Problem 1:

a) What is the value of \( \int_{\gamma} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta \) for \( f(z) = \frac{(2z - 1)^7}{z^3} \) where \( \gamma \) is the unit circle, oriented counter-clockwise? Clearly state the general result you are using.

b) Decide how many solutions (counted with multiplicity) there are to the equation \( 3z^4 - z^3 + 6z^2 + 1 = 0 \) in the region \( 1 < |z| < 2 \). Explain your answer.

Problem 2:

a) Let \( h(z) = \frac{z}{z - \sin z} \). Show that \( h(z) \) has a pole of order 2 at \( z = 0 \) and compute the coefficient \( a_{-2} \) (where \( h(z) = \sum_{j=-2}^{\infty} a_j z^j \)). Expand the equation \( h(z)(z - \sin z) = z \) into power series and solve for \( a_{-1}, a_0 \) by matching terms. Why is this computation justified?

b) Compute the following integral. Justify your steps.
\[
\int_{-\infty}^{+\infty} \frac{x^2}{(1 + x^2)^2} \, dx.
\]

Problem 3:

a) Let \( f(z) \) be an entire holomorphic function. Suppose the real part \( \text{Re} f(z) \) is bounded on all of \( \mathbb{C} \). Show that \( f(z) \) is constant. Carefully state any theorems you are using. Hint: What happens when you compose with the exponential function?

b) Let \( f(z) \) be holomorphic and non-constant in a region \( \Omega \) and suppose that \( f'(z_0) = 0 \) at some \( z_0 \in \Omega \). Let \( w_0 = f(z_0) \). Show that the equation \( f(z) = w \) has more than one solution (counted with multiplicity) for points \( z \) sufficiently close to \( z_0 \) and \( w \) sufficiently close to \( w_0 \). Hint: What is the multiplicity of the zero of \( f(z) - w_0 \) at \( z = z_0 \)?

Problem 4:

a) Define what it means for a holomorphic function to have a pole at the point at infinity.

b) Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire holomorphic function from the complex plane to itself with a pole at infinity. Prove that \( f \) is a polynomial. You may not use the general theorem that a meromorphic function on the extended complex plane is a rational function. You are being asked to prove a special case of that theorem.