Homework 1 Solutions
(If you find any errors, please send an e-mail to farana at stanford dot edu)

Problem 1

Let \((M, d)\) be a metric space. Prove that

\[
    d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \\
    d''(x, y) = \min\{d(x, y), 1\}
\]

define metrics on \(M\). Prove that \(d'\) and \(d''\) are bounded by 1.

Solution. We first study the properties of \(d'\):

1. As \(d\) takes values in \([0, \infty)\), given that it is a metric, it is clear that \(d'\) takes values in \([0, \infty)\).

2. Notice that \(d'(x, y) = 0\) if and only if \(d(x, y) = 0\) if and only if \(x = y\), the last equivalence because \(d\) is a metric.

3. As \(d\) is symmetric, given that it is a metric, it is clear that \(d'\) is symmetric.

4. Let \(x, y, z \in M\) be arbitrary. As \(d\) is a metric we know that \(d(x, z) \leq d(x, y) + d(y, z)\). Using the fact that the map \(x \mapsto 1/(1 + x)\) is increasing for \(x \in [0, \infty)\) we have:

\[
    d'(x, z) = \frac{d(x, z)}{1 + d(x, z)} \\
    \leq \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\
    = \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\
    \leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\
    = d'(x, y) + d'(y, z)
\]

5. For any \(x, y \in M\) we have:

\[
    d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, y)}{d(x, y)} = 1
\]

We now study the properties of \(d''\):

1. As \(d\) takes values in \([0, \infty)\), given that it is a metric, it is clear that \(d''\) takes values in \([0, \infty)\).
2. Notice that \( d''(x, y) = 0 \) if and only if \( d(x, y) = 0 \) if and only if \( x = y \), the last equivalence because \( d \) is a metric.

3. As \( d \) is symmetric, given that it is a metric, it is clear that \( d' \) is symmetric.

4. Let \( x, y, z \in M \) be arbitrary. As \( d \) is a metric we know that \( d(x, z) \leq d(x, y) + d(y, z) \). Notice then that:

\[
\begin{align*}
  d''(x, z) &= \min\{d(x, z), 1\} \\
  &\leq \min\{d(x, y) + d(y, z), 1\} \\
  &\leq \min\{d(x, y), 1\} + \min\{d(y, z), 1\} \\
  &= d''(x, y) + d''(y, z)
\end{align*}
\]

5. For any \( x, y \in M \) we have:

\[
  d''(x, y) = \min\{d(x, y), 1\} \leq 1
\]

**Problem 2**

Let \( H^\infty \) denote the set of all real sequences \( \{a_n\} \) such that \( |a_n| \leq 1 \) for every positive integer \( n \). \( H^\infty \) is called the Hilbert cube.

(a) Let \( \{a_n\}, \{b_n\} \in H^\infty \). Prove that the series

\[
\sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}
\]

converges.

**Solution.** We compute:

\[
\sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n} \leq \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{2^n} \leq \sum_{n=1}^{\infty} \frac{2}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} < \infty
\]

where in the last line we used the fact that the geometric series of \( 1/2 \) converges.

(b) Prove that

\[
d(\{a_n\}, \{b_n\}) = \sum_{i=1}^{\infty} \frac{|a_n - b_n|}{2^n}
\]

defines a metric on \( H^\infty \).

**Solution.** We study the properties of \( d \):

(a) It is clear that \( d \) takes values in \([0, \infty)\). The values of \( d \) are finite because of Part (a).
(b) Notice that \( d(x, y) = 0 \) if and only if \( |a_n - b_n| = 0 \) for all \( n \in \mathbb{N} \) if and only if \( a_n = b_n \) for all \( n \in \mathbb{N} \) if and only if \( \{a_n\} = \{b_n\} \) in \( H^\infty \).

(c) It is clear that \( d \) is symmetric.

