Math 116 - Homework 4 Solutions

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Problem 1 Solution  \( \frac{f}{g} \) is bounded by 1 everywhere on \( \mathbb{C} - \{ \text{zeros of } g \} \). This means that if \( g \) has a zero at \( z_0 \), \( \frac{f}{g} \) will have removable singularity there. Therefore, it is an entire bounded function, so constant.

Problem 2 Solution  If \( \sin \pi z \) vanishes, then the numerator has to vanish, meaning that
\[
e^{i\pi z}(1 - e^{-2i\pi z}) = 0
\]
Set \( z = x + iy \), then
\[
1 = e^{-2i\pi z} = e^{2y\pi}e^{-2i\pi x}
\]
So \( y = 0 \) and \( x \in \mathbb{Z} \). The zeros are of order 1 since the derivative of \( \sin \pi z \) is \( \pi \cos \pi z \), which does not vanish on \( \mathbb{Z} \).

Since \( \sin \pi z \) has simple zeros, \( \frac{1}{\sin \pi z} \) has simple poles at the integers. At \( z = n \), we can compute
\[
\text{Res}_n \frac{1}{\sin \pi z} = \lim_{z \to n} \frac{z - n}{\sin \pi z} = \frac{(-1)^n}{\pi}
\]
The last step is due to L’Hospital.

Problem 3 Solution  The poles are at \( z = e^{i k \pi/4} \) where \( k = 1, 3, 5, 7 \). The residues at \( e^{i\pi/4} \) and \( e^{3i\pi/4} \) are \( \frac{\sqrt{2} - \sqrt{2}i}{8} \) and \( -\frac{\sqrt{2} - \sqrt{2}i}{8} \). Thus the integral we are calculating is the same as
\[
2\pi i \left( -\frac{\sqrt{2} - \sqrt{2}i}{8} + \frac{\sqrt{2} - \sqrt{2}i}{8} \right) + \lim_{R \to \infty} \int_{\gamma_R} \frac{1}{1 + z^4} dz
\]
where \( \gamma_R \) is the semi circle center at 0 ranging from \( \pi \) to 0.

\[
\left| \lim_{R \to \infty} \int_{\gamma_R} \frac{1}{1 + z^4} dz \right| \leq \lim_{R \to \infty} \frac{1}{R^3 - 1} \pi R = 0
\]
thus the original integral equals to
\[
2\pi i \left( -\frac{\sqrt{2} - \sqrt{2}i}{8} + \frac{\sqrt{2} - \sqrt{2}i}{8} \right) = \frac{\pi}{\sqrt{2}}
\]

Problem 4 Solution  Consider the integral of \( f(z) = \frac{e^{iz}}{z^2 + a^2} \). Let \( \gamma_1(R) \) be the segment from \(-R\) to \( R \) and \( \gamma_2(R) \) be the upper semicircle from \(-R\) to \( R \). Then by residue theorem we have
\[
\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx = 2\pi i (\text{Res}_{ai} f(z)) + \lim_{R \to \infty} \int_{\gamma_2(R)} f(z) dz
\]
First we calculate the residues, which is \( e^{-a}/2 \). I claim that \( \lim_{R \to \infty} \int_{\gamma_2(R)} f(z)dz \) goes to 0. Use replace \( z \) with \( R e^{-i\theta} \), fix \( \epsilon > 0 \), let \( R \) be large enough so that \( R^2/(R^2-a^2) < 2 \)

\[
\left| \int_{\gamma_2(R)} f(z)dz \right| = \left| \int_{0}^{\pi} \frac{Re^{i\theta}e^{-R\sin \theta}}{R^2e^{i\theta} + a^2} Re^{i\theta} d\theta \right|
\leq \int_{0}^{\pi} \left| \frac{R^2e^{-R\sin \epsilon}}{R^2 - a^2} \right| d\theta + \int_{\pi-\epsilon}^{\pi} \left| \frac{R^2e^{-R\sin \epsilon}}{R^2 - a^2} \right| d\theta
\leq 4\epsilon + 2\pi e^{-R\sin \epsilon}
\]

So the limit goes to 0. We see that \( \int_{-\infty}^{-\infty} \frac{e^iz}{z^2+a^2}dz = i\pi e^{-a} \). Since the integral we study is the imaginary part of \( \int_{\gamma_1(R)} f(z)dz \), we are done.

