Traveling waves for the Keller-Segel system with Fisher birth terms

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Abstract

We consider the traveling wave problem for the one dimensional Keller-Segel system with a birth term of either a Fisher/KPP type or with a truncation for small population densities. We prove that there exists a solution under some stability conditions on the coefficients which enforce an upper bound on the solution and $\dot{H}^1(\mathbb{R})$ estimates. Solutions in the KPP case are built as a limit of traveling waves for the truncated birth rates (similar to ignition temperature in combustion theory).

We also discuss some general bounds and long time convergence for the solution of the Cauchy problem and in particular linear and nonlinear stability of the non-zero steady state.

Key-words: Chemotaxis; Traveling waves; Keller-Segel system; Reaction diffusion systems; Nonlinear stability.
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1 The main result

The growth of bacterial colonies undergoes complex biomechanical processes which underly the variety of shapes exhibited by the colonies. Usually cells divide and undergo active motion resulting in fronts of bacteria that are propagating. These fronts may be unstable leading to various patterns that have been studied for a long time, such as, for instance, spiral waves [16], aggregates [18] and dentrites [1, 10]. At least three elementary biophysical processes play commonly a central role in these patterns, and have been used in all modeling: (i) cell division which induces the growth of the colony, (ii) random cell motion – for instance, bacteria can swim in a liquid medium thanks to flagella, and (iii) chemoattraction through different molecules that the cells may release in their environment and that diffuse, leading to some kind of (possibly long distance) communication. Our purpose here is to study the existence of traveling waves and the linear and nonlinear stability of the steady states for a simple model combining these three effects. The macroscopic model describes the density of bacteria, denoted by $u(t, x)$ below, and the chemoattractant concentration $v(t, x)$ in the medium. It is a variant of the Keller-Segel system that has been widely studied in various contexts, see [5, 12, 19, 20] and references therein.

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We consider the one-dimensional Keller-Segel system with a Fisher-KPP birth term (we will refer to it as the Keller-Segel-Fisher system)

$$\begin{cases} u_t - u_{xx} + \chi (u v_x)_x = u(1-u), \\ -d v_{xx} + v = u. \end{cases} \tag{1}$$

Here the notation u_t or u_x means time or space derivatives. The boundary conditions for u and v are

$$v(-\infty) = u(-\infty) = 1, \quad v(+\infty) = u(+\infty) = 0, \tag{2}$$

that is, there are no bacteria on the left. The two parameters χ and d are, respectively, the sensitivity of the cells to chemoattraction, and the diffusion coefficient of the chemoattractant. The traveling wave solutions moving with a speed c (which becomes a new unknown of the problem) for (1) are special solutions of the form $u(x-ct)$ and $v(x-ct)$ that satisfy

$$\begin{cases} -cu' - u'' + \chi (u v')' = u(1-u), \\ -d v'' + v = u, \end{cases} \tag{3}$$

together with the boundary conditions (2). We prove the following result.

Theorem 1.1 *Let $\chi > 0$ and $d > 0$ satisfy*

$$\chi < \min(1, d). \tag{4}$$

Then there exists a traveling wave solution (c_*, u, v) of (3) with the boundary conditions(2) and a constant $K(d,\chi)$, such that the functions $u(x)$ and $v(x)$, and the speed c_* satisfy

$$0 < u(x), v(x) \le \Big(1 - \frac{\chi}{d}\Big)^{-1}, \tag{5}$$

$$\int u(x)\big(1-u(x)\big)^2 dx + \int \big|u'(x)\big|^2 + \int \big|v'(x)\big|^2 dx \le K(d,\chi), \tag{6}$$

$$2 \le c_* \le 2 + \frac{\chi\sqrt{d}}{d-\chi}. \tag{7}$$

Writing the second equation as a convolution $v = K_d \star u$, one may see this system as a Fisher equation with a nonlocal drift. Reaction-diffusion with non-local reaction or diffusion terms has been recently investigated (see [4, 7, 9, 11]), but this is not the case, as far as we know, for a nonlocal drift term. Nonlocal terms may make the homogeneous positive state unstable and then create periodic stable patterns. In this paper, we need some conditions on the coefficients, such as (4), that imply the stability of the state $u = v \equiv 1$.

Other situations where traveling waves appear in chemotaxis have been considered in the literature. For instance, [13] consider a source term for chemoattractant in the equation on v, [8] consider existence of traveling fronts by a linearization analysis (for small bacterial diffusion). There are also other related models of biological interest, see for instance the case of haptotaxis in [17]. We also refer to these papers for further references on the subject of fronts and waves for cell population as well as to [21, 22] for the general theory of traveling waves.

Our strategy for the proof of Theorem 1.1 is as follows. We introduce a smooth monotonic cut-off function $g_0(u)$ such that $g_0(u) = 0$ for $u \le 1$ and $g_0(u) = 1$ for $u \ge 2$ and set $g(u) = g_0\big((u-\theta_0)/\theta_0\big)$ – this function has a cut-off $\theta_0 \in (0,1)$. Consider a regularized system

$$\begin{cases} -cu' - u'' + \chi\big(g(u) u v'\big)' = g(u)u(1-u), \\ -d v'' + v = u, \end{cases} \tag{8}$$

with the same boundary conditions (2). The system with the cut-off is of an independent interest – the cut-off means that bacteria feel the chemoattractant and reproduce only if their density exceeds a critical threshold value. Mathematically, the role of the cut-off is very similar to that of the ignition temperature in the combustion theory [14]. The first step in the proof of Theorem 1.1 is to construct a traveling wave solution $\big(c(\theta_0), u(x;\theta_0), v(x;\theta_0)\big)$ of (8) for $\theta_0 > 0$ – as we have mentioned, this result is of an independent interest. We do this for $\theta_0 > 0$ sufficiently small and also obtain uniform in θ_0 bounds on $c(\theta_0)$ and $u(x;\theta_0)$, $v(x;\theta_0)$.

Proposition 1.2 *Let $\chi > 0$ and $d > 0$ satisfy*

$$\frac{1}{\chi} > \frac{1}{d} + 1. \tag{9}$$

Then there exists $\alpha_0 > 0$ so that for all $\theta_0 \in (0, \alpha_0)$ there exists a traveling wave solution $\big(c(\theta_0), u(x;\theta_0), v(x;\theta_0)\big)$ of (8), (2). In addition, there exists a constant $K > 0$ which does not depend on θ_0 so that we have the following uniform bounds:

$$\begin{cases} 0 < u(x;\theta_0), v(x;\theta_0) \le \Big(1 - \dfrac{\chi}{d}\Big)^{-1}, \qquad 0 < \dfrac{1}{K} \le c(\theta_0) \le K < +\infty, \\ \displaystyle\int u(x;\theta_0)\big(1 - u(x;\theta_0)\big)^2 dx + \int \big|u'(x;\theta_0)\big|^2 dx + \int \big|v'(x;\theta_0)\big|^2 dx \le K. \end{cases} \tag{10}$$

Here and throughout the paper we denote by C and K generic constants which may depend on χ and d but not on the cut-off θ_0 or the size a of the approximating finite interval which appears later in the proof. We recall that in the case of a single equation with no chemoatractant coupling ($\chi = 0$) the speed $c(\theta_0)$ is unique for $\theta_0 > 0$ [14].

In this general framework it seems difficult to relax the size condition (9) and to achieve the more general condition (4) that we use in Theorem 1.1. This is possible if we introduce two modifications in the above procedure. First, we consider another regularization of the system :

$$\begin{cases} -cu' - u'' + \chi\big(g(u)uv'\big)' = g(u)u(1-u), \\ -dv'' + v = g(u)u, \end{cases} \tag{11}$$

that is, the chemoattractant source also now has a small density cut-off. Second, we tune the truncation function appropriately – we now choose it with the following properties:

$$\begin{cases} g(u) = 0 \quad \text{for } u \le \theta_0, \qquad g' \ge 0, \qquad g(u) = 1 \quad \text{for } u \ge 1, \\ g(u) + ug'(u) \le 1 + \alpha(\theta_0) \quad \text{with } \alpha(\theta_0) \xrightarrow[\theta_0 \to 0]{} 0, \\ g(u) \text{ increases to 1 for } u \in (0,1) \text{ as } \theta_0 \to 0. \end{cases} \tag{12}$$

The reader can esily check that these conditions are satisfied by the function

$$g(u) = 1 + 2\alpha(1 + \ln(u) - u)$$

with $\alpha(\theta_0)$ normalized so that $g(\theta_0) = 0$.

