Research Program

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Combinatorial representation theory

My primary research interest is in the interplay between combinatorics and algebraic structures. By employing combinatorial tools such as symmetric functions, partitions, tableaux, graphs, posets, and crystal bases, one can gain significant insight on algebraic and geometric structures such as groups, algebras and rings; and, conversely, the corresponding structure theory can often lead to surprising combinatorial results.

Combinatorial representation theory uses combinatorial objects to understand the actions of algebraic structures on vector spaces. A vector space V together with an action by an algebraic structure A is called an A-module. It is natural to ask whether V has any subspace closed under the A-action, because if such a subspace exists, then V in fact decomposes into a direct sum of two A-modules. This type of decomposition suggests that there is some set of "smallest" A-modules, called irreducible modules, that form building blocks from which to construct all other A-modules. Some important problems in combinatorial representation theory therefore include

- Enumerating the irreducible modules using some known family of combinatorial objects,
- Finding a combinatorial formula for their dimensions as vector spaces,
- Constructing the irreducible modules relative to some natural set of operations,
- Giving combinatorial interpretations for how many times a given irreducible module appears in the decomposition of a larger module.

While this area of research has obvious uses in algebra and combinatorics, it also has many applications in geometry, number theory, probability, and physics. In particular, I have recently been exploring applications of representation theory to random walks in probability.

My research focuses largely on groups of Lie type and their associated algebras. In particular, this statement will examine the following three topics.

1. **Finite groups of Lie type:** a family of finite matrix groups that share an underlying geometric structure,
2. **Unipotent Hecke algebras:** a family of algebras associated to groups of Lie type,
3. **Supercharacter theory:** a coarser approximation to representation theory with applications to the study of random walks.

I will give a brief description of the each subject, outline some of my results, and describe some directions for future research, including possibilities for collaboration with undergraduates (problems especially suited are marked with $(*)$).

1. Finite groups of Lie type

In general, groups of Lie type are a large family of groups that share a Lie theoretic or geometric structure. A fundamental example of a finite group of Lie type is the finite general linear group

$$\mathrm{GL}_n(\mathbb{F}_q) = \left\{ \begin{array}{c} n \times n \text{ invertible matrices with entries} \\ \text{in the finite field } \mathbb{F}_q \text{ with } q \text{ elements} \end{array} \right\}.$$

Other examples include the finite unitary group, the finite symplectic group, and the finite orthogonal group.

While one can think of many finite groups of Lie type as matrix groups over finite fields, the following two interpretations become invaluable when working with them or applying them to other subjects.

- From a Lie theoretic point of view one may obtain a finite group of Lie type by "exponentiating" a Lie algebra,
- From a geometric point of view one can obtain finite groups of Lie type by considering fixed points of algebraic varieties under a Frobenius homomorphism (essentially a map that takes every matrix entry to the qth power for some prime power q).

One of the dreams in combinatorial representation theory is to combinatorially construct the irreducible modules for all finite groups of Lie type.

In the case of $\mathrm{GL}_n(\mathbb{F}_q)$, the mathematically nonsensical equation

$$\lim_{q\to 1} \mathrm{GL}_n(\mathbb{F}_q) = S_n$$

has long been a guiding principle, because it suggests that the combinatorics governing $\mathrm{GL}_n(\mathbb{F}_q)$ is fundamentally the same as that of the symmetric group S_n. This intuition was, to a large extent, formalized by Green [Gr] when he constructed an astonishing bijective correspondence between

$$\{\text{Modules of } \mathrm{GL}_n(\mathbb{F}_q)\} \overset{\text{characteristic}}{\longleftrightarrow} \text{A ring of symmetric functions,}$$

which allowed us to understand the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ via the representation theory of the symmetric group. Among other results, this characteristic map leads to combinatorial formulas for the dimensions of the irreducible $\mathrm{GL}_n(\mathbb{F}_q)$-modules in terms of partitions.

