^n$.

46. $\det(A^{-1}) = \dfrac{1}{\det(A)}$

48. $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

so $\quad \det(A)\det(D) - \det(B)\det(C) = 0 \cdot 0 - 0 \cdot 0 = 0.$

But $\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1.$

$-1 \neq 0.$