# KPP invasions in periodic media: lecture notes for the Toulouse KPP school

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## 1 Introduction to the Fisher-KPP equation

### The Freidlin-Gärtner formula

In these notes we will consider equations of the form

$$u_t - \Delta u = \mu(x)u - u^2, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

with a smooth function $\mu(x)$ that is 1-periodic in all variables $x_j$, $j = 1, \ldots, n$. This equation is known as the Fisher-Kolmogorov-Petrovskii-Piskunov, or Fisher-KPP equation, and was introduced in 1937 by Fisher, and KPP, in their two respective papers, Fisher's paper focusing on numerical and "applied tools" analysis, and KPP giving a rigorous mathematical treatment. Both papers were pioneering in many respects, and are true classics of applied mathematics and applied analysis. The Fisher-KPP equation is at the first sight a simple combination of the diffusion equation

$$u_t = \Delta u,$$

and an ordinary differential equation

$$\frac{du}{dt} = \mu u - u^2.$$

This simplicity is not totally deceptive but nevertheless, this model provided mathematics rich enough to survive almost eighty years of attack and is still capable of providing new and surprising results.

Before going to the origins of the model, let us describe the particular mathematical question we will focus on: what happens to solutions of (1.1) with compactly supported initial data $u(0, x) = u_0(x)$ such that $0 \le u_0(x) \le 1$. A crucial role in the final result on the

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time evolution will be played by the periodic eigenvalue problem for the Schrödinger operator with the potential $\mu(x)$:

$$\begin{aligned}&-\Delta\phi-\mu(x)\phi=\lambda\phi,\\&\phi(x)\text{ is 1-periodic in all its variables.}\end{aligned}\tag{1.2}$$

A classical result of the spectral theory for second order elliptic operators (a good basic reference is, as usual, [6]) is that this eigenvalue problem is self-adjoint, has a purely discrete spectrum $\lambda_k$, $k\in\mathbb{N}$, with

$$\lim_{k\to+\infty}\lambda_k=+\infty,$$

and all eigenvalues of (1.2) are real. The Krein-Rutman theorem [5], together with the comparison principle, implies that there is a unique eigenvalue $\lambda_1$ that corresponds to a positive eigenfunction $\phi_1$ (all other eigenfunctions change sign). Moreover, $\lambda_1$ is a simple eigenvalue, and it is is the smallest eigenvalue of (1.2). It is called the principal eigenvalue of (1.2), and has a variational characterization in terms of the Rayleigh quotient:

$$\lambda_1=\inf_{\psi\in H^1(\mathbb{T}^n)}\frac{\displaystyle\int_{\mathbb{T}^n}(|\nabla\psi|^2-\mu(x)\psi^2)dx}{\displaystyle\int_{\mathbb{T}^n}|\psi(x)|^2dx}.\tag{1.3}$$

Here $\mathbb{T}^n=[0,1]^n$ is the $n$-dimensional torus (the unit period cell of $\mu(x)$), and $H^1(\mathbb{T}^n)$ is the set of all 1-periodic functions in the Sobolev space $H^1$. Our main assumption about the function $\mu(x)$ will be that

$$\lambda_1<0.\tag{1.4}$$

This condition holds, for instance, if the (continuous) function $\mu(x)$ is non-negative and not identically equal to zero in $\mathbb{T}^n$: this can be seen by simply taking the test function $\psi(x)\equiv 1$ in (1.4). However, in general, we allow $\mu(x)$ to change sign. Assuming (1.4), we will show the following: first, the steady equation

$$-\Delta u=\mu(x)u-u^2,\quad x\in\mathbb{R}^n,\tag{1.5}$$

posed in the whole space, has a unique positive bounded solution $p(x)$. Moreover, $p(x)$ is 1-periodic in all variables. Second, any solution of the Cauchy problem for (1.1) with a nonnegative, bounded and compactly supported initial data $u_0(x)$ (that is positive on some open set) will tend to $p(x)$ as $t\to+\infty$, uniformly on every compact subset of $\mathbb{R}^n$. For example, when $\mu(x)\equiv 1$, then $p(x)\equiv 1$, and this result says that $u(t,x)\to 1$ as $t\to+\infty$, uniformly on compact sets in $x$.

Finally, and this is the core of these lectures, we will prove the following propagation result. For each unit vector $e\in\mathbb{R}^n$, $|e|=1$, consider the solution of the linear equation

$$v_t-\Delta v=\mu(x)v,\quad x\in\mathbb{R}^n,\tag{1.6}$$

of the form

$$v(t,x)=e^{-\lambda(x\cdot e-ct)}\phi(x),\tag{1.7}$$

with a positive 1-periodic function $\phi(x)$. Such exponential solutions are extremely important in the theory for the nonlinear problem. We will see that for each direction $e \in \mathbb{S}^{n-1}$ they exist only for $c \geq c_*(e)$, where $c_*(e)$ is the smallest possible propagation speed of such exponential (it does depend on the choice of $e$). If we set

$$w_*(e) = \inf_{|e'|=1,(e\cdot e')>0} \frac{c_*(e')}{(e \cdot e')}, \tag{1.8}$$

then solutions of the nonlinear problem (1.1) with a nonnegative bounded and compactly supported initial data $u_0(x)$ obey the following asymptotics: for each $w \in (0, w_*(e))$, and each $x \in \mathbb{R}^n$, we have

$$\lim_{t\to+\infty} \sup_{r\in[0,w]} |u(t, x + rte) - p(x + rte)| = 0, \tag{1.9}$$

and for each $w \in (w_*(e), +\infty)$ we have

$$\lim_{t\to+\infty} \sup_{r\geq w} u(t, x + rte) = 0. \tag{1.10}$$

That is, if we observe the solution $u(t, x)$ along the ray in the direction $e \in \mathbb{S}^{n-1}$, $u(t, x)$ is close to $p(x)$ at distances much smaller than $w_*(e)t$, and $u(t, x)$ is close to zero at distances much larger than $w_*(e)t$. The remarkable fact is that the invasion speed $w_*(e)$ is completely determined by the linear problem (1.6)! This is a reflection of a general principle that we will see repeatedly[1] "every question for the KPP equation can be understood from an appropriate version of the linearized equation". The more precise asymptotics for the location of the transition between these two regions is known as the "Bramson correction" and is somewhat more delicate.