From a more geometric point of view, G. Lusztig [Lu] provides an enumeration for the irreducible modules of all finite groups Lie type, but without the corresponding structure given by symmetric functions. Even today, a characteristic map analogous to Green's still remains elusive for most other finite groups of Lie type.

In [TVa, TVb], C.R. Vinroot and I describe how to construct the corresponding map in the case of the finite unitary group (based on the combined work of [En, HS, Ka, LS]), and give some of the combinatorial and representation theoretic applications. Our goal is to continue this program, understanding characteristic maps for other finite groups of Lie type. In particular, Problem 1, below, describes a possible next step in this program, and Problem 2 gives one of many possible applications that rely on the characteristic map.

Problem 1 *Find a characteristic map for the finite general symplectic groups.*

The characteristic map of the unitary group already suggests some of the difficulties inherent in extending the characteristic map to other groups of Lie type, but we think there is at least hope for "large" finite groups of Lie type such as the general symplectic group (a group that is used frequently in number theory). This work will involve understanding how to adapt Lusztig's enumeration of the irreducible modules, using Jordan decomposition to enumerate the conjugacy classes, and determining how the combinatorics of the Weyl group interacts with the representation theory of the general symplectic group.

Problem 2 (∗) *Using the characteristic map of the finite unitary group, construct a "q-partition algebra" for the finite unitary group and study the corresponding combinatorial identities.*

A convenient byproduct of the this work is that as we develop characteristic maps, representation theory suggests a variety of combinatorial problems that can be attacked by motivated undergraduates. This problem relates to an ongoing project with T. Halverson in which we examine the combinatorics of the q-partition algebra [HT]. This algebra is a q-deformation of the partition algebra, which is an important diagram algebra related to set partitions [Ma]. A fundamental part of getting started on our project was understanding the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ from a combinatorial point of view. Since [TVa, TVb] give us control over the representation theory of the finite unitary group, it is natural to construct the unitary q-partition algebra and study the corresponding combinatorial consequences.

2. Unipotent Hecke algebras

Unipotent Hecke algebras are a family of algebras that arise naturally in the study of groups of Lie type (special cases in the literature include [CS, Yo]), and generalize the classical Iwahori Hecke algebra.

If the underlying group is the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ over a finite field with q elements, then there is a unipotent Hecke algebra $\mathcal{H}_\mu$ for every partition μ of the number n. Each algebra has a natural basis indexed by sets N_μ of monomial matrices (permutation matrices where the 1's can be replaced by arbitrary field elements), so that

$$\mathcal{H}_\mu = \mathbb{C}\text{-span}\{T_v \mid v \in N_\mu\}.$$

Some of my key results in [Tha, Thb, Thc], include

- An explicit description of the sets N_μ in terms of matrices with fixed row and column sums,
- An RSK correspondence (combinatorial bijection) between the index set N_μ and pairs of multi-tableaux,
- An algorithm for computing the coefficients c_{uv}^w of

  $$T_u T_v = \sum_{w \in N_\mu} c_{uv}^w T_w$$

  using local combinatorial braid relations. This last result holds for general type, but has a particularly nice description in the $\mathrm{GL}_n(\mathbb{F}_q)$ case,
- A construction of the irreducible modules in the case where $\mu = (1^n)$.

Some of my future projects in this vein include

Problem 3 *Understand unipotent Hecke algebras when the underlying field is p-adic.*

Much of my work on unipotent Hecke algebras does not require a finite field, so the general structure of the corresponding algebras over other fields should be similar. In the same way that unipotent Hecke algebras generalize the Iwahori Hecke algebra, these algebras would generalize affine Hecke algebras, algebras that are useful in the study of crystal basis theory and number theory.

Problem 4 (∗) *Construct the representation theory of unipotent Hecke algebras of the finite unitary group.*

By double centralizer theory, [TVa] suggests that the combinatorics of these algebras is indexed by domino multi-tableaux. Thus, working out the representation theory here should lead to nice combinatorial results on domino tableaux (which are of independent interest in the literature). Since much of the work involves combinatorial enumerations and basic linear algebra, this problem would also be quite suitable for undergraduate collaboration.