Proposition 1.3 *Assume that the cut-off g satisfies the properties (12) and that χ and d satisfy the condition (4). Then there exists $\alpha_0 > 0$ so that for all $\theta_0 \in (0, \alpha_0)$ there exists a traveling wave solution $\big(c(\theta_0), u(x;\theta_0), v(x;\theta_0)\big)$ of (11), with the boundary conditions (2) which satisfies the estimates (5), (6) and*

$$K \le c(\theta_0) \le 2 + (1+\alpha)\frac{\chi\sqrt{d}}{d - \chi}, \tag{13}$$

with a constant $K > 0$ which does not depend on $\theta_0 \in (0, \alpha_0)$.

This proposition allows us to pass to the limit $\theta_0 \to 0$ and obtain a traveling wave solution of the original problem (3) without a cut-off as stated in Theorem 1.1 and with the smallness condition (4) on the chemotaxis . The traveling waves for a positive cut-off $\theta_0 > 0$ in Propositions 1.2 and 1.3 are constructed by first building an approximate solution on a finite interval $-a \le x \le a$ and then passing to the limit $a \to +\infty$, the strategy originated in [3].

By construction, the traveling wave solutions in Theorem 1.1 satisfy a nonlinear stability property with respect to the perturbations of the birth term, under condition (4). This condition arises several times in our proof but we do not know if it is sharp: it implies the less restrictive condition $\chi < d$ which provides us with the maximum principle for u, but it is also instrumental in deriving the other fundamental a priori estimates in (10). It is interesting that the linear stability condition of the steady states solutions $(1,1)$ of (1) is much weaker than (4). To see that, we linearize the problem in the neighborhood of $(1,1)$ and write

$$u = 1 + U, \qquad v = 1 + V, \qquad \text{where } \ U, V \ll 1.$$

One finds the linearized equations

$$\begin{cases} U_t - U_{xx} + \chi V_{xx} = -U, \\ -dV_{xx} + V = U. \end{cases} \tag{14}$$

Taking the Fourier transfrom we obtain

$$\begin{cases} \widehat{U}(k)_t + k^2\widehat{U} - \chi k^2 \widehat{V} = -\widehat{U}, \\ dk^2\widehat{V} + \widehat{V} = \widehat{U}, \end{cases}$$

and since $\widehat{V}$ can be explicitely computed in terms of $\widehat{U}$, we reduce it to

$$\widehat{U}_t + \left[k^2 + 1 - \frac{\chi k^2}{1 + dk^2}\right]\widehat{U} = 0.$$

This equation is *linearly stable* if and only if :

$$k^2 + 1 - \frac{\chi k^2}{1 + dk^2} \ge 0 \qquad \text{for all } k \in \mathbb{R}.$$

Setting $X = k^2 \ge 0$, we find the equivalent condition $1 + X(d + 1 - \chi) + dX^2 \ge 0$ for $X \ge 0$, which in turn is equivalent to

$$\chi \le (1 + \sqrt{d})^2. \tag{15}$$

In this case the steady state $(1,1)$ is linearly stable. When this condition is violated as in [8] unstable patterns arise. Condition (4) is of course stronger than (15) and even the sufficient condition $\chi < d$ for the uniform upper bound on u and v in (10) is still stronger than (15). This leaves the question of the optimal condition for the existence of traveling waves open.

The organization of this paper is as follows. We first consider the problem with cut-off on an interval $[-a, a]$ and prove the existence by the homotopy argument in Section 2. We also establish the main estimates in this section. In Section 3 we remove the cut-off and let the interval length a tend to infinity, the main difficulty being to show that the states $(1,1)$ and $(0,0)$ are indeed connected by the solution obtained by this procedure. In the last section we establish some general bounds on the solution of the Cauchy problem and prove that the homogeneous solution is stable as soon as it is linearly stable.

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2 The problem on a finite interval $[-a, a]$

Our approach follows the traditional methods, see [3] and [22], for instance, that we adapt to our specific situation. In particular, as usual, specific difficulties arise in showing that the speed c is controlled from below and above, and that the states $u = 1$ and $u = 0$ are indeed reached at infinity (see [2, 6, 15] for an example where this question is left open in the construction of travelling waves for a reactive Boussinesq system).

The finite interval approximation

In order to prove Proposition 1.2, we first construct an approximation (c_a, u_a, v_a) (we drop θ_0 in the notation for the traveling wave for the moment) on a finite interval $-a \le x \le a$:

$$\begin{cases} -c_a u_a' - u_a'' + \chi(g(u_a)u_a v_a')' = g(u_a)u_a(1-u_a), \\ -dv_a'' + v_a = u_a. \end{cases} \tag{16}$$

The boundary conditions for u_a are

$$u_a(-a) = 1, \qquad u_a(a) = 0. \tag{17}$$

Instead of imposing the boundary conditions for v_a at $x = \pm a$, we extend u_a to the whole real line as

$$\bar{u}_a(x) = \begin{cases} 1, & x < -a, \\ u_a(x), & -a \le x \le a, \\ 0, & x \ge a, \end{cases} \tag{18}$$

and then we set

$$v_a(x) = \int_{-\infty}^{\infty} K_d(|x-\xi|)\bar{u}_a(\xi)d\xi, \qquad K_d(\xi) = \frac{e^{-|\xi|/\sqrt{d}}}{2\sqrt{d}}, \qquad \int K_d(\xi)d\xi = 1. \tag{19}$$

The function $v_a(x)$ is defined for all $x \in \mathbb{R}$ and satisfies

$$-dv_a'' + v_a = \bar{u}_a, \quad v_a(-\infty) = 1, \; v_a(+\infty) = 0. \tag{20}$$

Three consequences of the representation formula (19) are the bounds

$$\begin{aligned} &|v_a(x)| \le \|u_a\|_\infty, \\ &|v_a'(x)| = \frac{1}{2d}\left|\int e^{-|x-\xi|/\sqrt{d}}\mathrm{sgn}(\xi - x)u_a(\xi)d\xi\right| \le \frac{1}{\sqrt{d}}\|u\|_\infty, \\ &|v_a''(x)| \le \frac{C}{d}\|u\|_\infty, \end{aligned} \tag{21}$$

which we frequently use.

In order to ensure that the solution u_a has a non-trivial limit as $a \to +\infty$, we normalize it so that

$$\max_{x \ge 0} u_a(x) = \theta_0. \tag{22}$$

This constraint indirectly fixes the speed c_a. It follows from the maximum principle and (22) that $u_a(0) = \theta_0$ and thus u_a satisfies the boundary value problem on $[0, a]$:

$$-c_a u_a'(x) - u_a''(x) = 0 \quad 0 \le x \le a, \qquad u_a(0) = \theta_0, \; u_a(a) = 0.$$

Proposition 2.1 *With the assumption (9), there exists a solution (c_a, u_a, v_a) of (16), (17), (19), (22) with non-negative functions $u_a \geq 0$ and $v_a \geq 0$, which in addition satisfies the uniform bounds (10).*

The rest of this section is devoted to the proof of this proposition which uses the homotopy argument. Accordingly, we introduce the homotopy parameter $\tau \in [0,1]$ and consider a family of problems

$$\begin{cases} -c_{\tau,a}u'_{\tau,a} - u''_{\tau,a} + \chi\tau(g(u_{\tau,a})u_{\tau,a}v'_{\tau,a})' = \tau g(u_{\tau,a})u_{\tau,a}(1-u_{\tau,a}), \\ -dv''_{\tau,a} + v_{\tau,a} = \tau u_{\tau,a}, \end{cases} \tag{23}$$

together with the boundary conditions (17), the relation (19) (with the right side multiplied by the factor τ) and normalization (22). To simplify the notation we drop the subscript τ below.