(d) Let \( \{a_n\}, \{b_n\}, \{c_n\} \in H^\infty \) be arbitrary. We compute:

\[
d(\{a_n\}, \{c_n\}) = \sum_{i=1}^{\infty} \frac{|a_n - c_n|}{2^n} \\
\leq \sum_{i=1}^{\infty} \frac{|a_n - b_n| + |b_n - c_n|}{2^n} \\
= \sum_{i=1}^{\infty} \frac{|a_n - b_n|}{2^n} + \sum_{i=1}^{\infty} \frac{|b_n - c_n|}{2^n} \\
= d(\{a_n\}, \{b_n\}) + d(\{b_n\}, \{c_n\})
\]

Problem 3

Let \( H^\infty \) be the Hilbert cube as defined in Problem 2 (Exercise 35.9).

i) Consider a sequence \( \{x^{(n)}\} \) of points in \( H^\infty \). Show that \( \{x^{(n)}\} \) converges to \( x \in H^\infty \) if and only if it converges coordinate-wise, i.e. if and only if, for each \( i \), the sequence of real numbers \( x^{(n)}_i \) which are the \( i \)th coordinates of \( x^{(n)} \) converges to the \( i \)th coordinate of \( x \).

**Solution.** Suppose first that \( \{x^{(n)}\} \to x \) when \( n \to \infty \). It follows that \( d(x^{(n)}, x) \to 0 \) when \( n \to \infty \). Notice that this implies \( |x^{(n)}_i - x_i| \to 0 \) when \( n \to \infty \) for all \( i \in \mathbb{N} \), else \( d(x^{(n)}, x) \) would be bounded below by some positive constant. It follows that \( x^{(n)}_i \to x_i \) when \( n \to \infty \) for all \( i \in \mathbb{N} \).

Suppose now that \( x^{(n)}_i \to x_i \) when \( n \to \infty \) for all \( i \in \mathbb{N} \). We know that the series \( \sum_{i=1}^{\infty} 1/2^i \) is convergent, and so its tail satisfies \( \sum_{i=m}^{\infty} 1/2^i \to 0 \) when \( m \to \infty \). Let \( \epsilon > 0 \) be arbitrary. Let \( m \in \mathbb{N} \) be big enough so that \( \sum_{i=m+1}^{\infty} 1/2^i \leq \epsilon/4 \). As \( \{1, \ldots, n-1\} \) is finite and \( x^{(n)}_i \to x_i \) when \( n \to \infty \) for all \( i \in \{1, \ldots, n-1\} \) we can find \( N \in \mathbb{N} \) such that \( |x^{(n)}_i - x_i| \leq 2^n \epsilon/2m \) for all \( n \geq N \) and all \( i \in \{1, \ldots, n-1\} \). Notice then that for all \( n \geq N \) we have:

\[
d(\{x^{(n)}\}, \{x\}) = \sum_{i=1}^{\infty} \frac{|x^{(n)}_i - x_i|}{2^n} \\
= \sum_{i=1}^{m} \frac{|x^{(n)}_i - x_i|}{2^n} + \sum_{i=m+1}^{\infty} \frac{|x^{(n)}_i - x_i|}{2^n} \\
\leq \sum_{i=1}^{m} \frac{\epsilon}{2m} + 2 \cdot \sum_{i=m+1}^{\infty} \frac{1}{2^n} \\
\leq \frac{\epsilon}{2} + 2 \cdot \frac{\epsilon}{2} \\
\leq \epsilon
\]
It follows that \(d(x^{(n)}, x) \to 0\) when \(n \to \infty\). We conclude that \(\{x^{(n)}\} \to x\) when \(n \to \infty\).

ii) Consider a sequence \(\{x^{(n)}\}\) of points in \(H^\infty\). Show that it has a convergent subsequence (limit in \(H^\infty\)).