This propagation result was discovered by Freidlin and Gärtner [10] who proved it with probabilistic tools. Since then its scope was considerably extended and at least four additional methods of proof are known:
(i) Probabilistic proofs using large deviation methods, due to Freidlin [11].
(ii) Viscosity solution methods (Evans and Souganidis [7, 8]).
(iii) Monotone dynamical systems methods in the discrete setting (Weinberger [22]).
(iv) PDE methods that we adapt in the present notes to the relatively simple case we consider (Berestycki, Hamel and Nadin [2]).

## Origins of the model

Let us briefly discuss how the Fisher-KPP equation comes about. The original motivation in [9] and [14] was by problems in genetics, while Freidlin and Gärtner motivated their study of (1.1) as a general model for concentration waves in a periodic medium. There is also a nice interpretation of this equation in terms of population dynamics. Let a population of animals, or bacteria, or even some flora be described in terms of its local density $u(t, x)$. That is,

[1]As any other general principle, it can be easily disproved, and, in particular, the meaning of the word "appropriate" occasionally requires a serious thought for a particular question about the solutions of the KPP equation.

$u(t,x)dx$ is the number of individuals present at time $t$ in an infinitesimal volume $dx$ around a point $x$ – the total number of individuals present in a given domain $\Omega$ at a time $t$ is

$$\int_\Omega u(t,x)dx.$$

This description assumes implicitly that the number of individuals is large, or equivalently, they are not too sparse – probably, one should not describe the population of camels in a desert in this way. The individuals multiply and disappear. In other words, in the absence of a spatial displacement, the population density evolves as

$$\frac{du}{dt} = \mu(x)u - u^2 = (\mu(x) - u)u. \tag{1.11}$$

Here, $x$ is the spatial position, and $\mu(x)$ is the local growth rate at $x$ for small $u$. These equations are uncoupled at different points $x$. The negative term in the right side of (1.11) accounts for the fact that there are limited resources – too many individuals present at one point prevent population growth due to competition. The threshold value at which the growth becomes negative in this model is $u = \mu(x)$. Hence, $\mu(x)$ can be both interpreted as the growth rate for small $u$, and as the carrying capacity of the population. We will allow $\mu(x)$ to be negative in some regions – these would reflect a "hostile" environment as opposed to a "favorable" domain where $\mu(x) > 0$. As we will see, the overall balance between the "good" and "bad" regions that measures the chances of survival is measured by the principal eigenvalue $\lambda_1$ that we have mentioned before.

An aspect missing in (1.11) is movement of the individuals, displacements and migrations. Assume for the moment that there is no growth of the population but the species may disperse. If the chances of entering a small volume $dx$ around $x$ from position $y$ are $k(x,y)$ then the balance equation for the population density is

$$\frac{\partial u(t,x)}{\partial t} = \int k(x,y)u(t,y)dy - \left(\int k(y,x)dy\right) u(t,x). \tag{1.12}$$

The first term on the right accounts for individuals entering the volume $dx$ from all other positions $y$ and the negative term accounts for those leaving $dx$. Assume now that the transition kernel $k(x,y)$ is localized and radially symmetric:

$$k(x,y) = \frac{1}{\varepsilon^n} r\left(\frac{|x-y|}{\varepsilon}\right),$$

and the mean drift is zero:

$$\int xr(x)dx = 0.$$

Then, expanding (1.12) in $\varepsilon$ we obtain, in the leading order:

$$\frac{\partial u}{\partial t} = D\varepsilon^2 \Delta u, \tag{1.13}$$

with the diffusion coefficient

$$D = \int |x|^2 r(x)dx.$$

**Exercise 1.1** This formal procedure is not difficult to make rigorous – it is, essentially, the PDE version of the convergence of a discrete time random walk on a lattice to a Brownian motion when the lattice step and the time step are scaled appropriately. Make this connection in a careful fashion.

Putting (1.11) and (1.13) together (with the appropriate time rescaling in (1.13) to get rid of the $\varepsilon^2$ factor and setting $D = 1$) gives the Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \Delta u + \mu(x)u - u^2, \tag{1.14}$$

that we will study in this chapter. A much more detailed explanation of the modeling issues is given in Murray's books [16, 17]. One may also obtain this equation in a more careful fashion using birth-death processes in probability theory.

A completely different point of view of where the Fisher-KPP equation comes from, and why it is important, comes from the probability theory [15]. Consider a branching Brownian motion that starts at $x = 0$ and branches at an exponentially distributed rate. This means that the particle starts at $x = 0$ and performs a Brownian motion until a random time $\tau_1$, with the probability distribution function

$$P(\tau_1 > t) = e^{-t}.$$

At this time the particle splits into two particles, each of them continues to perform an independent Brownian motion, until an exponentially distributed time when it splits into two, and the process is repeated – note that the times at which individual particles branch are independent. Then, at any given time $t > 0$ we will have a random number $N_t$ of particles $X_1(t) \le X_2(t) \le \cdots \le X_{N_t}(t)$ (the number $N_t$ depends on $t$ as well), with $X_{N_t}(t)$ being the rightmost particle at this time. The remarkable fact is that the probability distribution function of the rightmost particle

$$u(t,x) = \mathrm{Prob}\left[X_{N_t}(t) \ge x\right]$$

satisfies the Fisher-KPP equation

$$u_t = \frac{1}{2}u_{xx} + u - u^2,$$

with the initial data $u(0,x) = \mathbb{1}_{x\le 0}$. We will not pursue this connection here, but the reader should be aware that a very rich probabilistic literature on the subject exists that also provides a rich intuition for the behavior of the solutions of the Fisher-KPP equation.

## Pushed and pulled fronts

The equation

$$u_t = \Delta u + \mu u - u^2 \tag{1.15}$$

is an example of a more general equation of the form

$$u_t = \Delta u + f(u). \tag{1.16}$$

Such equation is said to be of the Fisher-KPP type if the nonlinearity $f(u)$ satisfies the following assumption:

$$f \in C^1[0,1], \quad f(0) = f(1) = 0, \quad f(u) > 0 \text{ for all } u \in (0,1), \tag{1.17}$$
$$f(u) \le f'(0)u. \tag{1.18}$$

The crucial assumption above is (1.18) – it means that the fastest rate of growth $f(u)/u$ is close to $u = 0$. We will see that it implies that solutions of the Cauchy problem for the Fisher-KPP type problems are governed by what happens far ahead of the bulk of the solution, where $u$ is small and it grows the fastest. Because of that the solutions are said to be pulled.