A uniform upper bound for the traveling speed

We begin with an upper bound for the speed.

Lemma 2.2 *If $d > \chi$, then any solution of (17), (19), (22), (23) satisfies*

$$0 \leq u_a(x), v_a(x) \leq \left(1 - \frac{\chi}{d}\right)^{-1}, \quad |v'_a(x)| \leq C, \tag{24}$$

with the constant $C > 0$ which depends only on d and χ. In addition, there exists a constant $a_0(\theta_0) > 0$, and a constant $K > 0$ which depends only on d and χ but not on a, $\tau \in [0,1]$, or $\theta_0 \in (0,1)$ so that for all $a > a_0(\theta_0)$ we have

$$c_a \leq K < +\infty. \tag{25}$$

Proof. Let us re-write the equation (23) for u_a as

$$\begin{aligned} -c_a u'_a - u''_a + \tau\chi g(u_a)v'_a u'_a + \tau\chi g'(u_a)u_a v'_a u'_a &= \tau g(u_a)u_a(1-u_a) - \frac{\tau\chi}{d}g(u_a)u_a(v_a - u_a) \\ &= \tau g(u_a)u_a\left(1 - u_a + \frac{\chi}{d}u_a - \frac{\chi}{d}v_a\right) \leq \tau g(u_a)u_a\left(1 - u_a + \frac{\chi}{d}u_a\right). \end{aligned} \tag{26}$$

The last inequality holds if $v_a \geq 0$. As $g(u) = 0$ for $u \leq 0$, it follows that u_a can not attain an interior negative minimum on $(-a, a)$ and thus $u_a \geq 0$, which, in turn, implies that $v_a \geq 0$ and (26) indeed holds. It also follows from (26) that u_a can not attain an interior maximum at a point where $u_a \geq (1-\chi/d)^{-1}$. Therefore, we have $0 \leq u_a \leq (1-\chi/d)^{-1}$ and hence the same bound holds for the function v_a. The bound for $|v'_a(x)|$ in (24) is then a consequence of (21).

Next, we show that the speed c_a is uniformly bounded from above by using the super-solution argument. The function $u_a(x)$ satisfies the inequality

$$-c_a u'_a - u''_a + \tau\chi[g(u_a) + g'(u_a)u_a]v'_a u'_a \leq \tau u_a,$$

which follows from (26) and the condition $\chi/d < 1$. Let us set $\psi_M(x) = Me^{-x}$, then the function ψ_M satisfies

$$\begin{aligned} -c_a\psi'_M - \psi''_M + \tau\chi[g(u_a) + g'(u_a)u_a]v'_a\psi'_M &= \big(c_a - 1 - \tau\chi g(u_a)v'_a - \tau\chi g'(u_a)u_a v'_a\big)\psi_M \\ &\geq (c_a - 1 - K_0)\psi_M, \end{aligned}$$

with the constant $K_0 = K_0(\chi, d)$, which is independent of a, τ and θ_0, chosen so that (using (21))

$$\chi[g(u_a) + g'(u_a)u_a]|v'_a| \leq \frac{\chi}{\sqrt{d}}\|g(\sigma) + g'(\sigma)\sigma\|_\infty\|u_a\|_\infty := K_0. \tag{27}$$

This is possible because of the uniform bounds in (24) and since for $u \notin (\theta_0, 2\theta_0)$ we have $g'(u) = 0$ while for $u \in (\theta_0, 2\theta_0)$ the following estimate holds:

$$|g'(u)u| = \frac{u}{\theta_0} g_0' \left(\frac{u - \theta_0}{\theta_0} \right) \leq 2 \max_{1 \leq u \leq 2} |g_0'(u)|.$$

Now, assume by contradiction that

$$c_a > 2 + K_0. \tag{28}$$

Then ψ_M satisfies

$$-c_a \psi_M' - \psi_M'' + \tau\chi[g(u_a) + g'(u_a)u_a]v_a'\psi_M' \geq \psi_M \geq \tau\psi_M.$$

Note that the upper bound on $u_a(x)$ in (24) implies that $\psi_M(x) > u_a(x)$ for $M \geq e^a/(1 - \chi/d)$. Let us define

$$M_0 = \inf\{M : \ \psi_M(x) > u_a(x) \text{ for all } x \in [-a, a]\},$$

then $M_0 > 0$ and, in addition, $\psi_{M_0}(x) \geq u_a(x)$ for all $x \in [-a, a]$ and there exists $x_0 \in [-a, a]$ such that $\psi_{M_0}(x_0) = u_a(x_0)$. However, the difference $\psi_M(x) - u_a(x)$ may not attain an interior minimum at x_0 and $\psi_M(a) > 0 = u_a(a)$. Therefore, we have $x_0 = -a$ and thus $M_0 = e^{-a}$. As a consequence, $\theta_0 = u_a(0) \leq \psi_{M_0}(0) = e^{-a}$, which is a contradiction if a is sufficiently large. We conclude that (28) is impossible and thus (25) holds with $K = 2 + K_0$. □

A lower bound for the traveling speed

Now, we need a lower bound for c_a and an upper bound for $\|u_a'\|_2$.

Lemma 2.3 *With the assumptions of Lemma 2.2 and (9), there exists a constant $a_0(\theta_0) > 0$ and $K > 0$ which depends only on d and χ but not on $a > a_0$, $\theta_0 \in (0, 1)$ and $\tau \in [0, 1]$ so that for all $a > a_0$ and $\theta_0 < 1/3$ we have*

$$c_a \geq \frac{\sqrt{\tau}}{K} - \frac{K\theta_0}{a}, \tag{29}$$

$$\tau \int_{-a}^{a} g(u_a)u_a(1 - u_a)^2 dx + \int_{-a}^{a} |u_a'(x)|^2 dx + \int_{-a}^{a} |v_a'(x)|^2 dx \leq K. \tag{30}$$

Proof. Start with

$$-c_a u_a' - u_a'' + \tau\chi(g(u_a)u_a v_a')' = \tau g(u_a)u_a(1 - u_a), \tag{31}$$

and integrate on $[-a, a]$:

$$c_a - u_a'(a) + u_a'(-a) - \tau\chi v_a'(-a) = \tau \int g(u_a)u_a(1 - u_a). \tag{32}$$

Now, multiply (31) by u_a and integrate:

$$\frac{c_a}{2} + u_a'(-a) + \int_{-a}^{a} |u_a'|^2 - \tau\chi v_a'(-a) - \tau\chi \int_{-a}^{a} g(u_a)u_a u_a' v_a' = \tau \int_{-a}^{a} g(u_a)u_a^2(1 - u_a).$$

Combining the last two equalities, we get

$$\frac{c_a}{2} - u_a'(a) - \int_{-a}^{a} |u_a'|^2 + \tau\chi \int g(u_a)u_a u_a' v_a' \tau \int_{-a}^{a} g(u_a)(u_a - u_a^2)(1 - u_a).$$

This can be written as

$$\tau\int_{-a}^{a} g(u_a)u_a(1-u_a)^2+\int_{-a}^{a}|u_a'|^2+u_a'(a)=\frac{c_a}{2}+\tau\chi\int_{-a}^{a} g(u_a)u_a u_a' v_a'. \tag{33}$$

However, on the interval $(0,a)$ we have $g(u_a)=0$, and we can find u_a explicitly:

$$u_a(x)=\theta_0\frac{e^{-c_a x}-e^{-c_a a}}{1-e^{-c_a a}}, \tag{34}$$

so that

$$u_a'(a)=-\frac{c_a\theta_0 e^{-c_a a}}{1-e^{-c_a a}}-\frac{c_a\theta_0}{e^{c_a a}-1}.$$

Note that for $c_a>0$ we have

$$0\le\frac{c_a}{e^{c_a a}-1}\le\frac{1}{a},$$

while for $c_a<0$ we have

$$0\le\frac{c_a}{e^{c_a a}-1}=\frac{a|c_a|}{a(1-e^{-|c_a|a})}\le\frac{1+|c_a|a}{a}=\frac{1}{a}+|c_a|.$$

Therefore, for all $c_a\in\mathbb{R}$ we have

$$|u_a'(a)|\le\frac{1}{a}\theta_0+|c_a|\theta_0. \tag{35}$$

We note that the special case $c_a=0$ that we did not treat above can be easily considered separately.