**Problem 5 Solution** We will use the following identity for \( z = e^{i\theta}, \theta \in [0,2\pi] \): \( \frac{z^{1/2}}{2} = \cos \theta \). Let \( C \) be the unit circle, we can rewrite the integral as

\[
\left| \int_{\gamma_2(R)} f(z)dz \right| = \left| \int_{0}^{2\pi} \frac{Re^{i\theta}e^{-R\sin \theta}}{2(e^{i\theta} + a)} Re^{i\theta} d\theta \right|
\leq \int_{0}^{\pi} \left| \frac{R^2e^{-R\sin \epsilon}}{R^2 - a^2} \right| d\theta + \int_{\pi-\epsilon}^{\pi} \left| \frac{R^2e^{-R\sin \epsilon}}{R^2 - a^2} \right| d\theta
\leq 4\epsilon + 2\pi e^{-R\sin \epsilon}
\]

Multiply by \( 2\pi i \) and \(-i\) and we get the equation we want.

**Problem 6 Solution** Use the Contour in figure 10 page 105. Let \( f(z) = \frac{\ln z}{z^2+a^2} \). Let \( \gamma_1(R) \) be the big arc, \( \gamma_2(\epsilon) \) be the small arc, \( \gamma_3(\epsilon, R) \) be the line segment from \( \epsilon \) to \( R \) and \( \gamma_4(-R, -\epsilon) \) be the line segment from \(-R \) to \(-\epsilon \).

The integral on the contour equals to

\[
2\pi i \text{Res}_{a} f(z) = \frac{\pi \ln a + \pi^2 i}{a}
\]

We want to show that as \( R \) goes to \( \infty \) and \( \epsilon \) to \( 0 \), the integral on \( \gamma_1 \) and \( \gamma_2 \) vanishes. The real part of the integral on \( \gamma_4 \) will be equal to the integral on \( \gamma_3 \), which is the integral we want to compute. Thus, if the integral on the arcs vanishes, we shall have that the integral we want to compute is half of the real part of the integral of \( f(z) \) on the contour after taking the limit:

\[
\int_{0}^{\infty} \frac{\ln x}{x^2+a^2}dx = \frac{\pi \ln a}{2a}
\]

Now study the integral on the arcs

\[
\left| \int_{\gamma_4(R)} f(z)dz \right| \leq \pi R \left| \frac{\pi i + \ln R}{R^2-a^2} \right|
\]

Since \( \ln R/R \) goes to 0 as \( R \) goes to infinity, we are done with \( \gamma_1 \). Similar on \( \gamma_2 \)

\[
\left| \int_{\gamma_3(\epsilon)} f(z)dz \right| \leq \frac{|\pi i + \ln \epsilon|}{a^2-\epsilon^2} \pi \epsilon
\]

Note that the denominator is bounded away from 0 if \( \epsilon \) is small. For the numerator, \( \epsilon \ln \epsilon \) goes to zero as \( \epsilon \) goes to 0, so we are done.
**Problem 7 Solution**  Consider the function
\[
f(z) = \frac{\pi \cot \pi z}{(u + z)^2}
\]
then we shall have
\[
\lim_{R_n \to \infty} \int_{C_{R_n}} f(z) \, dz = 2\pi i \sum \text{Res}_z f(z)
\]
where \(C_{R_n}\) is a square with center at 0 with edges of length \(R_n\), parallel to the axes when \(R_n = n + 1/2\).

Thus we have to identify the poles of \(f(z)\), which are the poles of \(\cot \pi z\) and the point \(-u\). The poles of \(\cot \pi z\) are \(\mathbb{N}\). At an integer, the poles are simple since the limit of \(\frac{e^{\pi u}}{\sin \pi z}\) as \(z \to n\) is \(\frac{1}{2}\). Thus the residues there are \(\frac{4}{(u + \pi n)^2}\).

The pole at \(-u\) is a double pole. Take the derivative we have that the residue is equal to \(-\csc(\pi u)^2\pi^2\). Thus if we can prove that the integral goes to zero as \(R_n\) goes to \(\infty\) then we are done. Let \(\gamma_1, \gamma_3\) be the horizontal lines in the square (when \(\gamma_1\) is the upper edge) and \(\gamma_2, \gamma_4\) are the vertical lines. Note that \(\cot \pi z\) has bounded modulus on the the square. To see this, we can write
\[
|\cot \pi z| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| = \left|\frac{e^{i2\pi z} + 1}{e^{i2\pi z} - 1}\right| = \left|\frac{e^{2i\pi x}e^{-2\pi y} + 1}{e^{2i\pi x}e^{-2\pi y} - 1}\right|
\]
Note that
\[
\left|\frac{e^{-2\pi y} - 1}{e^{-2\pi y} + 1}\right|
\]
is a bounded function in terms of \(y \in \mathbb{R}\). So on \(\gamma_2\) and \(\gamma_4\) we can bound \(|\cot z|\) since \(e^{2\pi i x} = -1\). On \(\gamma_3\), since \(y = n + 1/2\), \(e^{2\pi y}\) is small and we can say that
\[
\left|\frac{e^{2i\pi x}e^{-2\pi y} + 1}{e^{2i\pi x}e^{-2\pi y} - 1}\right| \leq \left|\frac{e^{-2\pi y} + 1}{e^{-2\pi y} - 1}\right|
\]
On the other hand, on \(\gamma_1\) \(y = -n - 1/2\) and \(e^{-2\pi y}\) is big, we have
\[
\left|\frac{e^{2i\pi x}e^{-2\pi y} + 1}{e^{2i\pi x}e^{-2\pi y} - 1}\right| \leq \left|\frac{e^{-2\pi y} + 1}{e^{-2\pi y} - 1}\right|
\]