In order to appreciate the difference between "pulled" Fisher-KPP fronts and "pushed" fronts for other types of nonlinearity $f(u)$, consider a reaction-diffusion equation

$$u_t = D\Delta u + f(u), \tag{1.19}$$

with a nonlinearity $f(u)$ that is not of the Fisher-KPP type but rather satisfies

$$f \in C^1[0,1], \quad f(u) = f(1) = 0, \text{ for all } 0 \le u \le \theta_0, \text{and } f(u) > 0 \text{ for all } u \in (\theta_0, 1), \tag{1.20}$$

with some $\theta_0 > 0$. Here, the rate of growth vanishes if $u$ is small – a very small population can not grow. Such nonlinearities are known as the ignition type, and $\theta_0$ is known as the ignition temperature because they are commonly used in the combustion literature. Let us consider solutions of (1.19) with the initial data $u_0(x)$ such that $0 < u_0(x) < 1$ that decays as $|x| \to +\infty$. Consider the "extreme" case $D = 0$, then if $f(u)$ is of the ignition type, we will have

$$u(t,x) \equiv u_0(x), \quad \text{if } 0 < u_0(x) \le \theta_0,$$

and

$$u(t,x) \to 1, \text{ as } t \to +\infty \text{ if } \theta_0 < u_0(x).$$

This means that there is no propagation – solution tends to one only on the (bounded) set where $u_0(x) > \theta_0$. On the other hand, if $f(u)$ is if the Fisher-KPP type, then

$$u(t,x) \to 1, \text{ as } t \to +\infty \text{ for all } x \in \mathbb{R}.$$

This is another sense in which Fisher-KPP propagation is pulled – it may occur even in the absence of diffusion, while for the ignition type nonlinearities diffusion is necessary to make solutions propagate.

# 2 The steady solution as the long time limit for the Cauchy problem

It is reasonable to expect that if solutions of (1.14) converge as $t \to +\infty$ to a certain limit $p(x) \neq 0$, this function should satisfy the steady problem[2]

$$\begin{gathered} -\Delta p = \mu(x)p - p^2, \quad x \in \mathbb{R}^n, \\ p(x) > 0 \text{ for all } x \in \mathbb{R}^n \text{ and } p(x) \text{ is bounded.} \end{gathered} \tag{2.1}$$

[2] Another reasonable possibility is that the limit is a solution of the time-dependent problem that is defined for all times, positive and negative, of which a steady solution is just one example.

The condition $p(x) > 0$ comes from the maximum principle: it is natural to assume that $p(x) \geq 0$ as it is "physically" a population density, and then the maximum principle implies that $p(x) > 0$ for all $x \in \mathbb{R}^n$. In this section we will investigate existence of such steady solutions.

This question is not immediately obvious even in the homogeneous case: is $u \equiv 1$ the only non-negative (not identically equal to zero) bounded solution of (we consider the one-dimensional case for utmost simplicity)

$$\Delta u + u(1-u) = 0, \quad x \in \mathbb{R}, \tag{2.2}$$

in the whole space? The maximum principle immediately implies that $u(x)$ can not have a maximum $x_m$ such that $u(x_m) \geq 1$, whence either $u(x) \equiv 1$, or $0 < u(x) < 1$ for all $x \in \mathbb{R}$. For the same reason, in the latter case $u(x)$ can not have a local minimum, thus it has to vanish as $x \to -\infty$ or $x \to +\infty$. We assume without loss of generality that this happens as $x \to +\infty$. Consider then the function $\psi_\varepsilon(x) = \varepsilon \sin(\pi x/R)$ defined on the interval $[0, R]$. It satisfies $\psi_\varepsilon(0) = \psi_\varepsilon(R) = 0$, and

$$\Delta \psi_\varepsilon + \psi_\varepsilon(1-\psi_\varepsilon) = -\frac{\pi^2}{R^2}\varepsilon\psi_\varepsilon + \varepsilon\psi_\varepsilon - \varepsilon^2\psi_\varepsilon^2 = \varepsilon\psi_\varepsilon(1 - \frac{\pi^2}{R^2} - \varepsilon\psi_\varepsilon) > 0,$$

if $R$ is sufficiently large and $\varepsilon > 0$ is sufficiently small. In that case, $\psi_\varepsilon(x)$ is a subsolution to (2.2). Let us choose $\varepsilon > 0$ so small that $\psi_\varepsilon(x) < u(x)$ for $0 \leq x \leq R$, and look at the shifts $\psi_{\varepsilon,z}(x) = \psi_\varepsilon(x - z)$, defined for $z \leq x \leq z + R$. As $u(x) \to 0$ as $x \to +\infty$, there exists

$$z_0 = \sup\{z : \ \psi_{\varepsilon,z}(x) \leq u(x) \text{ for all } z \leq x \leq z + R.$$

It is immediate to see that $\psi_{\varepsilon,z_0}(x) \leq u(x)$ for all $x \in [z_0, z_0+R]$ but there exists $x_0 \in (z, z+R)$ such that $\psi_{\varepsilon,z_0}(x_0) = u(x_0)$, that is, the graphs of $u(x)$ and $\psi_{\varepsilon,z_0}(x)$ touch at $x_0$. This is a contradiction to the fact that $u(x)$ is a solution to (2.2), and $\psi_{\varepsilon,z_0}$ is a sub-solution. Therefore, it is impossible that $u(x) \to 0$ as $x \to \pm\infty$, whence $u(x) \equiv 1$, and (2.2) has a unique bounded positive solution.

We go back to the general case (2.1) – the technicalities will be less trivial, but the heart of the analysis will be exactly as above. One of the main points here is that we impose neither periodicity nor any decay conditions on $p(x)$ as $|x| \to +\infty$, but only require that $p(x)$ is positive and bounded. Let us recall that we denote by $\lambda_1$ the principal eigenvalue of

$$\begin{aligned} &-\Delta\phi - \mu(x)\phi = \lambda_1\phi, \\ &\phi(x) \text{ is 1-periodic in all its variables, } \phi(x) > 0, \end{aligned} \tag{2.3}$$

and that the requirement that the eigenfunction $\phi(x)$ is positive identifies $\lambda_1$ uniquely because of the Krein-Rutman theorem. The next result explains the role of the principal eigenvalue rather succinctly.

**Theorem 2.1** *The problem (2.1) has a unique solution if $\lambda_1 < 0$ and no solutions if $\lambda_1 \geq 0$.*

The existence part of Theorem 2.1 has been known for a long time now but the uniqueness part is recent [3]. This result is important for two reasons: (1) it classifies all solutions to the steady problem, and (2) it is the key to understanding the long time behavior of the solutions to the corresponding Cauchy problem, as shown by the following theorem.