We may use the representation formula (19) for v to obtain

$$v'=\tau\, K_d*\bar{u}',\qquad \|v'\|_{L^2}\le\tau\,\|\bar{u}'\|_{L^2}\le\|u'\|_{L^2}. \tag{36}$$

Using this in (33) we obtain

$$\tau\int_{-a}^{a} g(u_a)u_a(1-u_a)^2+\int_{-a}^{a}|u_a'|^2\le\frac{c_a}{2}-u_a'(a)+\frac{\chi\tau}{1-\frac{\chi}{d}}\int_{-a}^{a}|u_a'|^2. \tag{37}$$

It follows that for $0\le\tau\le1$ we have, thanks to (35):

$$\tau\int_{-a}^{a} g(u_a)u_a(1-u_a)^2+M\int_{-a}^{a}|u_a'|^2\le\frac{c_a}{2}+|u_a'(a)|\le\frac{c_a}{2}+\frac{\theta_0}{a}+\theta_0|c_a|, \tag{38}$$

with, according to condition (9),

$$M=1-\frac{\chi}{1-\frac{\chi}{d}}>0.$$

In addition, as $u_a(-a)=1$ and $u_a(0)=\theta_0$, there exists a constant $K>0$ which does not depend on $\theta_0\in(0,1/3)$ such that

$$\left(\int g(u_a)u_a(1-u_a)^2\right)\left(\int|u_a'|^2\right)\ge K.$$

Therefore, provided that $a>a_0$ and $\theta_0\in(0,1/3)$, we have a lower bound for c_a:

$$c_a\ge c_0\sqrt{\tau}-\frac{C\theta_0}{a},$$

with the constants $c_0>0$ and $C>0$ which do not depend on the cut-off θ_0. This is the bound in (29), while the bounds in (30) follow from the upper bound (25) for the speed, (36) and (38). □

The homotopy argument

We may now finish the proof of Proposition 2.1 using a homotopy argument. The a priori bounds obtained in Lemmas 2.2 and 2.3 allow us to use the Leray-Schauder topological degree argument to prove existence of solutions to the problem (16), (17), (19) with the normalization (22) on the bounded interval $D_a = (-a, a)$. This method of construction of traveling wave solutions goes back to [3]. We introduce a map (we suppress the subscript a now, resurrecting the subscript τ for the homotopy parameter)

$$\mathcal{K}_\tau : (c, u, v) \to (\theta_\tau, U_\tau, V_\tau)$$

as the solution operator of the linear system

$$\begin{cases} -cU_\tau' - U_\tau'' + \tau\chi(g(u)U_\tau v')' = \tau g(u)u(1-u), \\ -dV_\tau'' + V_\tau = \tau\bar{u}. \end{cases} \tag{39}$$

The boundary conditions for U_τ are as in (17)

$$U_\tau(-a) = 1, \quad U_\tau(a) = 0, \tag{40}$$

while V_τ is given explicitly as before by

$$V_\tau(x) = \tau \int_{-\infty}^{\infty} K_d(|x-\xi|)\bar{u}(\xi)d\xi, \quad K_d(\xi) = \frac{e^{-|\xi|/\sqrt{d}}}{2\sqrt{d}}, \tag{41}$$

where $\bar{u}(x)$ is again the extension of $u(x)$ to the whole real line as in (18).

The number θ_τ is defined by

$$\theta_\tau = \theta_0 - \max_{x\geq 0} u(x) + c.$$

The operator $\mathcal{K}_\tau$ is a mapping of the Banach space $X = \mathbb{R} \times C^{1,\alpha}(D_a) \times C^{1,\alpha}(D_a)$, equipped with the norm $\|(c, u, v)\|_X = \max(|c|, \|u\|_{C^{1,\alpha}(D_a)}, \|v\|_{C^{1,\alpha}(D_a)})$, onto itself. A solution $s_\tau = (c_\tau, u_\tau, v_\tau)$ of the finite interval problem (16), (17), (19), (22) is a fixed point of $\mathcal{K}_\tau$ and satisfies $\mathcal{K}_\tau s_\tau = s_\tau$, and vice versa: a fixed point of $\mathcal{K}_\tau$ provides a solution. Hence, in order to establish the existence of a solution to (16), (17), (19) together with the normalization (22), it suffices to show that the kernel of the operator $\mathcal{F}_\tau = \mathrm{Id} - \mathcal{K}_\tau$ is not trivial. The standard elliptic regularity theory implies that the operator $\mathcal{K}_\tau$ is compact and depends continuously on the parameter $\tau \in [0, 1]$. Thus we may apply the Leray-Schauder topological degree theory. Let us introduce a ball $B_M = \{\|(c, u, v)\|_X \leq M\}$. Then Lemmas 2.2 and 2.3 show that the operator $\mathcal{F}_\tau$ does not vanish on the boundary ∂B_M with M sufficiently large for any $\tau \in [0, 1]$. It remains only to show that the degree $\deg(\mathcal{F}_1, B_M, 0)$ in $\bar{B}_M$ is not zero. However, the homotopy invariance property of the degree implies that $\deg(\mathcal{F}_\tau, B_M, 0) = \deg(\mathcal{F}_0, B_M, 0)$ for all $\tau \in [0, 1]$. Moreover, the degree at $\tau = 0$ can be computed explicitly as the operator $\mathcal{F}_0$ is given by

$$\mathcal{F}_0(c, u, v) = (\max_{x\geq 0} u(x) - \theta_0, u - u_0^c, v).$$

Here the function $u_0^c(x)$ solves

$$\frac{d^2u_0^c}{dx^2} + c\frac{du_0^c}{dx} = 0, \quad u_0^c(-a) = 1, \quad u_0^c(a) = 0$$

and is given by

$$u_0^c(x) = \frac{e^{-cx} - e^{-ca}}{e^{ca} - e^{-ca}}.$$

The mapping $\mathcal{F}_0$ is homotopic to

$$\Phi(c, u, v) = (\max_{x \geq 0} u_0^c(x) - \theta_0, u - u_0^c, v)$$

that in turn is homotopic to

$$\tilde{\Phi}(c, u, v) = (u_0^c(0) - \theta_0, u - u_0^{c_*^0}, v),$$

where c_*^0 is the unique number so that $u_0^{c_*}(0) = \theta_0$. The degree of the mapping $\tilde{\Phi}$ is the product of the degrees of each component. The last two have degree equal to one, and the first to -1, as the function $u_0^c(0)$ is decreasing in c. Thus $\deg \mathcal{F}_0 = -1$ and hence $\deg \mathcal{F}_1 = -1$ so that the kernel of $\mathrm{Id} - \mathcal{K}_1$ is not empty. This finishes the proof of Proposition 2.1. □

3 Identification of the limit as $a \to +\infty$

In this section we first pass to the limit $a \to +\infty$ constructing traveling waves with a positive cut-off $\theta_0 > 0$. In the second step we remove the cut-off and obtain traveling waves for the Fisher-KPP birth rate. At this stage we only prove a loose lower bound on c_*, the more precise bound stated in Theorem 1.1 is proved in Section 4.

Passage to the whole line with a cut-off

We now prove Proposition 1.2.