\textbf{Solution.} Notice that the sequence \(\{x^{(n)}_i\}\) indexed by \(i \in \mathbb{N}\) is bounded for all \(i \in \mathbb{N}\). The Bolzano-Weierstrass theorem in \([0, 1]\) allows us to find convergent subsequences of these sequences. But we want to be careful on how we obtain such convergent subsequences. The argument that follows is called \textit{diagonal argument} and is used multiple times in real analysis. Start with \(i = 1\). Using the Bolzano-Weierstrass theorem in \([0, 1]\) we find a sequence \(n^{(1)}_k \nearrow \infty\) of integers such that \(x^{(n^{(1)}_k)}_1 \to x_1\) when \(k \to \infty\) for some \(x_1 \in [0, 1]\). Now we use the Bolzano-Weierstrass theorem in \([0, 1]\) over the sequence \(x^{(n^{(1)}_k)}_2\) to find a subsequence \(n^{(2)}_k \nearrow \infty\) of \(n^{(1)}_k\) such that \(x^{(n^{(2)}_k)}_2 \to x_2\) when \(k \to \infty\) for some \(x_2 \in [0, 1]\). We proceed inductively in this way for all \(i \in \mathbb{N}\), finding subsequences \(n^{(i+1)}_k \nearrow \infty\) of \(n^{(i)}_k\) such that \(x^{(n^{(i+1)}_k)}_i \to x_i\) when \(k \to \infty\) for some \(x_{i+1} \in [0, 1]\). Now we consider the sequence of integers \(\{n^{(k)}_k\}\). Notice that \(\{n^{(k)}_k\}\) eventually becomes a subsequence of all of the sequences \(\{n^{(i)}_k\}\). In particular we have \(x^{(n^{(k)}_k)}_i \to x_i\) for all \(i \in \mathbb{N}\). Letting \(x = \{x_i\}\) we see that \(x \in H^\infty\) and by Part i) that \(x^{(n^{(k)}_k)} \to x\) when \(k \to \infty\).

\textbf{Problem 4}

Show that the subspaces \(\ell^1, \ell^2 \subseteq \mathbb{R}^\infty\) satisfy the inclusion \(\ell^1 \subseteq \ell^2\). Show, by finding an example, that the containment is strict. Find a sequence of points \(\{x^{(n)}\} \in \ell^1 \subseteq \ell^2\) that converge in \(\ell^2\) but do not converge in \(\ell^1\).

\textbf{Solution.} Let \(\{a_n\} \in \ell^1\). In particular we have \(a_n \to 0\) when \(n \to \infty\). Then there exists \(N \in \mathbb{N}\) such that \(|a_n| \leq 1\) for all \(n \geq N\). Now we compute:

\[
\sum_{i=1}^{\infty} |a_n|^2 = \sum_{i=1}^{N-1} |a_n|^2 + \sum_{i=N}^{\infty} |a_n|^2 \leq \sum_{i=1}^{N-1} |a_n|^2 + \sum_{i=N}^{\infty} |a_n| < \infty
\]

We conclude that \(\{a_n\} \in \ell^2\). This proves the inclusion \(\ell^1 \subseteq \ell^2\).

The containment is strict because the sequence \(\{a_n\}\) given by \(a_n = 1/n\) belongs to \(\ell^2\) but not to \(\ell^1\) as was seen in Homework 2.

We construct \(\{x^{(n)}\}\) a sequence of elements in \(\ell^1\) that converge in \(\ell^2\) but do not converge in \(\ell^1\). Let \(x^{(n)} \in \ell^1\) be given by \(x^{(n)}_k = 1/k\) if \(k \leq n\) and \(x^{(n)}_k = 0\) if \(k \geq n + 1\). Let \(x \in \ell^2\) be given by \(x_k = 1/k\) for all \(k \in \mathbb{N}\). Notice that \(x^{(n)} \to x\) in \(\ell^2\) when \(n \to \infty\) because the series \(\sum_{k \in \mathbb{N}} 1/k^2\) is...
convergent. On the other hand \( x^{(n)} \not\to x \) in \( \ell^1 \) because the series \( \sum_{k\in\mathbb{N}} 1/k \) diverges to \(+\infty\).

Problem 5

Let \( \{a_n\} \in \ell^1 \) and \( \{b_n\} \in \ell^\infty \). Prove that \( \{a_nb_n\} \in \ell^1 \).

Solution. As \( \{b_n\} \in \ell^\infty \) there exists \( M > 0 \) such that \( |b_n| \leq M \) for all \( n \in \mathbb{N} \). Using this fact and the fact \( \{a_n\} \in \ell^1 \) we get:

\[
\sum_{n=1}^{\infty} |a_nb_n| = \sum_{n=1}^{\infty} |a_n||b_n| \leq \sum_{n=1}^{\infty} |a_n|M = M \sum_{n=1}^{\infty} |a_n| < \infty
\]

We conclude that \( \{a_nb_n\} \in \ell^1 \).