**Theorem 2.2** *Let $u(t,x)$ be the solution of the initial value problem*

$$\frac{\partial u}{\partial t} = \Delta u + \mu(x)u - u^2, \quad t > 0,\ x \in \mathbb{R}^n, \qquad (2.4)$$
$$u(0,x) = u_0(x),$$

*with a bounded non-negative function $u_0(x)$ such that $u_0(x) \not\equiv 0$, and let $\lambda_1$ be the principal eigenvalue of (2.3). Then if $\lambda_1 < 0$ we have*

$$u(t,x) \to p(x) \text{ as } t \to +\infty, \qquad (2.5)$$

*uniformly on compact sets $K \subset \mathbb{R}^n$. On the other hand, if $\lambda_1 \geq 0$ then*

$$u(t,x) \to 0 \text{ as } t \to +\infty, \qquad (2.6)$$

*uniformly in $\mathbb{R}^n$.*

In the rest of this section we will prove these two theorems – the proof of Theorem 2.1, in particular, is not short but it utilizes various tools that are interesting in their own right.

## Triviality of the steady solutions when $\lambda_1 \geq 0$

Let us first explain what happens if $\lambda_1 > 0$. Let $\phi(x)$ be the corresponding (periodic) principal eigenfunction of (2.3), and let $u(t,x)$ satisfy the time-dependent problem (2.4). As $\phi(x)$ is periodic and positive, its minimum is positive. Hence, as $u_0(x)$ is bounded, we can find $M > 0$ so that at $t = 0$ we have

$$u(0,x) = u_0(x) \leq \sup_{x \in \mathbb{R}^n} u_0(x) \leq M \min_{x \in \mathbb{T}^n} \phi(x) \leq M\phi(x). \qquad (2.7)$$

The function

$$\psi(t,x) = Me^{-\lambda_1 t}\phi(x)$$

satisfies

$$\psi_t = \Delta\psi + \mu(x)\psi, \qquad (2.8)$$

which means that $\psi(t,x)$ is a super-solution to (2.4):

$$\psi_t > \Delta\psi + \mu(x)\psi - \psi^2. \qquad (2.9)$$

This, together with the inequality (2.7), by virtue of the parabolic maximum principle, implies that for all $t \geq 0$ we have

$$u(t,x) \leq \psi(t,x) = Me^{-\lambda_1 t}\phi(x) \leq M\|\phi\|_{L^\infty(\mathbb{T}^n)} e^{-\lambda_1 t}. \qquad (2.10)$$

It follows that

$$u(t,x) \to 0 \text{ as } t \to +\infty, \qquad (2.11)$$

uniformly in $\mathbb{R}^n$, and, in particular, precludes the existence of non-trivial bounded solutions to (2.1).

The argument in the case $\lambda_1 = 0$ is similar albeit with a nice additional step. In this situation, the eigenfunction is a periodic function $\phi(x) > 0$ such that

$$-\Delta\phi = \mu(x)\phi. \tag{2.12}$$

By the same token as before, we know that for any solution of (2.4) with a bounded initial data $u_0(x) \geq 0$ we can find a constant $M > 0$ so that

$$u(0, x) \leq M\phi(x). \tag{2.13}$$

The parabolic maximum principle[3] implies that this inequality holds for all $t \geq 0$:

$$u(t, x) \leq M\phi(x), \text{ for all } x \in \mathbb{R}^n.$$

Let now $M_k$ be the smallest constant $M$ so that we have

$$u(k, x) \leq M\phi(x) \text{ for all } x \in \mathbb{R}^n, \tag{2.14}$$

at the time $t = k$. The sequence $M_k$ is non-increasing: since $M_k\phi(x)$ is a super-solution, (2.14) together with the strong maximum principle guarantees that (with the strict inequality)

$$u(k + 1, x) < M_k\phi(x), \tag{2.15}$$

which implies that $M_{k+1} \leq M_k$. Let us now show that the strong maximum principle implies that this inequality is strict: $M_{k+1} < M_k$. It suffices to verify this for $k = 1$: assume that $M_2 = M_1$. Then there exists a sequence $x_k$ such that

$$u(2, x_k) \geq \left(M_1 - \frac{1}{k}\right)\phi(x_k). \tag{2.16}$$

Let us define the translates

$$v_k(t, x) = u(t, x + x_k), \quad \phi_k(x) = \phi(x + x_k).$$

The parabolic regularity theory implies that the shifted functions $v_k(t, x)$ and $\phi_k(x)$ are uniformly bounded in $C^{2,\alpha}_{loc}$ for $1 \leq t \leq 2$, hence we may extract a subsequence $k_n \to +\infty$ so that the limits

$$\bar{v}(t, x) = \lim_{n\to+\infty} v_{k_n}(t, x), \quad \bar{\phi}(x) = \lim_{n\to+\infty} \phi_{k_n}(x)$$

exist. The shifted coefficients $\mu_k(x) = \mu(x+x_k)$ also converge after extracting a subsequence, locally uniformly to a limit $\bar{\mu}(x)$. The limits satisfy

$$\frac{\partial\bar{v}}{\partial t} = \Delta\bar{v} + \bar{\mu}(x)\bar{v} - \bar{v}^2, \quad 1 \leq t \leq 2, \ x \in \mathbb{R}^n, \tag{2.17}$$

and

$$-\Delta\bar{\phi} = \bar{\mu}(x)\bar{\phi}. \tag{2.18}$$

[3] Recall that $\phi(x)$ is a super-solution to the problem (2.4) that $u(t, x)$ satisfies.

In addition, we have $\bar{v}(t=1,x) \leq M_1\bar{\phi}(x)$ for all $x \in \mathbb{R}^n$, and $\bar{v}(t=2,x=0) = M_1\bar{\phi}(0)$. This contradicts the strong maximum principle since $\bar{\phi}$ is a strict super-solution to (2.17). Therefore, the sequence $M_n$ is strictly decreasing.

