MATH 171 
Spring 2017 
Solutions to the Midterm Problems 

Problem 1: 

a) Define what it means for a sequence in a metric space $M$ to be a Cauchy sequence. Prove that, if a Cauchy sequence has a subsequence which converges to a point $x \in M$, then the entire Cauchy sequence converges to $x$.

soln: A sequence is a Cauchy sequence if, for every $\epsilon > 0$, there is an $N$ so that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Suppose a subsequence $x_{n_i}$ converges to $x$. Given $\epsilon > 0$ there is an $N_1 > 0$ so that $d(x_{n_i}, x) < \frac{\epsilon}{2}$ for all $i \geq N_1$. There is also an $N_2$ so that $d(x_n, x_m) < \frac{\epsilon}{2}$ for all $n, m \geq N_2$. Let $N$ equal the larger of $N_1, N_2$. Then, for any $m \geq N$, choose $i \geq N$. Since $\{x_{n_i}\}$ is a subsequence, $n_i \geq i \geq N$. By the triangle inequality $d(x_m, x) \leq d(x_m, x_{n_i}) + d(x_{n_i}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So the entire sequence converges to $x$.

b) Define what it means for a subset $X$ of a metric space $M$ to be complete. Prove that, if $M$ is a complete metric space and $X \subset M$ is a closed subset of $M$, then $X$ is complete. You may use properties of closed sets that you know as long as you state them clearly.

soln: A subset $X \subset M$ is complete if every Cauchy sequence $\{x_n\}, x_n \in X$ converges, with limit $x \in X$. If $M$ is complete, the sequence converges with limit $x \in M$. Since $X$ is closed, it contains all its limit points, so $x \in X$.

Problem 2: 

Let $M_1$ and $M_2$ be two metric spaces and let $f$ be a map from $M_1$ to $M_2$.

a) State what it means for $f$ to be continuous at a point $x \in M_1$, both in terms of the $\epsilon - \delta$ definition and in terms of convergent sequences.

soln: The $\epsilon - \delta$ definition says that $f$ is continuous at $x \in M_1$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ so that $d_{M_2}(f(x), f(y)) < \epsilon$ whenever $d_{M_1}(x, y) < \delta$. The convergent sequence definition for continuity is that $f$ is continuous at $x \in M_1$ if, for every $x_n \to x$ in $M_1$, we have $f(x_n) \to f(x)$ in $M_2$.

b) Show that $f$ is continuous if and only if for all subsets $X$ of $M_1$, we have $f(X) \subset \overline{f(X)}$. (Hint: It might be easier to prove the contrapositive of the “if” direction.) Give an example of a continuous $f$ and an $X$ so that the containment is strict; i.e., $f(X) \neq \overline{f(X)}$. 

Define the taxi cab and sup norms on $\mathbb{R}^\infty$. Denote by $l^1, l^\infty$ the metric spaces whose elements are vectors in $\mathbb{R}^\infty$ for which the respective norms are finite and where the metric is induced by the norms.

**Problem 3:**

a) Suppose that a subset $X \subset M$ of a metric space $M$ is connected. Prove that the closure $\overline{X} \subset M$ is connected.

**soln:** If $\overline{X}$ were not connected, there would be open sets, $U, V \subset M$ so that $\overline{X} \subset U \cup V$, $\overline{X} \cap U \neq \emptyset$, $\overline{X} \cap V \neq \emptyset$, and $U \cap V \cap \overline{X} = \emptyset$. Since $X \subset \overline{X}$, we would also have $X \subset U \cup V$ and $U \cap V \cap X = \emptyset$. Furthermore, $X \cap U \neq \emptyset$. To see this, if $x \in \overline{X}$, there is $\{x_n\}$, $x_n \in X$ converging to $x$. If $x \in \overline{X} \cap U$, there is an open ball around $x$ contained in $U$ ($U$ is open), which thus contains all of the elements in the sequence, for large enough $n$. Similarly, $X \cap V \neq \emptyset$. This implies that $X$ is not connected, a contradiction.

b) Let $M$ be a compact metric space. Suppose that $U_k$ is a “growing sequence” of open subsets of $M$, i.e., $U_k \subset U_{k+1}$ for all $k$, and suppose further that $U_k \neq M$ for any $k$. Prove that $\bigcup_{k=1}^\infty U_k \neq M$.

**soln:** If $\bigcup_{k=1}^\infty U_k = M$, the $\{U_k\}$ would be an open cover for $M$. Since $M$ is compact, there would be a finite subcover. Let $K$ be the largest index for the finite set of $\{U_k\}$ in the subcover. Since $U_k \subset U_{k+1}$ for all $k$ the union of the elements in the subcover simply equals $U_K$. But $U_K \neq M$ by hypothesis so we obtain a contradiction.