Problem 6

Let \( \{a_n\} \) be a sequence such that \( \{a_nb_n\} \in \ell^1 \) for every sequence \( \{b_n\} \in \ell^1 \). Prove that \( \{a_n\} \in \ell^\infty \). Show (by example) that the above statement is false if \( \ell^\infty \) is replaced by \( c_0 \).

Solution. Suppose by contradiction that \( \{a_n\} \not\in \ell^\infty \). Then we can find a sequence \( n_k \nearrow \infty \) such that \( |a_{n_k}| > 2^k \). Let \( \{b_n\} \) be given by \( b_{n_k} = 1/2^k \) for all \( k \in \mathbb{N} \) and 0 elsewhere. Notice that \( \{b_n\} \in \ell^1 \). Now we compute:

\[
\sum_{n=1}^{\infty} |a_nb_n| \geq \sum_{k=1}^{\infty} |a_{n_k}b_{n_k}| = \sum_{k=1}^{\infty} |a_{n_k}| \frac{1}{2^k} > \sum_{k=1}^{\infty} 1 = \infty
\]

This contradicts the fact \( \{a_nb_n\} \in \ell^1 \). We conclude that \( \{a_n\} \in \ell^\infty \).

It is straightforward to check that the sequence \( \{a_n\} \) given by \( a_n = 1 \) for all \( n \in \mathbb{N} \) provides a counterexample for the claim obtained when \( \ell^\infty \) is replaced with \( c_0 \).

Problem 7

Let \((M,d)\) be a metric space, and let \(d'\) and \(d''\) be defined as in Exercise 35.7 (Problem 1). Let \( \{a_n\} \) be a sequence in \( M \) and let \( a \in M \). Prove that the following statements are equivalent.

(a) \( \{a_n\} \) converges to \( a \) in \((M,d)\).

(b) \( \{a_n\} \) converges to \( a \) in \((M,d')\).

(c) \( \{a_n\} \) converges to \( a \) in \((M,d'')\).
Solution. Suppose that \( a_n \to a \) when \( n \to \infty \) in \( (M,d) \). It follows that \( d(a_n,a) \to 0 \) when \( n \to \infty \). Directly from the definition of \( d' \) we see that \( d'(a_n,a) \to 0 \) when \( n \to \infty \). It follows that \( a_n \to a \) when \( n \to \infty \) in \( (M,d') \). Suppose now that \( a_n \to a \) when \( n \to \infty \) in \( (M,d') \). It follows that \( d'(a_n,a) \to 0 \) when \( n \to \infty \). Directly from the definition of \( d' \) we see that this can only happen if \( d(a_n,a) \to 0 \) when \( n \to \infty \). It follows that \( a_n \to a \) when \( n \to \infty \) in \( (M,d) \).

Problem 8

Let \( X \) be a subset of a metric space \( M \). We say that a point \( x \in M \) is an accumulation point of \( X \) if there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} x_n = x \) and \( x_n \neq x \) for every positive integer \( n \). We let \( X^a \) denote the set of accumulation points of \( X \).

(a) Prove that \( X \) is closed if and only if \( X^a \subseteq X \).

Solution. It is straightforward to check from the definitions that the set of limit points of \( X \) is exactly the union of \( X \) with its accumulation points. It follows at once that \( X \) is closed if and only if every limit point of \( X \) belongs to \( X \) if and only if \( X^a \subseteq X \).

(b) Prove the following form of the Bolzano-Weierstrass theorem: If \( X \) is a bounded infinite subset of \( \mathbb{R} \), the \( X^a \neq \emptyset \).