Let now

$$\bar{M} = \lim_{k\to+\infty} M_k. \tag{2.19}$$

We need to show that $\bar{M} = 0$, in order to conclude that $u(t,x) \to 0$ as $t \to +\infty$, uniformly in $x \in \mathbb{R}^n$. As in the previous step, choose $x_k$ so that

$$v(k, x_k) \geq (M_k - \frac{1}{k})\phi(x_k),$$

and define the translates

$$v_k(t,x) = v(k+t, x_k + x), \quad \phi_k(x) = \phi(x + x_k), \tag{2.20}$$

as well as $\mu_k(x) = \mu(x + x_k)$. Note that

$$v(k-1, x) \leq M_{k-1}\phi(x), \tag{2.21}$$

for all $x \in \mathbb{R}^n$. Once again, the parabolic regularity theory implies that the sequences $v_k(t,x)$, $\phi_k(x)$ and $\mu_k(x) = \mu(x+x_k)$ (after extraction of a subsequence) converge as $k \to +\infty$, locally uniformly, to the respective limits $\bar{v}(t,x)$, $\bar{\phi}(x)$ and $\bar{\mu}(x)$ that satisfy, in this case,

$$\frac{\partial \bar{v}}{\partial t} = \Delta\bar{v} + \bar{\mu}(x)\bar{v} - \bar{v}^2, \quad -\infty < t < +\infty,\ x \in \mathbb{R}^n, \tag{2.22}$$

and

$$-\Delta\bar{\phi} = \bar{\mu}(x)\bar{\phi}. \tag{2.23}$$

That is, $\bar{v}(t,x)$ is a global in time solution, defined for positive and negative $t$. In addition, the normalization (2.20) implies that

$$\bar{v}(0,0) = \bar{M}\bar{\phi}(0), \tag{2.24}$$

while we also have, from (2.21):

$$\bar{v}(-1,x) \leq \bar{M}\bar{\phi}(x), \quad x \in \mathbb{R}^n. \tag{2.25}$$

The parabolic strong maximum principle implies that then $\bar{v}(t,x) \equiv \bar{M}\bar{\phi}(x)$ which is only possible if $\bar{M} = 0$. Therefore, $\bar{M} = 0$, and

$$u(t,x) \to 0 \text{ as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}^n,$$

also when $\lambda_1 = 0$.

## Existence of the periodic steady solutions when $\lambda_1 < 0$

We now turn to the most interesting case $\lambda_1 < 0$. We need to show that then a non-trivial steady solution $p(x)$ of (2.1) exists, and, moreover, solution of the parabolic problem converges to it as $t \to +\infty$, locally uniformly in $x$. The crucial idea throughout will be to use compactly supported sub-solutions to the Fisher-KPP equation that come from eigenfunctions of the linearized problem with either the periodic boundary conditions (in the existence part of the proof) or the Dirichlet boundary condition on sufficiently large balls (in the uniqueness part of the proof).

Let $\phi(x)$ be the positive periodic eigenfunction of

$$-\Delta\phi - \mu(x)\phi = \lambda_1\phi. \tag{2.26}$$

Consider the function $\phi_\varepsilon(x) = \varepsilon\phi(x)$. A simple but very important observation is that for $\varepsilon > 0$ sufficiently small we have

$$-\Delta\phi_\varepsilon - \mu(x)\phi_\varepsilon = \lambda_1\phi_\varepsilon \le -\phi_\varepsilon^2, \tag{2.27}$$

that is, $\phi_\varepsilon(x)$ is a sub-solution for the steady nonlinear problem. More precisely, this inequality holds as soon as

$$\varepsilon < -\frac{\lambda_1}{\max_{x\in\mathbb{T}^n}\phi(x)}, \tag{2.28}$$

and it is here that we need the assumption $\lambda_1 < 0$. On the other hand, the constant function $w(x) \equiv M$ satisfies

$$-\Delta w - \mu(x)w = -\mu(x)M \ge -M^2, \tag{2.29}$$

as soon as

$$M \ge \max_{x\in\mathbb{T}^n}\mu(x). \tag{2.30}$$

Therefore, we have both a sub-solution $\phi_\varepsilon(x)$ (with an $\varepsilon$ that satisfies (2.28)) and a super-solution $w(x)$ (with $M$ that satisfies (2.30)) for the steady problem (2.1). With these in hand, a true solution of (2.1) can be constructed using a standard iteration scheme. First, choose a number $N > -2\lambda_1$ and restate (2.1) as

$$-\Delta p(x) - \mu(x)p(x) + Np(x) = Np(x) - p^2. \tag{2.31}$$

The reason to add the term $Np(x)$ on the left is to make sure that all eigenvalues of the periodic problem

$$-\Delta\phi - \mu(x)\phi + N\phi(x) = \lambda\phi, \tag{2.32}$$

are strictly positive. In this case, the inhomogeneous elliptic problem

$$-\Delta\phi - \mu(x)\phi + N\phi(x) = f(x) \tag{2.33}$$

has a unique periodic solution $p(x)$ for any bounded periodic function $f(x)$. Moreover, if $f(x) > 0$ for all $x \in \mathbb{T}^n$ then the solution of (2.33) is also positive.

We set up the iteration scheme as follows: let $p_0 = \phi_\varepsilon(x)$ and for $k \geq 1$ let $p_k(x)$ be the periodic solution of

$$-\Delta p_k - \mu(x) p_k + N p_k(x) = N p_{k-1}(x) - p_{k-1}^2(x). \tag{2.34}$$

We claim that the sequence $p_k(x)$ is increasing pointwise in $x$:

$$p_{k+1}(x) \geq p_k(x), \text{ for all } k \geq 0 \text{ and all } x \in \mathbb{T}^n, \tag{2.35}$$

and satisfies

$$p_k(x) \leq \frac{N}{2} \text{ for all } k \geq 0 \text{ and all } x \in \mathbb{T}^n. \tag{2.36}$$

In order to prove the upper bound (2.36) we observe that $p_0(x) \leq N/2$ if $\varepsilon$ is sufficiently small, and then use induction: define $w_k(x) = N/2 - p_k(x)$, assume that $p_{k-1}(x) \leq N/2$ for all $x \in \mathbb{T}^n$, and write, with $\bar{\mu} = \sup_{x \in \mathbb{T}^n} \mu(x)$:

$$\begin{aligned}
-\Delta w_k - \mu(x) w_k + N w_k &= -\mu(x)\frac{N}{2} + \frac{N^2}{2} + \Delta p_k + \mu(x) p_k - N p_k \\
= -\mu(x)\frac{N}{2} + \frac{N^2}{2} - N p_{k-1} + p_{k-1}^2 &\geq -\bar{\mu}\frac{N}{2} + \frac{N^2}{2} - \frac{N^2}{4} > 0,
\end{aligned}$$

as long as $N > 2\bar{\mu}$. This proves that $w_k(x) > 0$, hence (2.36) holds. The reason for the pointwise monotonicity of the sequence $p_k(x)$ is that $p_0$ is a sub-solution for (2.32). The proof is by induction: set

$$z_k(x) = p_k(x) - p_{k-1}(x), \quad k \geq 1,$$

then $z_1$ satisfies

$$\begin{aligned}
-\Delta z_1 - \mu(x) z_1 + N z_1 &= -\Delta p_1 - \mu(x) p_1 + N p_1 + \Delta p_0 + \mu(x) p_0 - N p_0 \\
&= N p_0 - p_0^2 - \lambda_1 p_0 - N p_0 = -\lambda_1 p_0 - p_0^2 > 0.
\end{aligned} \tag{2.37}$$