**Problem 4:** Define the taxi cab and sup norms on $\mathbb{R}^\infty$. Denote by $l^1, l^\infty$ the metric spaces whose elements are vectors in $\mathbb{R}^\infty$ for which the respective norms are finite and where the metric is induced by the norms.

**soln:** If $x = \{x_1, x_2, x_3, \cdots\} \in \mathbb{R}^\infty$, $\|x\|_{\inf} = \sup_{i=1,2,\ldots} |x_i|$ and $\|x\|_1 = \sum_{i=1}^\infty |x_i|$.

a) Prove that, in $l^1$, convergence of vectors implies coordinate-wise convergence.

**soln:** If $\{x^{(n)}\}$ is a sequence in $l^1$ converging to $x \in l^1$, then for any $\varepsilon > 0$, there is an $N$ so that $\|x - x^{(n)}\| = \sum_{i=1}^\infty |x_i - x_i^{(n)}| < \varepsilon$ for all $n \geq N$. But, for any $i$, $|x_i - x_i^{(n)}| \leq \|x - x^{(n)}\|$ so each of the coordinate sequences converges.
b) Consider the sequence of vectors in $\mathbb{R}^\infty$ defined by \( \frac{1}{S_n} (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots) \) where \( S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \). (In other words, the vector entries are the first \( n \) elements of the harmonic sequence, divided by the \( n \)th partial sum of the harmonic series, followed by all zeroes.) Decide whether or not this sequence converges in $l^\infty$, in $l^1$. If so, find the limit; if not, explain why not. In either case, prove your answer.

soln: The first coordinate of each of the vectors in the sequence equals \( \frac{1}{S_n} \). Since \( S_n \to +\infty \), this coordinate converges to 0. But \( |x^{(n)}_{i+1}| \leq |x^{(n)}_i| \) for each \( n \) and all \( i \), so the sup of the coordinates goes to 0. Hence, in $l^\infty$, the sequence converges to 0. In particular, it converges coordinate-wise to 0. So, if the sequence were to converge in $l^1$, it would have to converge to 0 there also, by part a). But the norm of each of the vectors in the sequence is 1, so this is impossible.

Problem 5:

Consider the metric space $M$ whose points are the subset \([1, \infty) \subset \mathbb{R}\) of the reals but whose non-standard metric is defined by: \( d(x, y) = |\frac{1}{x} - \frac{1}{y}| \).

a) Show that the subset $X$ of \([1, \infty)\) consisting of the positive integers, $X = \{1, 2, 3, \ldots\}$, is closed and bounded but not compact with respect to this metric. You may use, without proof, any facts about compactness or about closed sets that you know as long as they are clearly stated.

soln: The set $X$ is bounded because \( d(n, m) = |\frac{1}{n} - \frac{1}{m}| \leq 2 \) since \( n, m \geq 1 \). (Note that this actually shows that all of $M$ is bounded.) To show that $X$ is closed, we will show that any sequence $x_n \in X$ that converges in $M$ is eventually constant, so the limit will again be in $X$. In general, if a sequence \( \{y_n\} \), where $y_n \in M$, converges to $y \in M$, we see that the sequence, viewed in $\mathbb{R}$ with its usual metric, is bounded above. This is because, if $y_n > 2y$, we have that \( d(y, y_n) = |\frac{1}{y} - \frac{1}{y_n}| > |\frac{1}{y} - \frac{1}{2y}| = |\frac{1}{2y}| \). Choosing $\epsilon < |\frac{1}{2y}|$, we conclude that $y_n \leq 2y$ for sufficiently large $n$. But \( |\frac{1}{y} - \frac{1}{y_n}| = \frac{|y_n - y|}{y y_n} \geq \frac{|y_n - y|}{2y^2} \). Since $y$ is fixed, convergence to $y$ in the metric on $M$ implies convergence in $\mathbb{R}$ with its usual metric. But distinct integers are distance at least 1 apart in the usual metric, so the only convergent sequences in $X$ are eventually constant as claimed. The sequence $x_n = n$ is unbounded with the usual metric, as is every subsequence, so it has no convergent subsequences. Hence, $X$ is not compact.

b) Prove that $M = [1, \infty)$ with this metric is not complete.

soln: The sequence $x_n = n$ is Cauchy because for all $n, m \geq N$ we have \( d(n, m) \leq \frac{1}{N} \). Given $\epsilon > 0$ we can choose $N$ so that $N > \frac{1}{\epsilon}$. However, we saw in part a) that the sequence doesn’t converge; in fact, no subsequence of it converges. Thus M is not complete.