Solution. As \( X \) is infinite we can construct a sequence \( \{a_n\} \) of distincts points of \( X \). As this sequence is bounded, the usual Bolzano-Weierstrass theorem gives us a sequence of integers \( n_k \not\to \infty \) such that \( a_{n_k} \to a \) for some \( a \in \mathbb{R} \). Notice that \( a \) could potentially equal some of the \( a_{n_k} \). But as the \( a_{n_k} \) are all different, this happens only once, and so we can consider the sequence \( a_{n_k} \) after that moment to see that \( a \in X^a \). We conclude that \( X^a \neq \emptyset \).

Problem 9

Let \( X \subseteq M \) be any subset of a metric space \( M \).

i) Show that the interior \( X^o \) of \( X \) equals the union of all open sets \( Y \subseteq M \) such that \( Y \subseteq X \).

Solution. Consider first \( x \in X^o \). Then \( B_\epsilon(x) \subseteq X \) for some \( \epsilon > 0 \). Now \( B_\epsilon(x) \) is an open set of \( M \) such that \( B_\epsilon(x) \subseteq X \), so it is contained in the union in consideration. As \( x \in B_\epsilon(x) \),
$x$ belongs to the union in consideration. Suppose now that $x$ belong to the union in consideration. The $x \in Y$ for some open set $Y$ of $M$ such that $Y \subseteq X$. As $Y$ is open and $x \in Y$ it follows that $B_\epsilon(x) \subseteq Y$ for some $\epsilon > 0$. But then $B_\epsilon(x) \subseteq Y \subseteq X$ so that by definition $x \in X^\circ$. We conclude that $X^\circ$ equals the union of all open sets $Y \subseteq M$ such that $Y \subseteq X$.

ii) Show that the closure $\overline{X}$ of $X$ equals the intersection of all closed sets $C \subseteq M$ such that $X \subseteq C$.

**Solution.** Consider first $x \in \overline{X}$. Then there exists a sequence $\{x_n\}$ of points of $X$ such that $x_n \to x$ when $n \to \infty$. Let $C$ be a closed set of $M$ such that $X \subseteq C$. Notice that $x_n$ are points of $C$ and $x_n \to x$ when $n \to \infty$. As $C$ is closed it follows that $x \in C$. We see then that $x$ belongs to the intersection in consideration. We now want to show that the intersection in consideration is contained in $\overline{X}$. Notice that we would be done if we are able to show that $\overline{X}$ is closed. Let $x \in M$ be a limit point of $\overline{X}$, i.e. there exists a sequences $x^{(i)}$ of points of $\overline{X}$ such that $x^{(i)} \to x$ when $i \to \infty$. As $x^{(i)} \in \overline{X}$ we can find sequences $x_n^{(i)}$ for all $i \in N$ such that $x_n^{(i)} \to x^{(i)}$ when $n \to \infty$. From this sequences we can build a sequence of points of $X$ converging to $x$. Let $\epsilon_k = 1/k$. As $x^{(i)} \to x$ when $i \to \infty$ we can find $i_k \in N$ such that $d(x^{(i_k)}, x) \leq \epsilon_k/2$. As $x_n^{(i_k)} \to x^{(i_k)}$ when $n \to \infty$ we can find $n_k \in N$ such that $d(x_n^{(i_k)}, x_{n_k}) \leq \epsilon_k/2$. The triangle inequality then shows that $d(x, x_{n_k}) \leq \epsilon_k$. Let $x_k = x_{n_k}$. Notice that $x_k \in X$ for all $k \in N$. From the construction we see that $d(x, x_k) = \epsilon_k = 1/k \to 0$ when $k \to \infty$, i.e. $x_k \to x$ when $k \to \infty$. It follows that $x \in \overline{X}$. We conclude that $\overline{X}$ is closed. From this we deduce that the intersection in consideration is contained in $\overline{X}$. We conclude that $\overline{X}$ equals the intersection of all closed sets $C \subseteq M$ such that $X \subseteq C$.

**Problem 10**

If $X$ is a subset of a metric space $M$, we define the *boundary* of $X$ to be the set $\partial X = \overline{X} \cap (X')^c$. Let $M$ be a metric space. Prove the following:

(a) $\partial X$ is closed for all $X, X \subseteq M$.

**Solution.** By Problem 9 we know that $\overline{X}$ and $(X')^c$ are closed. It follows that $\partial X = \overline{X} \cap (X')^c$ must be closed.