3. Supercharacter theory

The character theory of a group is a simplified version of the representation theory. Given a G-module V, the associated character $\chi : G \to \mathbb{C}$ is a map where $\chi(g)$ is the trace of the element $g \in G$ as a linear transformation of V. It turns out that two modules are the same if and only if their characters are equal; and a module is irreducible if and only if its character cannot be written as a sum of two other characters.

While character theory is a powerful tool in group theory, Diaconis and Saloff-Coste [DS] have also developed a program that uses character theoretic information to understand mixing times in random walks. Random walks on groups come up surprisingly often in applications, including the East model in physics, pseudo-random number generation in computer science, and card shuffling. In fact, one of the motivations in studying supercharacter theory is to simplify character theory without losing information about interesting random walks.

Supercharacter theory is a coarser version of character theory developed by [An, DI, Ya] that often is more tractable than character theory itself. For example, if

$$U_n = \left\{ \begin{array}{c} n \times n \text{ upper-triangular matrices with} \\ \text{ones on the diagonal, and entries in } \mathbb{F}_q \end{array} \right\},$$

then it is well-known that the character theory of U_n is "wild." On the other hand, [ADS] applied the explicit supercharacter theory of U_n to study the East model – a model used in physics to understand glass transition and super-cooled liquids.

In [DT], Diaconis and I study a family of groups that generalize U_n, called pattern groups. In particular, we associate a group $U_{\mathcal{P}}$ to every finite poset $\mathcal{P}$. It turns out that the structure of the poset has a great influence on the supercharacter theory of the group. Our main results include

- An explicit character formula for the irreducible supercharacters,
- Simple conditions on $\mathcal{P}$ that determine when the supertheory is in fact the normal character theory.

This summer we expanded our research group to include two undergraduates, Eric Marberg and Vidya Venkateswaran, who are studying larger families of posets for more explicit descriptions.

There is still much to be done in the area of supercharacter theory.

Problem 5 (∗) *Analyze random walks on pattern groups using the supertheory.*

We have an intriguing interplay between representation theory and probability. As we learn to work with the representation theory of a given group (or some approximation thereof), we can then proceed to study the corresponding random walks. Conversely, any model coming from a random walk on a group inspires a study of the corresponding representation theory. I find both sides of this divide compelling avenues for future research.

Problem 6 *Make more explicit the connection between the supercharacter theory of pattern groups and the character theory of the global group* $\mathrm{GL}_n(\mathbb{F}_q)$.

A large part of the representation theory of finite groups of Lie type has involved "hiding" the U_n part of the representation theory and seeing how far one can get. However, with an actual understanding of the supercharacter theory of pattern groups, one might be able to get a richer contribution to the representation theory of finite groups of Lie type. There should also be a more explicit connection between supercharacters of finite unipotent groups and Kawanaka's generalized Gelfand-Graev characters [Ka] (see also [BK] for generalized Gelfand-Graev characters in another context). This question ties together all three parts of this statement, since from this point of view, unipotent Hecke algebras are a first approximation towards using supercharacter theory to understand finite groups of Lie type.

Problem 7 (∗) *Study the relationship between poset combinatorics and pattern group theory.*

Pattern groups are a group version of the poset incidence algebras, which can be used to study posets. There are many properties of posets that translate to group theoretic properties and vice-versa. For example, the center of $U_{\mathcal{P}}$ is "generated" by the 2-chains consisting of maximal and minimal elements in the poset. However, there are many properties for which we do not know of a corresponding interpretation (mainly because no-one has tried yet). Some possible poset properties of interest include the Möbius function, the size of the largest anti-chain, etc. These projects could range from honors theses for undergraduates to publishable papers.

Problem 8 *Find supercharacter theories for other groups of Lie type.*

Pattern groups are naturally associated with $\mathrm{GL}_n(\mathbb{F}_q)$. It would be interesting to study supercharacter theories for other finite groups of Lie type. C. André and A. Neto [AN] give one approach by inducing linear characters from subgroups. However, this theory still seems to be derived from the U_n supercharacter theory, and there may be some more type specific approaches that adapt [DI] to more general situations.

