Problem 1 Solution  We know that
\[ \frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \]
Plug in \( z = \frac{1}{2} \). Since that is not a zero of \( \sin \pi z \), we can invert all the factors to get
\[ \frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{n^2 - \frac{1}{4}}{n^2 - \frac{1}{4}} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} \]

Problem 2 Solution
1. Define \( \log(1 + z) = \sum_{n=1}^{\infty} (\frac{-1}{n})^{n+1} z^n \). Such definition gives the desired logarithmic property and it is analytic on the unit disk. Note that \( \prod (1 + a_n) \) converges to nonzero value if and only if \( \sum \log(1 + a_n) \) converges. So now we are studying the convergence of \( \sum \log(1 + a_n) \) and \( \sum a_n \). We can write
\[ \sum \log(1 + a_n) = \sum (a_n - a_n^2/2 + O(a_n^3)) \]
Since \( \sum |a_n|^2 \) converges, we have that \( \lim \sum O(a_n^3) \) converges too. Hence \( \lim \sum (1 + a_n) \) converges if and only if \( \sum a_n \) does.

2. Let \( a_n = (-1)^n \frac{1}{\sqrt{n}} \) for \( n \geq 2 \). \( \sum a_n \) converges by alternating series. However since
\[ \sum \log(1 + a_n) = \sum (a_n - a_n^2/2 + O(a_n^3)) \]
and we have that \( \sum -a_n^2 \) diverges to \( \infty \) but \( \lim \sum O(a_n^3) \) and \( \sum a_n \) converge, \( \lim (1 + a_n) \) does not converge.

3. Let \( a_{2k} = \frac{1}{k} - \frac{1}{\sqrt{k}} \) and \( a_{2k+1} = \frac{1}{\sqrt{k}} \) for \( k \geq 2 \) and define \( a_0 = a_1 = a_2 = a_3 = 0 \). Then the series diverges since we are summing up \( \sum 1/k \). However the product will converge since the subsequence
\[ \prod_{n=1}^{\infty} (1 + a_n) = \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k} - \frac{1}{\sqrt{k}} \right) \left( 1 + \frac{1}{\sqrt{k}} \right) = \prod \left( 1 + \frac{1}{k^{1/2}} \right) \]
which converges since
\[ \sum \frac{1}{k^{3/2}} \]

Problem 3 Solution  For \( z = 0 \) the equality is obvious. Let \( z \neq 0 \) and \( a_N = \prod_{k=1}^{N} \cos(z/2^k) \). Then by trig identity we have
\[ \sin(\frac{z}{2N})a_N = \frac{\sin(z)}{2^N} \]
Thus if we can show that
\[ \frac{1}{z} = \lim_{N} \frac{1}{N} \frac{\prod_{k=1}^{N}}{\sin(\frac{z}{2^N})} \]
we are done. This is a standard L'Hopital argument, taking the derivative top and bottom we got the limit as
\[ \lim_{N} e^{-\log 2z} \frac{(-\log 2)}{[\cos(\frac{z}{2^N}) e^{-\log \frac{2z}{(\log 2)]} z]} = \frac{1}{z} \]
**Problem 4 Solution**  Let \( a_N = \prod_{k=0}^{N} (1 + z^{2^k}) \) then by the property \((1 - z)(1 + z) = 1 - z^2\) we have that
\[
a_N = \frac{1 - z^{2N}}{1 - z}
\]
Since \(|z| < 1\) we have that \(a_N \to \frac{1}{1-z}.\)

**Problem 5 Solution**  Let \( f \) be a meromorphic function. By the definition of meromorphic function, we have that its poles do not accumulate. Let \( \{ p_n \} \) be its poles, with the correct order. i.e if \( f(z) \) has a pole at \( z_0 \) of order \( k \) then there should be \( p_{j_k} = \cdots = p_{j_k} = z_0 \) and no other \( p \)'s with this value. By Weierstrass factor we can create an entire function \( g \) so that \( g \) vanishes on the poles of \( f \) with a multiplicity at a point equals to the order of the pole at this point. Thus \( f \cdot g \) is entire and \( f = \frac{f}{g}. \)

For the second part, use Weierstrass factor to create \( f \) and \( g \) that vanishes exactly on \( \{ a_n \} \) and \( \{ b_n \} \), then \( \frac{f}{g} \) is what we want.

**Problem 6 Solution**  Suppose \( f \) is locally injective, then for any \( z \in U \) there exists \( D \) so that \( z \in D \) and \( f \) is injective on \( D \). Thus by proposition 1.1 \( f' \) does not vanish on \( D \), in particular \( f'(z) \neq 0. \)
Since \( z \in U \) is arbitrary, we finish one direction.

Now suppose \( f'(z) \) is nowhere vanishing on \( U \). Fix \( z_0 \in U. \) At \( z_0 \) we can write \( f(z) \) as
\[
f(z) = f(z_0) + f'(z_0)(z - z_0) + g(z)
\]
where \( g(z) = O((z - z_0)^2). \)

Choose two small circle \( C_2 \) around \( z_0, C_2 \) inside of \( C_1 \), so that the following two statement hold simultaneously

1. \( \sup |f(z) - f(z_0)| := \lambda \) where the sup is taken inside of \( C_2 \).
2. For any complex \( w \) so that whenever \( |w| \leq \lambda \) we have \( |f'(z_0)(z - z_0) - w| > |g(z)| \) on \( C_1 \).