The last inequality above holds by virtue of (2.28), and, once again, requires that $\lambda_1 < 0$. It follows that $z_1 \geq 0$ – as discussed above, just below (2.33). Next, assume that $z_j(x) \geq 0$ for all $x \in \mathbb{T}^n$ and all $j = 1, \ldots, k$. The function $z_{k+1}(x)$ satisfies

$$\begin{aligned}
-\Delta z_{k+1} - \mu(x) z_{k+1} + N z_{k+1} &= -\Delta p_{k+1} - \mu(x) p_{k+1} + N p_{k+1} + \Delta p_k + \mu(x) p_k - N p_k \\
&= N p_k - p_k^2 - N p_{k-1} + p_{k-1}^2 = N z_k - (p_{k-1} + p_k) z_k > 0.
\end{aligned} \tag{2.38}$$

We used the induction assumption $z_k \geq 0$ and the upper bound (2.36) for $p_k$ in the last step. Once again, we conclude that $z_{k+1}(x) \geq 0$ for all $x \in \mathbb{T}^n$. Thus, the sequence $p_k(x)$ is, indeed, increasing. Therefore, it converges pointwise in $x$ to a limit profile $p(x)$ that satisfies

$$\phi_\varepsilon(x) \leq p(x) \leq \frac{N}{2}, \tag{2.39}$$

and

$$-\Delta p - \mu(x) p + N p = N p - p^2, \tag{2.40}$$

which is nothing but (2.1). Condition (2.39) is very important – it ensures that $p(x) \not\equiv 0$, and also prevents $p(x)$ from blowing up. We have, thus, established that when $\lambda_1 < 0$ this equation has a non-trivial steady periodic solution, finishing the proof of the existence part of Theorem 2.1.

## Uniqueness of a bounded solution when $\lambda_1 < 0$

Next, we show that the periodic solution of (2.1) that we have just constructed is unique in the class of bounded solutions. That is, if $s(x)$ is another bounded (not necessarily periodic) solution of

$$-\Delta s = \mu(x)s - s^2 \tag{2.41}$$
$$s(x) \text{ is bounded, and } s(x) > 0 \text{ for all } x \in \mathbb{R}^n,$$

then $s(x)$ coincides with the periodic solution $p(x)$ that we have constructed above. The proof follows [2] with some modifications from [20]. The crucial part in the proof of uniqueness is played by the following lemma.

**Lemma 2.3** *Any solution of (2.41) is bounded from below by a positive constant:*

$$\inf_{x\in\mathbb{R}^n} s(x) > 0. \tag{2.42}$$

Let us first explain why uniqueness of the solution of (2.41) follows from this lemma. Let $p(x)$ and $s(x)$ be two solutions. As $s(x)$ is bounded, and

$$\inf_{x\in\mathbb{R}^n} p(x) > 0,$$

we may define $r_0$ as the smallest $r$ such that $s(x) \le rp(x)$:

$$r_0 = \inf\{r : \ \ s(x) \le rp(x), \ \ \text{for all } x \in \mathbb{R}^n\}.$$

We claim that $r_0 \le 1$. Indeed, the difference

$$v(x) = r_0 p(x) - s(x)$$

satisfies

$$-\Delta v - \mu(x)v = -r_0 p^2 + s^2,$$

and a simple computation shows that

$$\begin{aligned}&-\Delta v + (-\mu(x) + r_0 p(x) + s(x))v = r_0 pv + sv - r_0 p^2 + s^2\\ &= r_0 p(r_0 p - s) + s(r_0 p - s) - r_0 p^2 + s^2 = r_0(r_0 - 1)p^2(x).\end{aligned}$$

Therefore, if $r_0 > 1$ the function $v(x)$ satisfies

$$\begin{aligned}&-\Delta v + (-\mu(x) + r_0 p(x) + s(x))v = r_0(r_0 - 1)p^2(x) > c_0 = r_0(r_0 - 1)\inf_{x\in\mathbb{R}^n} p^2(x) > 0,\\ &\quad v(x) \ge 0 \text{ for all } x \in \mathbb{R}^n.\end{aligned} \tag{2.43}$$

As $v(x) \ge 0$, the strong maximum principle implies that $v(x) > 0$ for all $x \in \mathbb{R}^n$. Furthermore, if there is a sequence $x_k$ such that $|x_k| \to +\infty$, and

$$\lim_{k\to\infty} v(x_k) = 0,$$

this is also a contradiction to the strong maximum principle. Indeed, as we have seen several times before, the elliptic regularity theory implies that we may extract a subsequence $n_k \to +\infty$ so that the shifted functions $v_k(x) = v(x_k + x)$, $p_k(x) = p(x + x_k)$, $s_k(x) = s(x + x_k)$, and $\mu_k(x) = \mu(x + x_k)$ converge to the respective limits $\bar{v}(x)$, $\bar{p}(x)$, $\bar{s}(x)$ and $\bar{\mu}(x)$ as $k \to +\infty$ that satisfy

$$
\begin{aligned}
&-\Delta\bar{v} + (-\bar{\mu}(x) + r_0\bar{p}(x) + \bar{s}(x))\bar{v} > c_0 > 0,\\
&v(x) \geq 0 \text{ for all } x \in \mathbb{R}^n,
\end{aligned}
\tag{2.44}
$$

with $\bar{v}(0) = 0$, which is impossible. Hence, if $r_0 > 1$ we must have

$$\inf_{x\in\mathbb{R}^n} v(x) > 0,$$

which contradicts the minimality of $r_0$: as $p(x)$ is bounded from above, there will then exist $r' < r_0$ such that

$$s(x) \leq r'p(x) \text{ for all } x \in \mathbb{R}^n.$$

We conclude that $r_0 \leq 1$, meaning that $s(x) \leq p(x)$. The only property of the solution $p(x)$ we have used above is that there exist two constants $c_{1,2} > 0$ so that

$$0 < c_1 < p(x) < c_2 < +\infty \text{ for all } x \in \mathbb{R}^n.$$

Lemma 2.3 asserts that "the other" solution $s(x)$ obeys the same bounds (with different constants $c_{1,2}$). Hence, an identical argument implies that $p(x) \leq s(x)$, and it follows that $p(x) = s(x)$ establishing uniqueness of the solutions of (2.41).

