2.2

6. By Fact 2.2.1, $\text{proj}_L \begin{bmatrix}1\\1\\1\end{bmatrix} = \left(\vec{u}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right)\vec{u}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix}2\\1\\2\end{bmatrix}$:

$$\vec{u} = \tfrac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}, \text{ so that } \text{proj}_L \begin{bmatrix}1\\1\\1\end{bmatrix} = \tfrac{5}{3}\cdot\tfrac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}\frac{10}{9}\\ \frac{5}{9}\\ \frac{10}{9}\end{bmatrix}.$$

8. From Definition 2.2.2, we can see that this is a reflection about the line $x_1 = -x_2$.

26. a. $\begin{bmatrix}k & 0\\0 & k\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}2k\\-k\end{bmatrix} = \begin{bmatrix}8\\-4\end{bmatrix}$. So $k = 4$ and $A = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}$.

b. This is the orthogonal projection onto the horizontal axis, with matrix $B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

c. $\begin{bmatrix}a & -b\\b & a\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}-5b\\5a\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$. So $a = \frac{4}{5}, b = -\frac{3}{5}$, and $C = \begin{bmatrix}\frac{4}{5} & \frac{3}{5}\\-\frac{3}{5} & \frac{4}{5}\end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a rotation matrix.

d. Since the x_1 term is being modified, this must be a horizontal shear.

Then $\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1+3k\\3\end{bmatrix} = \begin{bmatrix}7\\3\end{bmatrix}$. So $k = 2$ and $D = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$.

e. $\begin{bmatrix}a & b\\b & -a\end{bmatrix}\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}7a+b\\7b-a\end{bmatrix} = \begin{bmatrix}-5\\5\end{bmatrix}$. So $a = -\frac{4}{5}, b = \frac{3}{5}$, and $E = \begin{bmatrix}-\frac{4}{5} & \frac{3}{5}\\\frac{3}{5} & \frac{4}{5}\end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a reflection matrix.

28. a. D is a scaling, being of the form $\begin{bmatrix}k & 0\\0 & k\end{bmatrix}$.

b. E is the shear, since it is the only matrix which has the proper form (Fact 2.2.5).

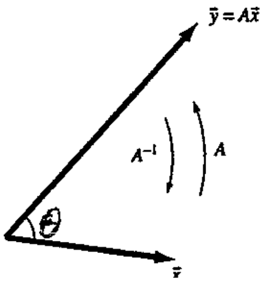
c. C is the rotation, since it fits Fact 2.2.3.

d. A is the projection, following the form given in Definition 2.2.1.

e. F is the reflection, using Definition 2.2.2.

44. By Exercise 1.1.13b, $A^{-1} = \begin{bmatrix}a & -b\\b & a\end{bmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix}a & b\\-b & a\end{bmatrix}$.

If A represents a rotation through θ followed by a scaling by r, then A^{-1} represents a rotation through $-\theta$ followed by a scaling by $\frac{1}{r}$.



2.3

4. Use Fact 2.3.5; the inverse is $\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}$.

12. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

14. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

48. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Note that Fact 2.3.4 applies to *square* matrices only.

52. Let $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\left[A \vdots \vec{b}\right] = \left[\begin{array}{ccc:c} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 3 & 6 & 1 \\ 1 & 4 & 8 & 0 \end{array}\right]$. We find that rref$\left[A \vdots \vec{b}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, which has an inconsistency in the third row.

2.4

6. $\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

14. $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $BC = [14\ 8\ 2]$, $BD = [6]$, $C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$,

$DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, $EB = [5\ 10\ 15]$, $E^2 = [25]$

26. $$\left[\begin{array}{c|c} \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}2&3\\4&5\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} \\ \hline \begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}2&3\\4&5\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} \end{array}\right] = \begin{bmatrix} \begin{bmatrix}1&2\\3&4\end{bmatrix} & \begin{bmatrix}3&5\\7&9\end{bmatrix} \\ \begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{bmatrix}1&2\\3&4\end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

46. Use the result of Exercise 45, with $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$;

$$A = BS^{-1} = \begin{bmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{bmatrix}$$

45. We want A such that $A\vec{v}_i = \vec{w}_i$, for $i = 1, 2, \ldots, m$, or $A[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_m] = [\vec{w}_1\ \vec{w}_2\ \ldots\ \vec{w}_m]$, or $AS = B$.

Multiplying by S^{-1} from the right we find the unique solution $A = BS^{-1}$.

70. Try to find a matrix $B = \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix}$ (where X is $n \times n$) such that

$$AB = \begin{bmatrix} I_n & \vec{v} \\ \vec{w} & 1 \end{bmatrix} \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix} = \begin{bmatrix} X + \vec{v}\vec{y} & \vec{x} + t\vec{v} \\ \vec{w}X + \vec{y} & \vec{w}\vec{x} + t \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}.$$

We want $X + \vec{v}\vec{y} = I_n$, $\vec{x} + t\vec{v} = \vec{0}$, $\vec{w}X + \vec{y} = \vec{0}$, and $\vec{w}\vec{x} + t = 1$.

Substituting $\vec{x} = -t\vec{v}$ into the last equation we find $-t\vec{w}\vec{v} + t = 1$ or $t(1 - \vec{w}\vec{v}) = 1$.

This equation can be solved only if $\vec{w}\vec{v} \neq 1$, in which case $t = \frac{1}{1-\vec{w}\vec{v}}$. Now substituting $X = I_n - \vec{v}\vec{y}$ into the third equation, we find $\vec{w} - \vec{w}\vec{v}\vec{y} + \vec{y} = \vec{0}$ or $\vec{y} = -\frac{1}{1-\vec{w}\vec{v}}\vec{w} = -t\vec{w}$.

We summarize: A is invertible if (and only if) $\vec{w}\vec{v} \neq 1$. In this case, $A^{-1} = \begin{bmatrix} I_n + t\vec{v}\vec{w} & -t\vec{v} \\ -t\vec{w} & t \end{bmatrix}$, where $t = \frac{1}{1-\vec{w}\vec{v}}$. The same result can be found (perhaps more easily) by working with $\text{rref}[A \vdots I_{n+1}]$, rather than partitioned matrices.

Multiplying both sides with A^{-1} we find that $A = I_n$: The identity matrix is the only invertible matrix with this property.