Having established the existence of a solution (c_a, u_a, v_a) of (16), (17), (19), (22) on a finite interval we now pass to the limit $a \to +\infty$ and show that (c_a, u_a, v_a) converges to a traveling wave (c, u, v). The L^2-bound for $u'(x)$ and $v'(x)$ in Lemma 2.3 together with the uniform bounds in Lemma 2.2 and the elliptic regularity imply that there exists a sequence $a_n \to +\infty$ so that $c_n = c_{a_n}$ converges to a limit $c_*(\theta_0)$ and the functions $u_n = u_{a_n}$ and $v_n = v_{a_n}$ converge locally uniformly together with their derivatives to the limits $u(x; \theta_0)$ and $v(x; \theta_0)$. The functions $u(x)$ and $v(x)$ satisfy (we drop the dependence on θ_0 in the notation):

$$\begin{cases} -c_* u' - u'' + \chi(g(u) u v')' = g(u) u (1 - u), \\ -d v'' + v = u, \end{cases} \tag{42}$$

and

$$v(x) = \int_{-\infty}^{\infty} K_d(|x - \xi|) u(\xi) d\xi, \quad K_d(\xi) = \frac{e^{-|\xi|/\sqrt{d}}}{2\sqrt{d}}. \tag{43}$$

Furthermore, the lower bound (2.3) yields that $c_*(\theta_0) \geq \frac{1}{K}$, where K is a positive constant that only depends on d and χ. In particular, c_* is positive.

It remains to prove that $u(x)$ and $v(x)$ satisfy the boundary conditions (2) and, because of (43), it is sufficient to verify them for the function $u(x)$ only. The L^2-bound for the gradient of u in Lemma 2.3 and elliptic regularity imply that the function $u(x)$ has limits as $x \to \pm\infty$:

$$u_l = \lim_{x \to -\infty} u(x), \quad u_r = \lim_{x \to +\infty} u(x).$$

The functions $u_a(x)$ are given by an explicit expression (34) on the interval $0 \leq x \leq a$. Therefore, the limit $u(x)$ is given by

$$u(x) = \theta_0 e^{-c_* x}, \text{ for all } x \geq 0. \tag{44}$$

As $c_* > 0$, it follows that $u_r = 0$.

Next, we show that $u_l = 1$ when θ_0 is sufficiently small. We first note that according to the maximum principle the function u_a can not attain a minimum at a point x where $u_a(x) \leq \theta_0$. Therefore, $u_a \geq \theta_0$ for $x \in (-a, 0)$ and thus $u_l \geq \theta_0$. On the other hand, the uniform bound

$$\int_{-a}^{a} g(u_a)u_a(1-u_a)^2 dx \leq K$$

in Lemma 2.3 implies that the limit $u(x)$ satisfies

$$\int_{-\infty}^{\infty} g(u)u(1-u)^2 dx \leq K. \tag{45}$$

Therefore, we have that either $u_l = 1$ or $u_l \in [0, \theta_0]$. The previous argument implies that the only two possibilities are $u_l = \theta_0$ and $u_l = 1$. Let us assume that $u_l = \theta_0$ and find a contradiction when θ_0 is sufficiently small. With this assumption we integrate the first equation in (42) once to get

$$c_*\theta_0 = \int_{-\infty}^{\infty} g(u)u(1-u)dx. \tag{46}$$

Multiplying the same equation by u and integrating leads to

$$\frac{c_*\theta_0^2}{2} + \int_{-\infty}^{\infty} |u'|^2 dx - \chi \int_{-\infty}^{\infty} g(u)uu'v'dx = \int_{-\infty}^{\infty} g(u)u^2(1-u)dx = c_*\theta_0 - \int_{-\infty}^{\infty} g(u)u(1-u)^2 dx. \tag{47}$$

Using the L^∞-bound for u and since $\|v'\|_2 \leq \|u'\|_2$ we get, still using condition (9),

$$\frac{c_*\theta_0^2}{2} + K \int_{-\infty}^{\infty} |u'|^2 dx + \int_{-\infty}^{\infty} g(u)u(1-u)^2 dx \leq c_*\theta_0, \tag{48}$$

with $K > 0$, as in the computation leading to (38). Note that since $u_l = u(0) = \theta_0$ and $u(x)$ can not attain a local minimum at a value below θ_0, the function $u(x)$ attains its maximum at some point x_M – otherwise, $g(u) \equiv 0$ and $c_* = 0$ which would be a contradiction. For the same reason, $u_M = u(x_M) > \theta_0$ since the integral in the right side of (46) is positive because $c_* > 0$. Observe that if $u_M > 1/2$ and $u_l = \theta_0 < 1/3$, then there exists $K_1 > 0$ which does not depend on θ_0 so that

$$\int_{-\infty}^{\infty} |u'|^2 + \int_{-\infty}^{\infty} g(u)u(1-u)^2 \geq K_1.$$

Therefore, as c_* is bounded from above, it follows from (48) that there exists $\alpha_0 > 0$ so that if $\theta_0 \in (0, \alpha_0)$ then $\theta_0 < u_M < 1/2$.

Next, assume that $\theta_0 \in (0, \alpha_0)$ and integrate the first equation in (42) between $-\infty$ and x_M to get

$$-c_*(u_M - \theta_0) + \chi g(u_M)u_M v'(u_M) = \int_{-\infty}^{x_M} g(u)u(1-u)dx. \tag{49}$$

As $u_M < 1/2$, the right side above is positive. In addition, we have $\|v'\|_{L^\infty} \leq C\|u\|_\infty = Cu_M$ and $g(u)$ satisfies

$$g(u) = g_0\left(\frac{u-\theta_0}{\theta_0}\right) \leq \frac{C(u-\theta_0)}{\theta_0}$$

for $u \geq \theta_0$. Then (49) implies

$$-c_*(u_M - \theta_0) + \frac{C\chi(u_M - \theta_0)u_M^2}{\theta_0} \geq 0.$$

Therefore, as $c_* > 0$ and $u_M > \theta_0$, we have

$$u_M^2 \geq K\theta_0 \text{ with } K > 0. \tag{50}$$

In particular, we have $u_M \geq 2\theta_0$ when θ_0 is sufficiently small. Let x_0 be the first point to the left of x_M such that $u(x_0) = u_M/2$, that is, $u(x) \in [u_M/2, u_M]$ for all $x \in (x_0, x_M)$ and $g(u(x)) = 1$ on this interval. Set $L = x_M - x_0$, then we have, using (48),

$$c_*\theta_0 \geq K\int_{x_0}^{x_M} |u'(x)|^2 + \int_{x_0}^{x_M} g(u)u(1-u)^2 \geq C\left[\frac{(u_M)^2}{L} + u_M L\right] \geq C u_M^{3/2}.$$

It follows that $u_M \leq C\theta_0^{2/3}$, which contradicts (50). This contradiction shows that $u_l = \theta_0$ is impossible when θ_0 is sufficiently small. Therefore, we have $u_l = 1$. This finishes the proof of Proposition 1.2. □

Proof of Proposition 1.3

We now indicate the additional arguments necessary to arrive to the statement of Proposition 1.3, that is, how existence of traveling waves can be deduced under the weaker restriction (4) on the chemotaxis parameter χ.

The entire proof above of Proposition prop-cutoff goes through with the general assumption (12) on g. We indicate now how we can take advantage of the property

$$g + \sigma g' \leq 1 + \alpha. \tag{51}$$

First, the upper bound on c_* in (7) follows clearly from the value K_0 computed in (27).

Now, we prove gradient and "reaction" bounds in (6). To do that we use equation (33) and the key point is to handle the right hand side more carefully with the help of (51): we split the integral as

$$\chi\int_{-a}^{a} \tau g(u_a)u_a u_a' v_a' = \chi\int_{-a}^{a} [\tau g(u_a)u_a - 1]u_a' v_a' + \chi\int_{-a}^{a} u_a' v_a'.$$

We treat separately the two terms on the right side.

Using the equation on v in (11), which now also has the small density cut-off, we have

$$\begin{aligned}\left(\chi\int_{-a}^{a} u_a' v_a'\right)^2 &\leq \chi^2\int_{-a}^{a}(u_a')^2\int_{-a}^{a}(v_a')^2 \leq \chi^2\int_{-a}^{a}(u_a')^2\int_{-a}^{a}[(\tau g(u_a) + u_a\tau g'(u_a))u_a']^2 \\ &\leq \chi^2(1+\alpha)^2\left(\int_{-a}^{a}(u_a')^2\right)^2.\end{aligned}$$

This term is nicely absorbed for $\chi < 1$ and α (or, equivalently, θ_0) small enough by the corresponding term in the left hand side of (33).