**The uniform lower bound: the proof of Lemma 2.3**

We now prove Lemma 2.3, the last ingredient in the proof of Theorem 2.1. An immediate trivial observation is that if $s(x)$ is a periodic solution of

$$
\begin{aligned}
&-\Delta s = \mu(x)s - s^2 &(2.45)\\
&s(x) \text{ is bounded and } s(x) > 0 \text{ for all } x \in \mathbb{R}^n,
\end{aligned}
$$

then, of course,

$$\inf_{x\in\mathbb{R}^n} s(x) > 0. \tag{2.46}$$

The main difficulty is, therefore, in dealing with general bounded solutions, that need not be periodic. To this end, we would like to get a nice subsolution for (2.45) that we would be able to put under $s(x)$ to give a lower bound for $s(x)$. As in the proof of existence of a solution to (2.45), a good candidate is $\phi_\varepsilon(x) = \varepsilon\phi(x)$, where $\phi(x)$ is the principal periodic eigenfunction of

$$
\begin{aligned}
&-\Delta\phi - \mu(x)\phi = \lambda_1\phi, &(2.47)\\
&\phi(x) > 0 \text{ for all } x \in \mathbb{T}^n.
\end{aligned}
$$

Recall that the function $\phi_\varepsilon(x)$ satisfies

$$-\Delta\phi_\varepsilon - \mu(x)\phi_\varepsilon + \phi_\varepsilon^2 = \lambda_1\phi_\varepsilon + \phi_\varepsilon^2 < 0, \tag{2.48}$$

provided that (compare to (2.28))

$$\varepsilon < -\frac{\lambda_1}{\max_{x\in\mathbb{T}^n}\phi(x)}. \tag{2.49}$$

The difficulty in using this subsolution is that it is periodic – how can we put it under $s(x)$ unless we already know that $s(x)$ is uniformly positive? Instead, we are going to use the principal Dirichlet eigenfunction in a ball $B(m,R)$ where $m\in\mathbb{Z}^d$ is an integer point, and $R$ is sufficiently large. Its advantage is that this eigenfunction is compactly supported so that a sufficiently small multiple of it can be put under any positive function. Let $\lambda_R$ be the principal Dirichlet eigenvalue in such ball. It does not depend on $m$ since the coefficient $\mu(x)$ is periodic, hence we set $m=0$ for the moment:

$$\begin{aligned} &-\Delta\psi_R(x)-\mu(x)\psi_R=\lambda_R\psi_R(x),\quad |x|<R,\\ &\psi_R(x)>0 \text{ for } |x|<R,\\ &\psi_R(x)=0 \text{ on } \{|x|=R\}. \end{aligned} \tag{2.50}$$

We have the following result, instructive in its own right.

**Proposition 2.4** *Let $\lambda_1$ be the principal periodic eigenvalue of the problem (2.47), and $\lambda_R$ be the principal Dirichlet eigenvalue of the problem (2.50), then*

$$\lim_{R\to+\infty}\lambda_R=\lambda_1. \tag{2.51}$$

This means that for such eigenvalue problems the periodic microstructure dominates and there is no averaging effect in the following sense: consider the rescaled eigenvalue problem with $y=x/R$, posed in the unit ball, for the function $\zeta_R(y)=\phi_R(Ry)$, and with $\lambda_R'=R^2\lambda_R$:

$$\begin{aligned} &-\Delta\zeta_R(y)-R^2\mu(Ry)\zeta_R(y)=\lambda_R'\zeta_R(y),\quad |y|<1,\\ &\zeta_R(x)>0 \text{ for } |y|<1,\\ &\zeta_R(x)=0 \text{ on } \{|y|=1\}. \end{aligned} \tag{2.52}$$

The potential $R^2\mu(Ry)$ in (2.52) has two competing effects: it is oscillatory that often leads to some averaging but it is also very strong. If the oscillatory nature of the potential would dominate over its strength, the principal eigenfunction for (2.52) would vary on the scale $O(1)$, and would not vary much on the scale of the period, which is $1/R$ in (2.52). Proposition 2.4 says that this is not the case, and the strength of the potential dominates over its oscillatory nature – this leads to a large eigenvalue $\lambda'(R)\approx R^2\lambda_1$, and the oscillations in the eigenfunction $\zeta_R(y)\approx\phi(Ry)$. This is what we mean by the dominance of the microstructure and lack of homogenization.

Here is how we use the above proposition. Let $\psi_R$ be the eigenfunction of (2.50) normalized so that

$$\sup_{|x|\le R}\psi_R(x)=1.$$

Set $\phi_{\varepsilon,R}=\varepsilon\psi_R(x)$, then, as in (2.48) we have

$$-\Delta\phi_{\varepsilon,R}-\mu(x)\phi_{\varepsilon,R}+\phi_{\varepsilon,R}^2<0, \tag{2.53}$$

as long as

$$\varepsilon < -\lambda_R. \tag{2.54}$$

Proposition 2.4 implies that there exists $R$ so that for all $R' > R$ we have $\lambda_{R'} < -\lambda_1/2$. Therefore, the function $\phi_{\varepsilon,R}$ is a sub-solution to (2.48) on the ball $B(0,R)$ for any $\varepsilon < -\lambda_1/2$. In addition, we know that for $\varepsilon$ sufficiently small we have $\phi_{\varepsilon,R} < p(x)$ for all $x \in B(0,R)$ simply because $p(x)$ is smooth and $p(x) > 0$ for all $x \in \mathbb{R}^n$. Let us now start increasing $\varepsilon$ until $p(x)$ and $\phi_{\varepsilon,R}$ touch:

$$\varepsilon_0 = \sup\{\varepsilon > 0 : \ p(x) \geq \varepsilon\phi_R(x) \text{ for all } x \in B(0,R)\}.$$