From (2) and Rouche, we have that \( f(z) - f(z_0) - w \) has only 1 zero inside of \( C_1 \). Thus for any \( w = f(z') - f(z_0) \) so that \( z' \) is inside of \( C_2 \), \( f(z) - f(z_0) - w \) has at most one zero. Thus \( f \) is injective inside of \( C_2 \)

**Problem 7 Solution**  By translation, we can assume \( z_0 = 0. \) Thus we can write \( F(z) = z^2(1 + G(z)) \) whereas \( G(z) = O(|z|^3). \) For a small enough neighborhood we can assume \( G(z) + 1 \) never vanishes, and we can define \( g(z) = z\sqrt{1 + G(z)} \). So \( g \) is holomorphic on this neighborhood and \( g'(z) = F \).

Since \( g'(0) = 0 \) would imply \( F''(0) = 0 \), we have that on a even smaller neighborhood of \( 0 \) we can make \( g'(0) \) nonzero, call this neighborhood \( B \). Then \( g \) is a conformal map from \( B \) to \( g(B) \). Since \( g(0) = 0 \) and \( g(B) \) is open, we can assume that there are small segments of \( X \) and \( Y \) axis intersecting at \( 0 \) and those two segments are contained in \( g(B) \). Call the segment on \( X \) axis \( L_1 \) and \( Y \) axis \( L_2 \).
Define
\[
\Gamma_n = g^{-1}(L_n)
\]
for \( n = 1, 2. \)
Thus \( F(\Gamma_1) = (L_1)^2 > 0 \) except for \( F(0) = 0. \) Similarly we have \( F(\Gamma_2) < 0 \) except for \( F(0) = 0. \)
\( \Gamma_n \) intersects orthogonally since \( L_1 \) and \( L_2 \) are orthogonal and \( g \) preserves angles.

**Problem 8 Solution**  Let \( F \) be a homeomorphism from \( U \) to \( V \). Let \( \gamma_1 \) and \( \gamma_2 \) be two curves in \( V \) with the same end points. Want to show they are homotopic.

Let \( \Gamma_n(x) = F^{-1}(\gamma_n(x)) \). Then we shall have a homotopy \( H(x, t) \) from \( \Gamma_1 \) to \( \Gamma_2 \).

Then define \( \Theta(x, t) \) so that
\[
\Theta(x, t) = F \circ H(x, t)
\]
Then \( \Theta(x, 0) = F \circ F^{-1}(\gamma_1) = \gamma_1 \) and similarly for \( t = 1 \). And since \( F \) is continuous \( \Theta \) is continuous.
Problem 9 Solution  We know that the unit disk is conformal equivalent to \( \mathbb{H} \), so we just have to find a surjection map from \( \mathbb{H} \) to \( \mathbb{C} \). Define \( D(z) = z - i \) and \( S(z) = z^2 \). I claim that \( S \circ D \) is surjective to \( \mathbb{C} \).

Indeed \( \mathbb{H} - i \) contains \( x \) axis. For any \( z \) we can write \( re^{i\theta} \) for \( \theta \in [0, 2\pi] \) and pick \( \sqrt{r}e^{i\theta/2} \) that squares to \( z \). Note that \( e^{i\theta/2} \) now is on the upper plane or \( x \) axis, thus we are done since the square map is surjective from \( \mathbb{H} \) to \( \mathbb{C} \).

Problem 10 Solution  Note that \( f \) indeed maps into the upper half plane since

\[
f(x + iy) = \frac{-1}{2}(x + iy + \frac{x - iy}{x^2 + y^2})
\]

so the imaginary part is

\[
\frac{(1 - y)y}{2(x^2 + y^2)}
\]

which is always positive.

To see that \( f \) is injective, let \( f(z_1) = f(z_2) \) for two distinct \( z_1, z_2 \), then

\[
f(z_1) = f(z_2) \iff z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2}
\]

\[
\iff z_1 - z_2 = \frac{z_1 - z_2}{z_1 z_2}
\]

\[
\iff z_1 z_2 = 1
\]

which is absurd since \( |z_1 z_2| < 1 \).

To see \( f \) is surjective, we can check that for \( w \in \mathbb{H} \), we have that \( f(z) = w \) is the same to solve

\[
z^2 + 2wz + 1 = 0
\]

For \( w \neq \pm 1 \), which is the case on \( \mathbb{H} \), we have that the quadratic has distinct two roots say \( z_1 \) and \( z_2 \).
Since \( z_1 + z_2 = -2w \) we have that one of the root has to have negative imaginary part, say \( z_2 \).

Write \( z_n = r_n e^{i\theta_n} \), since \( z_1 z_2 = 1 \) we have that

\[
z_1 = r_1 e^{i\theta} \quad z_2 = r_2 e^{-i\theta}
\]

where \( \theta \in (0, \pi) \) and \( r_1 r_2 = 1 \). Note that we have ruled out the case \( r_1, r_2 = 1 \), since that would result \( \theta = 0, \pi \) which means \( w = \pm 1 \).

Since \( z_1 + z_2 \) has imaginary part \( r_1 \sin \theta - r_2 \sin \theta \), and this number has to be negative, thus \( r_1 < r_2 \).
By the fact \( r_1 r_2 = 1 \) we have that \( z_1 \) is in the upper disk.