So we can claim that \(\cot z\) has bounded modulus among all the \(C_{R_n}\), let \(M\) be this upper bound. So for \(k = 1, 2, 3, 4\)
\[
\int_{\gamma_k} f(z) \, dz \leq R_n \frac{1}{R_n^2 - u^2} M
\]
which goes to 0 as \(n \to \infty\).

**Problem 8 Solution**  It is enough to show that \(|f(z)|\) is bounded on a punctured neighborhood of \(z_0\). What we have is
\[
|(z - z_0)f(z)| \leq A|z - z_0|^\varepsilon
\]
where the right hand side is bounded on a small neighborhood around 0. So \((z - z_0)f(z)\) is in fact holomorphic (removable singularity), which leads to the fact that if \(f(z)\) is not holomorphic at \(z_0\), then \(f(z)\) can only have a simple pole.

Write \(f(z)\) as \(\frac{a_{-1}}{(z - z_0)} + h(z)\) where \(h(z)\) is holomorphic. What we have is
\[
\left|\frac{a_{-1}}{(z - z_0)} + h(z)(z - z_0)\right| = |(z - z_0)^{1-\varepsilon}f(z)| \leq A
\]
for any \(z\) in the punctured neighborhood of \(z_0\), which is absurd, since the left hand side could still go to infinity if \(a_{-1} \neq 0\).
Problem 9 Solution  
If \( f \) has removable singularity at infinity, then \( f \) is constant and not injective. If \( f \) has a pole at infinity, then \( f \) is a polynomial. But \( f \) is injective, this forces \( f \) has degree 1. If \( f \) has degree more than 1, the fundamental theorem of algebra + injectivity force all the roots to be the same. Thus \( f = a(z - z_0)^k \) where \( k \geq 1 \). Then \( z_0 + \zeta_k \), where \( \zeta_k \) as \( k \)th root of unity, are all mapped to 1.

If \( f \) has an essential singularity at \( \infty \), by Casorati-Weierstrass \( f (\{|z| > R\}) \) is dense in the complex plane. By the open mapping theorem \( f (\{|z| < R/2\}) \) is open and \( f (\{|z| < R/2\}) \cap f (\{|z| > R\}) \neq \emptyset \), which is impossible if \( f \) is injective.

We see that \( f \) has to a polynomial of degree 1.

Problem 10 Solution  
We have that \( z^{1/3} \) is holomorphic in any region for which \( \log z \) is holomorphic. We let our region be \( \mathbb{C} \setminus \{z = iy \mid y \leq 0\} \), and there \( z^{1/3} = e^{\frac{i\theta}{3}} = r^{1/3}e^{i\theta/3} \). We have that \( f(z) = \frac{z^{1/3}}{1+z^2} \) has a pole at \( z = i \), and

\[
\text{Res}_if(z) = \frac{e^{i\pi/3}}{2i}
\]

We also have that \( \int_{-R}^{-\epsilon} f(z)dz = e^{i\pi/3} \int_{\epsilon}^{R} f(z)dz \). Now, on \( \gamma_R \) (the semicircle of radius \( R \)) \( f \) is bounded by \( |f(z)| \leq \frac{R^{1/3}}{R^2-1} \), so \( \left| \int_{\gamma_R} f(z) \right| \leq \frac{\pi R^{4/3}}{R^2-1} \) which goes to zero as \( R \to \infty \). On \( \gamma_\epsilon \) \( |f(z)| \leq \frac{R^{1/3}}{R^2-1} \) and again \( \int_{\gamma_\epsilon} f(z) \) vanishes as \( \epsilon \to 0 \). So we get:

\[
(1 + e^{i\pi/3}) \int_0^\infty f(z)dz = \frac{2\pi ie^{i\pi/3}}{2i}
\]

so \( \int_0^\infty f(z)dz = \frac{\pi}{\sqrt{3}} \)
