Problem 1 Solution  Let $\Omega$ define the region $\theta < \arg z < \phi$. Extend $f$ to $F$ defined on $\Omega$ in the following way

$$F(z) = \begin{cases} f(z) & |z| < 1 \\ 0 & |z| \geq 1 \end{cases}$$

The condition of the uniform convergence gives the uniform continuity of $F$. One can check that $F$ is analytic on $\Omega$ by Morera’s theorem. Indeed, for any triangle in $\Omega$, we only have to worry about those who intersects $\{|z| = 1\} \cap \Omega$. We can separate the triangle into two parts with the common boundary $\{|z| = 1\} \cap \Omega$. The part outside of $|z| < 1$ is zero everywhere; the other part has 0 integral since $f$ is holomorphic inside of the unit disk, and for each $\epsilon > 0$, there is a neighborhood of $\{|z| = 1\} \cap \Omega$, on which $|f| < \epsilon$. We see that the integral on the inner part will be less than $C \cdot \epsilon$, for some constant $C$ independent of $\epsilon$. Hence it must be zero.

One should also check the special case where the triangle intersect $\{|z| = 1\} \cap \Omega$ at one point. A similar argument as above will also apply. So, $F$ is 0 on $\Omega$ by analytic continuation, then so is $f$.

Problem 2 Solution  Consider the function

$$f(z) = \prod_{i=1}^{n}(z - w_i)$$

$f$ is a polynomial, thus holomorphic. $|f(0)| = 1$ thus by the maximal boundary principal there must be some point $w$ on the unit circle such that $|f(w)| \geq 1$. Otherwise $f$ achieves its maximal modulus inside of the disk, which means $f$ is constant, and this is absurd.

Note that $f(w_1) = 0$ on the boundary. Since $|f|$ on the boundary could be seen as continuous function from $[0, 2\pi]$ to $\mathbb{R}$ under the parametrization $z = re^{i\theta}$, thus by intermediate value theorem, $|f|$ will attain 1 somewhere on the circle.

Problem 3 Solution  For (a), let $\epsilon$ be small enough so that $f(z) + \epsilon g(z)$ never vanishes on $|z| = 1$. This is also true for any $\delta < \epsilon$. Consider the homotopy $f_t(z) = f(z) + (1 - t)\epsilon g(z)$. As $t$ varies, $f_t(z)$ is never zero on $|z| = 1$. Thus, if $C$ is the unit circle, the function $F$

$$F(t) = \frac{1}{2\pi i} \int_C \frac{f_t'}{f_t} dz$$

is a continuous function of $t$, which means that the winding number of $f_0$ will be 1, thus simple zero since it is holomorphic.

For the second part, suppose not, then there exists $\delta_n \to \delta$ where $\delta_n$ and $\delta$ are all positive number less than $\epsilon$. Furthermore we can choose all of $z_{\delta_n}$ have positive distance to $z_\delta$, say $\eta$. Let $C_{z_\delta}(\eta/2)$ be the circle centered at $z_\delta$ with radius $\eta/2$, then

$$F(\delta_n) = \frac{1}{2\pi i} \int_{C_{z_\delta}(\eta/2)} \frac{(f(z) + \delta_n g(z))'}{f(z) + \delta_n g(z)} = 0$$

for any $n$. However $F(t)$ is a continuous function for some $t$ on a small neighborhood $\delta$ such that the $f(z) + tg(z)$ never vanishes on $C_{z_\delta}(\eta/2)$. Thus $F(\delta_n) \to F(\delta) = 1$, which is a contradiction.
**Problem 4 Solution**  For (a), if $f$ is non-vanishing inside of $D$, then $f$ is constant by previous homework. Thus $f$ has a root inside of $D$.

Let $w_0 \in D$. Consider $g(z) = f(z) - w_0$. By Rouche’s Theorem, $g(z)$ has the same number of zeros as $f$.

**Problem 5 Solution**

1. $f(z)$ does not vanish on $|z| = 6$, since $6^8 > 5 \cdot 6^7 + 20$. By Rouche’s Theorem, $f(z)$ and $z^8$ have the same number of zeros inside the disk of radius 6, so $f(z)$ have 8 zeros there (including multiplicity).

2. When $|z| = 2$, $|z^3 - 3z^2| = 4|z - 3| > 1$. By Rouche’s Theorem, $f(z)$ and $z^2(z - 3)$ have the same number of zeros inside the disk of radius 3, so $f(z)$ have 2 zeros there (including multiplicity).