(b) $X \cup \partial X = \overline{X}$ for all $X, X \subseteq M$.

**Solution.** From the definitions we see that $X \cup \partial X \subseteq \overline{X}$. Now let $x \in \overline{X}$. We want to show that $x \in X \cup \partial X$. If $x \in X$ we are done. Suppose that $x \notin X$. We want to show that $x \in \partial X = \overline{X} \cap (X')^c$. As $x \in \overline{X}$ this amounts to showing that $x \in (X')^c$. Suppose by contradiction that this was not the case. Then we can find $\epsilon > 0$ such that $B_\epsilon(x) \cap X' = \emptyset$. This is equivalent to $B_\epsilon(x) \subseteq X$. But then $x \in X^\circ \subseteq X$ contradicting the assumption $x \notin X$. 

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It follows that $X \subseteq X \cup \partial X$. We conclude that $X \cup \partial X = \overline{X}$.

(c) $X \setminus \partial X = X^\circ$ for all $X$, $X \subseteq M$.

**Solution.** Let $x \in X \setminus \partial X$. As $\partial X = \overline{X} \cap (X')^-$ this is equivalent to $x \in X$ and $x \notin \cap (X')^-$. It follows that we can find $\varepsilon > 0$ such that $B_\varepsilon(x) \cap X' = \emptyset$. This is equivalent to $B_\varepsilon(x) \subseteq X'$. It follows that $x \in X^\circ$. This shows that $X \setminus \partial X \subseteq X^\circ$. Now let $x \in X^\circ$. We want to show that $x \in X \setminus \partial X$. As $\partial X = \overline{X} \cap (X')^-$ this amounts to showing that $x \notin (X')^-$. As $x \in X^\circ$ we can find $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X$. Notice then that $x \notin (X')^-$ as any sequence of points of $X'$ converging to $x$ must eventually fall inside $B_\varepsilon(x) \subseteq X$. This shows that $X^\circ \subseteq X \setminus \partial X$. We conclude that $X \setminus \partial X = X^\circ$.

(d) If $X$ is a proper nonempty subset of $\mathbb{R}$, then $\partial X \neq \emptyset$.

**Solution.** Let $X$ be a proper nonempty subset of $\mathbb{R}$. Suppose by contradiction that $\partial X = \emptyset$. By Part (b) we must have $\overline{X} = X \cup \partial X = X$, so that $X$ is closed. By Part (c) we must have $X^\circ = X \setminus \partial X = X$, so that $X$ is open. It follows that $X \subseteq \mathbb{R}$ is open and closed. The only open and closed subsets of $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$ (we will later call this property *connectedness*). As we are assuming $X$ be a proper nonempty subset of $\mathbb{R}$ we reach a contradiction.

Since we have not mentioned the notion of connectedness in class, nor proved that $\mathbb{R}$ is connected, here is a proof that doesn’t use this property. Suppose $X$ has no limit points; we will get a contraction. Let $x \in X$. By part (c), if it is not an interior point of $X$, the boundary is not empty and we are done. So we can assume that all points of $X$ are interior points. Reversing the roles of $X$ and its complement, we can assume that every point of $X'$ is an interior point of $X'$. Let $x \in X$. Consider the set of points $b \in \mathbb{R}$ such that there is an open interval of the form $(a, b) \subset X$ with $a < x < b$. Denote it by $Z$. Since $x$ is an interior point, it is nonempty. Suppose $Z$ is bounded above and let lub($Z$) = $c$. If $c \in X$, it can’t be an interior point of $X$ since that would imply that $c$ is not an upper bound for $Z$. If $c \in X'$ it can’t be an interior point of $X'$ or $Z$ would have a smaller upper bound. So $Z$ is not bounded above and $[x, \infty) \subset \mathbb{R}$. Similarly consider the set of points $a \in \mathbb{R}$ such that there is an open interval of the form $(a, b) \subset X$ with $a < x < b$. If it were bounded below, $X$ would have a boundary point. So $(-\infty, x] \subset X$. But this implies that $X = \mathbb{R}$, a contradiction.