For the other term, we introduce the function

$$h(u) = \int_1^u [\tau g(\sigma)\sigma - 1]\, d\sigma \text{ for } 0 \leq u \leq 1$$

and with $h(u) = 0$ for $u \geq 1$. Note that

$$0 \leq h(u) \leq \frac{1}{2}(1-u)^2, \qquad h(1) = 0.$$

We write

$$\chi \int_{-a}^{a} [\tau g(u_a)u_a - 1]u_a' v_a' = \chi \int_{-a}^{a} h(u_a)' v_a' = \chi \int_{-a}^{a} h(u_a)(-v_a)'' + \chi h(u_a) v_a'\big|_{x=a} - \chi h(u_a) v_a'\big|_{x=-a}$$
$$\leq \frac{\chi}{d} \int_{-a}^{a} h(u_a)\big(\tau g(u_a)u_a - v_a\big) \leq \tau \frac{\chi}{2d} \int_{-a}^{a} (1-u_a)^2 g(u_a)u_a,$$

because $v_a'(a) = \big(K_d' * \bar{u}_a\big)(a) \leq 0.$ for a sufficiently large. Consequently, one has:

$$\tau \int_{-a}^{a} g(u_a)u_a(1-u_a)^2 + \int_{-a}^{a} |u_a'|^2 + u_a'(a) = \frac{c_a}{2} + \tau\chi \int_{-a}^{a} g(u_a)u_a u_a' v_a' \tag{52}$$
$$\leq \frac{c_a}{2} + \chi(1+\alpha) \int_{-a}^{a} (u_a')^2 + \tau \frac{\chi}{2d} \int_{-a}^{a} (1-u_a)^2 g(u_a)u_a.$$

It follows that :

$$\tau\left(1 - \frac{\chi}{2d}\right) \int_{-a}^{a} g(u_a)u_a(1-u_a)^2 + (1 - \chi(1+\alpha)) \int_{-a}^{a} |u_a'|^2 + u_a'(a) \leq \frac{c_a}{2}, \tag{53}$$

and $u_a'(a)$ is still bounded by (35).

Thus if $\chi < \min(1, d)$ and θ_0 is small enough such that $\chi(1+\alpha) < 1$, the quantities of the left hand-side are controlled by that of the right-hand side and we can go on the proof and conclude as before. □

4 Removal of the cut-off

Here we remove the cut-off, letting the parameter θ_0 vanish, and prove Theorem 1.1. The traveling waves $(c(\theta_0), u(x;\theta_0), v(x;\theta_0))$, constructed in Proposition 1.3 for $\theta_0 > 0$, are translationally invariant and have the left and right limits $u_l = v_l = 1$, $u_r = v_r = 0$. Therefore, we may translate them and fix the shift so that $u(0;\theta_0) = 1/2$. The uniform estimates in the same proposition allow us to pass to the limit $\theta_{0,n} \to 0$ along a subsequence, so that the traveling wave speeds $c_n = c_*(\theta_{0,n})$ converge to a limit $c_* > 0$, and the functions $u(x;\theta_{0,n})$ and $v(x;\theta_{0,n})$ converge to the limits $u(x)$ an $v(x)$. We also have $g(u_n) \to \Psi(x)$ with $\Psi(x) \equiv 1$ on the set $\{u(x) \neq 0\}$. In addition, the limits satisfy the system (3):

$$-c_* u' - u'' + \chi(\Psi(x)uv')' = \Psi(x)u(1-u), \tag{54}$$
$$-dv'' + v = \Psi(x)u,$$

and the functions u and v are still related by (43). Moreover, as the function $p(u) = g(u)u$ is globally Lipschitz, the functions $u(x;\theta_{0,n})$ and $v(x;\theta_{0,n})$ are uniformly bounded in $C^{2,\alpha}(\mathbb{R})$ and thus so are the limits u and v. Therefore, we have $u > 0$ and thus $\Psi(x) \equiv 1$ and u and v actually satisfy the system (3):

$$-c_* u' - u'' + \chi(uv')' = u(1-u), \tag{55}$$
$$-dv'' + v = u.$$

It remains only to verify that u and v satisfy the boundary conditions (2) at infinity. As in the case with $\theta_0 > 0$ it suffices to ensure that the function $u(x)$ has the left and right limits $u_l = 1$ and

$u_r = 0$, respectively. Once again, existence of the limits at infinity follows from the L^2-bound on the gradient

$$\int_{-\infty}^{\infty} |u'(x)|^2 dx \le K,$$

and standard elliptic regularity estimates. Moreover, in the limit $\theta_0 \to 0$ the estimate (45) becomes

$$\int_{-\infty}^{\infty} u(1-u)^2 dx \le K < +\infty.$$

As a consequence, the only possible values for u_l and u_r are 0 and 1, hence, in order to show that $u_l = 1$ and $u_r = 0$ it suffices to show that $u_l > u_r$. Integrating the first equation in (55) we obtain

$$c_*(u_l - u_r) = \int u(1-u),$$

while multiplying the same equation by u and integrating leads to

$$\frac{c_*(u_l^2 - u_r^2)}{2} + \int |u'|^2 - \chi \int uu'v' = \int u^2(1-u) = c_*(u_l - u_r) - \int u(1-u)^2.$$

As before, we conclude that

$$\frac{c_*(u_l^2 - u_r^2)}{2} + \int u(1-u)^2 + M \int |u'|^2 \le c_*(u_l - u_r),$$

which may be re-witten as

$$\int u(1-u)^2 + M \int |u'|^2 \le c_*(u_l - u_r)\left(1 - \frac{u_l + u_r}{2}\right).$$

As $u(0) = 1/2$ the left side is strictly positive. Moreover, we have $c_* > 0$ and $(u_l + u_r)/2 \le 1$. As a consequence, $u_l > u_r$, thus $u_l = 1$, $u_r = 0$ and the proof of the existence partTheorem 1.1 is complete.

A lower bound for the traveling speed

We now obtain a more precise lower bound for the propagation speed c_* in Theorem 1.1. To do so, we consider a more general birth term $f(u)$ in place of $u(1-u)$ in equation (3). We do not expect more difficulties in the proof of the existence part of Theorem 1.1 as long as $f(u)$ is of the KPP type:

$$f(0) = f(1) = 0,\ f(u) > 0 \text{ for } 0 \le u \le 1,\ f(u) < 0 \text{ for } u \ge 1 \text{ and } f'(0) = \sup_{u \ge 0} \frac{f(u)}{u} > 0. \qquad (56)$$

Then, we have

Proposition 4.1 *Any traveling wave solution of (2)-(3) in $\dot{H}^1(\mathbb{R})$ with the nonlinearity f satisfying (56), and such that $u, v \ge 0$, and*

$$\int u(1-u)^2 dx < \infty \qquad (57)$$

satisfies $c \ge 2\sqrt{f'(0)}$.

Proof. Consider a traveling wave (c, u, v) and choose a sequence x_n that increases to $+\infty$ when $n \to \infty$. Set $u_n(x) = u(x + x_n)/u(x_n)$ and $v_n(x) = v(x + x_n)$, these functions satisfy

$$\begin{cases} -u_n'' - cu_n' + \chi(v_n' u_n)' = f\big(u(x_n)\, u_n\big)/u(x_n), \\ -dv_n'' + v_n = u(x + x_n). \end{cases} \tag{58}$$

Next, note that $u(x + x_n) \to 0$ uniformly in x as $n \to +\infty$. Indeed, choosing $A > 0$ large enough so that $u(x) \leq 1/2$ for $x \geq A$, we deduce from (57) that $u \in L^1(A, +\infty)$ and thus we may write

$$u^2(x) = \int_{-\infty}^{x} u\, u' \leq \Big(\int_{-\infty}^{A} u^2\Big)^{1/2} \Big(\int_{-\infty}^{A} u'^2\Big)^{1/2} \xrightarrow[A \to \infty]{} 0$$

so that $u(x_n) \to 0$ as $n \to +\infty$.