Problem 9 (∗) *Understand the supercharacter theory for any large family of posets.*

We have a computable character formula (using a combination of combinatorics and linear algebra), but working out any given class of examples is still hard work. On the other hand, every family we have examined has had a beautiful theory, making this another interesting problem for undergraduates.

References

[An] André, C. "Basic characters of the unitriangular group," *J. Algebra* **175** (1995), 287-319.

[AN] André, C; Neto, A. "Super-characters of finite unipotent groups of types B_n, C_n, and D_n," August 2006 preprint.

[ADS] Arias-Castro E; Diaconis, P; Stanley, R. "A super-class walk on upper-triangular matrices," *J. Algebra* **278** (2004), 739-765.

[BK] Brundan, J; Kleshchev, A. "Shifted Yangians and finite W-algebras," *Adv. Math.* **200** (2006): 136–195.

[DI] Diaconis, P; Isaacs, M. "Supercharacters and superclasses for algebra groups," To appear in *Trans. Amer. Math. Soc.*, 2006.

[DS] Diaconis, P; Saloff-Coste, L. "Comparison techniques for random walk on finite groups," *Ann. of Probab.* **21** (1993), 2131-2156.

[DT] Diaconis, P; Thiem, N. "Supercharacter formulas for pattern groups," arXiv math.RT/0610161, 2006.

[CS] Curtis, C; Shinoda, K. "Zeta functions and functional equations associated with the components of the Gelfand-Graev representations of a finite reductive group," *Representation theory of algebraic groups and quantum groups*, 121-139, *Adv. Stud. Pure Math.* **40** Math. Soc. Japan, Tokyo, 2004.

[En] Ennola, V. "On the characters of the finite unitary groups," *Ann. Acad. Sci. Fenn. Ser. A I No.* **323** (1963), 1–35.

[Gr] Green, J. "The characters of the finite general linear groups," *Trans. Amer. Math. Soc.* **80** (1955), 402-447.

[HT] Halverson, T; Thiem, N. "Combinatorics of the q-partition algebra," in preparation.

[HS] Hotta, R; Springer, T. "A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups," *Invent. Math.* **41** (1977), 113-127.

[Ka] Kawanaka, N. "Generalized Gel'fand-Graev representations and Ennola duality," in *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, 175-206, *Adv. Stud. Pure Math.* **6** North-Holland, Amsterdam, 1985.

[Lu] Lusztig, G. *Characters of reductive groups over a finite field*, Ann. of Math. Stud. **107**, Princeton University Press, Princeton: 1984.

[LS] Lusztig, G; Srinivasan, B. "The characters of the finite unitary groups," *J. Algebra* **49** (1977), 167-171.

[Ma] Martin, P. "The structure of the partition algebras," *J. Algebra* **183** (1996), 319-358.

[Tha] Thiem, N. *Unipotent Hecke algebras*, Thesis, University of Wisconsin, Madison, 2004.

[Thb] Thiem, N. "Unipotent Hecke algebras of $\mathrm{GL}_n(\mathbb{F}_q)$," *J. Algebra* **284** (2005), 559-577.

[Thc] Thiem, N. "A skein multiplication algorithm for unipotent Hecke algebras," to appear in *Trans. Amer. Math. Soc.*

[TVa] Thiem, N; Vinroot, C.R. "On the characteristic map of finite unitary groups," to appear in *Adv. Math.*

[TVb] Thiem, N; Vinroot, C.R. "Gelfand-Graev characters of the finite unitary group," arXiv math.RT/0611010, 2006.

[Ya] Yan, N. *Representation theory of the finite unipotent linear groups*, Unpublished Ph.D. Thesis, Department of mathematics, Pennsylvania State Unversity, 2001.

[Yo] Yokonuma, T. "Sur le commutant d'une reprsentation d'un groupe de Chevalley fini. II." *J. Fac. Sci. Univ. Tokyo Sect. I* **16** (1969), 65-81.