The right side in the equation on u_n in (58) is bounded by $f'(0)u_n$. Therefore we use elliptic regularity and, up to extraction of a subsequence, we know that $u_n \to u_\infty$ and $v_n \to v_\infty$ as $n \to \infty$ in $\mathcal{C}^2_{loc}(\mathbb{R})$. These functions satisfy

$$\begin{cases} -u_\infty'' - cu_\infty' + \chi(v_\infty' u_\infty)' = f'(0)u_\infty, \\ -dv_\infty'' + v_\infty = 0. \end{cases} \tag{59}$$

As v_∞ is non-negative and bounded, we necessarily have $v_\infty \equiv 0$.

Furthermore, as $u_\infty(0) = 1$ and $u_\infty \geq 0$, the maximum principle yields that $u_\infty > 0$. Thus we can explicitly solve the first equation and the solution can only be of the exponential type: $u_\infty(x) = \mu e^{-\lambda x}$. Inserting such a λ in the equation for u_∞ we find $-\lambda^2 + c\lambda = f'(0)$. Hence we have proved that necessarily $c \geq 2\sqrt{f'(0)}$. $\square$

5 Time evolution problem

We now consider the problem

$$\begin{cases} u_t - u_{xx} + \chi(uv_x)_x = u(1 - u), \\ -dv_{xx} + v = u, \\ u(t = 0) = u_0, \quad \text{with compact support}, \quad 0 \leq u_0(x) \leq (1 - \chi/d)^{-1}. \end{cases} \tag{60}$$

The maximum principle, as already used earlier, implies that we have the uniform bounds

$$0 \leq u(x), v(x) \leq \frac{d}{d - \chi}, \quad |v_x(t, x)| \leq K, \quad |v_{xx}(t, x)| \leq K.$$

Our goal in this section is to prove two kinds of results on this problem. Firstly we assume that χ satisfies the conditions of existence of traveling waves. Then, we derive some bounds expressing that in the long time limit, the solution converges to 1 on compact sets. Secondly we show that, under the (weaker) linear stability condition on χ, the state 1 is in fact nonlinearly asymptotically stable.

5.1 The long time limit of $u(t, x)$

We have the following

Theorem 5.1 *Assume $\chi \le \min(1,d)$. There exist $C>0$ and $\varepsilon_0>0$ so that for any $\varepsilon \in (0,\varepsilon_0)$ there exists a time t_0 so that for all $T>t_0$ the following holds. There exists a set $B \subset [T,2T]$ of exceptional times, with $|B| \le C/\varepsilon$ such that for all non-exceptional $t \in [T,2T]\cap B^c$ and all $p\in[0,1/2)$ we have*

$$|\{x:\ |1-u(t,x)| \ge \varepsilon^p\}| \le C\varepsilon^{1-2p}\int u(t,x)dx. \tag{61}$$

The constant $C>0$ in Theorem 5.1 does not depend on the time T. Therefore, the total set B of "bad" times between a (large) time T and $2T$ is bounded independent of T. The right side of (61) may be loosely interpreted as the size on the support of the function $u(t,x)$ (disregarding the fact that $u(t,x)$ has an infinite support). Thus, (61) may be interpreted as saying that for large times the fraction of the support of $u(t,x)$ where $u(t,x)$ is far from 1 is negligible, except for a (relatively) small set of bad times.

We first prove the following proposition.

Proposition 5.2 *Assume that (4) holds and let the initial data $u_0(x)\not\equiv 0$ be compactly supported, $0\le u_0(x)\le 1$. There exist two constants K_1 and K_2 which do not depend on the initial data, and a time t_0 so that*

$$K_1(t-t_0) \le \int u(t,x)dx \le K_2(t_0+t).$$

Proof. First, let u and v be solutions of (60) and consider the function $\psi(t,x)=Me^{-\lambda(x-\xi t)}$. It satisfies the inequality

$$\begin{aligned}\psi_t-\psi_{xx}+\chi v_x\psi_x+\chi v_{xx}\psi-\psi(1-\psi) &\ge \psi_t-\psi_{xx}+\chi v_x\psi_x-\frac{\chi}{d}(u-v)\psi-\psi\\ &\ge \psi_t-\psi_{xx}-K|\psi_x|-K\psi\left(\lambda\xi\psi-\lambda^2-K-K\lambda\right)\psi \ge 0.\end{aligned}$$

This last inequality holds provided that ξ is sufficiently large and λ is chosen appropriately. Therefore, we may also take M large enough so that $\psi_M(t,x)$ is a super-solution for $u(t,x)$. Similarly, $\phi_M(t,x)=Me^{\lambda(x+\xi t)}$ is a super-solution for u. Therefore, we have

$$u(t,x) \le \min\left(Me^{-\lambda(x-\xi t)}, Me^{\lambda(x+\xi t)}\right)$$

therefore, integrating in x

$$\int_{\mathbb{R}} u(t,x)\ dx \le C(t+t_0). \tag{62}$$

And the upper bounded statement of the Proposition is proved.

To obtain a lower bound on $\|u(t)\|_{L^1}$ we proceed as in the traveling wave case. We have

$$\frac{d}{dt}\int(u-\frac{u^2}{2}) = \int(u_x)^2+\int u(1-u)^2-\chi\int u_xv_xu. \tag{63}$$

The last integral on the right side may be estimated as

$$\chi\int uu_xv_xdx = \chi\int(u-1)u_xv_xdx+\chi\int u_xv_xdx.$$

The second term is bounded as

$$\left(\int u_xv_xdx\right)^2 \le \int u_x^2dx\int v_x^2dx \le \left(\int u_x^2dx\right)^2,$$

while the first one satisfies

$$\chi\int(u-1)u_xv_xdx=\frac{\chi}{2}\int\left((u-1)^2\right)_xv_xdx=\frac{\chi}{2d}\int(u-1)^2(u-v)dx\le\frac{\chi}{2d}\int(u-1)^2udx.$$

Using the last two inequalities in (63) leads to

$$\begin{aligned}\frac{d}{dt}\int(u-\frac{u^2}{2})&\ge\int(u_x)^2+\int u(1-u)^2-\chi\int u_x^2-\frac{\chi}{2d}\int(u-1)^2udx\\&\ge M\int(u_x)^2+M\int u(1-u)^2.\end{aligned}\tag{64}$$

Integrating in time and combining this with the upper bound in (62) we obtain

$$\int_0^T\int(u_x)^2+\int_0^T\int u(1-u)^2\le\int u(T,x)dx\le C(1+T).\tag{65}$$

Note that if at some time $t\in[0,T]$ there exists x_0 such that $u(t,x_0)>1/2$ then we have

$$M\int u_x^2(t,x)dx+M\int u(t,x)(1-u(t,x))^2dx\ge K.$$

On the other hand, if we have $0\le u(t,x)\le 1/2$ for all $x\in\mathbb{R}$ then

$$\int u(t,x)(1-u(t,x))^2dx\ge\frac{1}{4}\int u(t,x)dx.$$

Let $\mathcal{A}_T=\{t\in[0,T]:\ 0\le u(t,x)\le 1/2 \text{ for all } x\in\mathbb{R}\}$ – it follows from the above that there exists a constant $K>0$ so that

$$\int u(T,x)\ge\int_{\mathcal{A}_T^c}Kdx+K\int_{\mathcal{A}_T}\left(\int u(t,x)dx\right)dt.\tag{66}$$

As a consequence, the function

$$W(t)=\int u(t,x)dx$$

satisfies

$$W(T)\ge\int_{\mathcal{A}_T^c}Kdx+K\int_{\mathcal{A}}W(t)dt\ge K\int_0^T\min(1,W(t))dt,\tag{67}$$

for all $T\ge 0$. In addition, $W(t)$ is locally Lipschitz in time:

$$|W_t(t)|=\left|\int u(t,x)(1-u(t,x))dx\right|\le M\int u(t,x)dx\le C(1+t).$$

Therefore, in particular, there exists τ_0 so that $W(t)\ge W(0)/2>0$ for $0\le t\le\tau_0$ and thus there exists $k_0>0$ (which depends on the initial data) such that

$$W(T)\ge K\int_0^T\min(1,W(t))dt\ge K\int_0^{\tau_0}\min(1,W(t))dt\ge k_0.$$

Going back to (66) we see that

$$W(T)\ge K|\mathcal{A}_T^c|+k_0K|\mathcal{A}|\ge k_0KT.$$

In order to get rid of the dependence on the initial data observe that, as a consequence we have $W(T) \geq 1$ for all $T \geq t_0$ (the time t_0 does depend on the initial data). Hence, it follows from the second inequality in (67) that

$$W(T) \geq K \int_0^T \min(1, W(t))dt \geq K(T - t_0). \tag{68}$$

This finishes the proof of Proposition 5.2. □

Proof of Theorem 5.1. Theorem 5.1 is an easy consequence of Proposition 5.2 and its proof. Let us start with the inequality (64)

$$\frac{d}{dt} \int \left(u - \frac{u^2}{2}\right) dx \geq M \int u_x^2 dx + M \int u(1-u)^2 dx. \tag{69}$$

Consider the set $B \subset [T, 2T]$ of times $t \in [T.2T]$ such that

$$\int u(t,x)(1-u(t,x))^2 dx \geq \varepsilon \int \left(u(t,x) - \frac{u^2(t,x)}{2}\right) dx.$$

Let us set

$$Q(t) = \int \left(u(t,x) - \frac{u^2(t,x)}{2}\right) dx.$$

Exactly as in the proof of Proposition 5.2 we deduce that

$$C_1(t - t_0) \leq Q(t) \leq C_2(t_0 + t). \tag{70}$$

As $Q(t)$ is monotonically increasing in time, integrating (69) over B we obtain

$$Q(2T) \geq Q(T)e^{\varepsilon|B|}.$$

For $t > 10t_0$ it follows that

$$4C_2T \geq C_1Te^{\varepsilon|B|},$$

so that $|B| \leq K/\varepsilon$ with the constant K independent of $T > t_0$. On the other hand, for times $t \in [T, 2T] \cap B^c$ we have

$$\varepsilon^{2p} \left|\{x : \ |1 - u(t,x)| \geq \varepsilon^p\}\right| \leq C \int u(t,x)(1-u(t,x))^2 dx \leq C\varepsilon \int u(t,x)dx,$$

and (61) follows. □

5.2 Nonlinear asymptotic stability of the homogeneous state $(1, 1)$

In this section, we consider the Keller-Segel-Fisher system and we consider the stability of the state $(1, 1)$ as discussed in the introduction -see (14)–(15). Therefore, we set $u = 1 + U$ and $v = 1 + V$ and the system (60) writes

$$\begin{cases} U_t - U_{xx} + \chi(V_x U)_x = -U(1+U) - \chi V_{xx}, \\ -dV_{xx} + V = U, \\ U(t=0,x) = U_0(x) := u^0 - 1, \qquad x \in \mathbb{R}. \end{cases} \tag{71}$$

We prove that the linear stability of the homogeneous equilibrium state $(u = 1, v = 1)$ implies its nonlinear asymtotic stability. More precisely

Theorem 5.3 *For $\chi < (1+\sqrt{d})^2$, there is a positive constant $\delta > 0$ such that for any initial data $u_0 = 1 + U_0$ with $\int_{\mathbb{R}} U_0^2 < \delta$, then the solution u of the Cauchy problem (60) converges to 1 in the L^2 norm with an exponetial rate*

$$\int_{\mathbb{R}} (u(t,x)-1)^2 dx \to 0 \quad as \;\; t \to +\infty. \tag{72}$$

Proof of Theorem 5.3. For $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, we set $U(t,x) := u(t,x) - 1$, $V(t,x) := v(t,x) - 1$ and $\lambda := (1+\sqrt{d})^2 - \chi > 0$. Multiplying equation (71) by U and integrating over $\mathbb{R}$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} U^2 dx + \int_{\mathbb{R}} \left(U_x^2 - \chi V_x^2 + U^2\right) &= -\int_{\mathbb{R}} U^3 + \chi \int_{\mathbb{R}} U U_x V_x = -\int_{\mathbb{R}} U^3 - \frac{\chi}{2}\int_{\mathbb{R}} U^2 V_{xx} \\ = \left(\frac{\chi}{2d} - 1\right)\int U^3 - \frac{\chi}{2d}\int_{\mathbb{R}} U^2 V &\leq \left|\frac{\chi}{2d} - 1\right| \int |U|^3 + \frac{\chi}{2d}\int_{\mathbb{R}} U^2 |V|.\end{aligned} \tag{73}$$

The second term on the left side of (73) can be written as

$$\int_{\mathbb{R}} (U_x^2 - \chi V_x^2 + U^2) dx = \int_{\mathbb{R}} \left(\xi^2 + 1 - \frac{\chi \xi^2}{1+d\xi^2}\right) |\hat{U}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{P(\xi)}{(1+d\xi^2)(1+\xi^2)} (1+\xi^2)\hat{U}(\xi)^2 d\xi,$$

where P is a fourth order poynomial function which is positive since $\chi < (1+\sqrt{d})^2$.

As $p(\xi) = (1+d\xi^2)(1+\xi^2)$ is also a positive fourth order polynomial function, the quotient $P(\xi)/[(1+d\xi^2)(1+\xi^2)]$ has a positive infimum $\lambda > 0$. This gives:

$$\int_{\mathbb{R}} \left[(U_x)^2 - \chi V_x^2 + U^2\right] dx \geq \lambda \int_{\mathbb{R}} (1+\xi^2)\hat{U}(\xi)^2 d\xi \geq \lambda \int_{\mathbb{R}} \left(U_x^2 + U^2\right) dx. \tag{74}$$

Next, set $I(t) = \int_{\mathbb{R}} U^2 dx$, the above computation yields that

$$\frac{1}{2}\frac{d}{dt} I(t) + \lambda I(t) + \lambda \int_{\mathbb{R}} U_x^2 \leq \left|\frac{\chi}{2d} - 1\right| \int |U|^3 + \frac{\chi}{2d}\int_{\mathbb{R}} U^2 |V|. \tag{75}$$

We treat the two terms of the right side separately using Gagliardo-Nirenberg-Sobolev type inequalities :

$$\int_{\mathbb{R}} |U|^3 \leq C \left(\int_{\mathbb{R}} U_x^2\right)^{1/4} \left(\int_{\mathbb{R}} U^2\right)^{5/4} \leq \frac{\lambda}{2|\frac{\chi}{2d} - 1|}\int_{\mathbb{R}} U_x^2 + M \left(\int_{\mathbb{R}} U^2\right)^{5/3} \tag{76}$$

(the second inequality follows from the Minkowski inequality). In the same way we obtain

$$\begin{aligned}\int_{\mathbb{R}} |U|^2 |V| &\leq \left(\int_{\mathbb{R}} |U|^4 \int_{\mathbb{R}} V^2\right)^{1/2} \leq C_1 \left(\int_{\mathbb{R}} U_x^2\right)^{1/4} \left(\int_{\mathbb{R}} U^2\right)^{3/4} \left(\int_{\mathbb{R}} U^2\right)^{1/2} \\ &\leq \frac{\lambda d}{\chi}\int_{\mathbb{R}} U_x^2 + M' \left(\int_{\mathbb{R}} U^2\right)^{5/3},\end{aligned}$$

where M' is a constant that only depends on C', χ, d and λ. This finally gives:

$$\frac{1}{2}\frac{d}{dt} I(t) + \lambda I(t) + \lambda \int_{\mathbb{R}} U_x^2 \leq \lambda \int_{\mathbb{R}} U_x^2 + (M+M') I^{5/3}(t), \tag{77}$$

and thus:

$$\frac{1}{2}\frac{d}{dt} I(t) + \lambda I(t) \leq (M+M') I^{5/3}(t). \tag{78}$$

Set now $\delta = (\lambda/(M+M'))^{3/2}$. Then, for $I(0) < \delta$, the differential inequality (78) yields that $t \mapsto I(t)$ decreases. As it is a nonnegative function it converges to the equilibrium state $I \equiv 0$. Also, there is an exponential decay (with rate as close to 2λ as we wish) and the proof is complete. □

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