We claim that $\varepsilon_0 \geq -\lambda_R$. Indeed, otherwise $\phi_{\varepsilon,R}$ is a sub-solution and $p(x)$ is a solution, hence they can not touch without violating the maximum principle. Thus, we have $\varepsilon_0 \geq -\lambda_R$, that is, $\varepsilon_0$ is sufficiently large so that $\phi_{\varepsilon_0,R}$ is no longer a sub-solution. Therefore, we have shown that

$$p(x) \geq \left(-\frac{\lambda_1}{2}\right)\phi_R(x) \text{ for all } x \in B(0,R). \tag{2.55}$$

By considering a shifted ball $B(m,R)$ we see that, actually, we have a generalization of (2.55):

$$p(x) \geq \left(-\frac{\lambda_1}{2}\right)\phi_R(x-m) \text{ for all } x \in B(m,R), \text{ and all } m \in \mathbb{Z}^n. \tag{2.56}$$

It follows immediately that there exists a constant $c_0 > 0$ so that $p(x) > c_0$ for all $x \in \mathbb{R}^n$. Note that we may only shift $\phi_R$ by an integer $m$ – otherwise it would cease being a subsolution since $\mu(x)$ is not a constant. The proof of Lemma 2.3 is complete, as well as that of Theorem 2.1.

**Proof of Proposition 2.4**

Let us first recall the variational principles for the principal periodic and Dirichlet eigenvalues $\lambda_1$ and $\lambda_R$:

$$\lambda_1 = \inf_{v \in H^1(\mathbb{T}^n)} \frac{\displaystyle\int_{\mathbb{T}^n} (|\nabla v|^2 - \mu(x)v^2)dx}{\displaystyle\int_{\mathbb{T}^n} |v|^2 dx} \tag{2.57}$$

and

$$\lambda_R = \inf_{v \in H^1_0(B(0,R))} \frac{\displaystyle\int_{\mathbb{T}^n} (|\nabla v|^2 - \mu(x)v^2)dx}{\displaystyle\int_{\mathbb{T}^n} |v|^2 dx} \tag{2.58}$$

The difference between the two expressions is in the collection of test functions: 1-periodic $H^1$ functions in the case of $\lambda_1$ and $H^1_0(B(0,R))$ functions in the case of $\lambda_R$. Uniqueness of the principal eigenvalue shows that $\lambda_1$ is also the principal periodic eigenvalue on the larger torus $T_m = [0,m]^n$, hence $\lambda_1$ can be written as

$$\lambda_1 = \inf_{v \in H^1(T_m)} \frac{\displaystyle\int_{T_m} (|\nabla v|^2 - \mu(x)v^2)dx}{\displaystyle\int_{T_m} |v|^2 dx}. \tag{2.59}$$

That is, the infimum can be also taken over all $m$-periodic functions, for any positive integer $m \in \mathbb{N}$. Let us then take $m > 2R$, set the vector $e = (1, 1, \ldots, 1)$, and consider an $m$-periodic function $v_{R,m}$ (defined in the period cell $[0, m]^n$) that equals $\phi_R(x - (m/2)e)$ in the ball $B(me/2, R)$, and to zero everywhere else in $T_m = [0, m]^n$. Note that $B(me/2, R) \subset T_m$. The Rayleigh quotient of $v_{R,m}$ is exactly $\lambda_R$, hence

$$\lambda_1 \leq \lambda_R. \tag{2.60}$$

In order to establish the opposite bound, let $\phi_1$ be the 1-periodic eigenfunction and set

$$w_R(x) = \chi_R(x)\phi_1(x),$$

where $\chi_R(x)$ is a smooth cut-off function such that $0 \leq \chi_R(x) \leq 1$, $\chi_R(x) = 1$ for $|x| \leq R-1$, and $\chi_R(x) = 0$ for $|x| \geq R$. We may assume that $\|\chi_R\|_{C^2} \leq K$ with a constant $K$ that does not depend on $R$. The $L^2$-norm of the gradient of $w_R$ is

$$\begin{aligned}\int\limits_{B(0,R)} |\nabla w_R(x)|^2 dx &= \int_{B(0,R)} |\nabla \chi_R(x)\phi_1(x) + \chi_R(x)\nabla\phi_1(x)|^2 dx \\ &= \int\limits_{B(0,R)} (|\nabla\chi_R|^2|\phi_1(x)|^2 + 2(\phi_1(x)\chi_R(x)\nabla\chi_R(x)\cdot\nabla\phi_1(x))dx + \int\limits_{B(0,R)} |\chi_R(x)|^2|\nabla\phi_1(x)|^2 dx.\end{aligned}$$

As $\nabla\chi_R(x) = 0$ for $x$ outside the annulus $R - 1 \leq |x| \leq R$, the first term in the last line above is bounded by $CR^{n-1}$, and we have

$$\int_{B(0,R)} |\nabla w_R(x)|^2 dx = \int_{B(0,R)} |\nabla\phi_1(x)|^2 dx + O(R^{n-1}).$$

Furthermore, we can estimate, using the same idea:

$$\int_{B(0,R)} \mu(x)|w_R(x)|^2 dx = \int_{B(0,R)} \mu(x)|\phi_1(x)|^2 dx + O(R^{n-1}).$$

The notation above means that the integrals in the left and right side differ by expressions that can be bounded by $CR^{n-1}$. And, finally, we have, in the same way:

$$\int_{B(0,R)} |w_R(x)|^2 dx = \int_{B(0,R)} |\phi_1(x)|^2 dx + O(R^{n-1}).$$

The last observation is that, for instance,

$$\int_{B(0,R)} |\phi_1(x)|^2 dx = N_R \int_{[0,1]^n} |\phi_1(x)|^2 dx + O(R^{n-1}),$$

and similarly for the other integrals appearing in the Rayleigh quotient for $w_R$. Here $N_R$ is the number of disjoint $[0, 1]^n$ cubes that fit into the ball $B(0, R)$. We deduce that

$$\begin{aligned}\lambda_R &\leq \frac{\displaystyle\int_{B_R} |(|\nabla w_R(x)|^2 - \mu(x)|w_R(x)|^2)dx}{\displaystyle\int_{B_R} |w_R(x)|^2 dx} = \frac{\displaystyle\int_{[0,1]^n} |(|\nabla\phi_1(x)|^2 - \mu(x)|\phi_1(x)|^2)dx}{\displaystyle\int_{[0,1]^n} |\phi_1(x)|^2 dx} + O(R^{-1}) \\ &= \lambda_1 + O(R^{-1}).\end{aligned} \tag{2.61}$$

This estimate, together with (2.60) shows that

$$\lim_{R\to+\infty}\lambda_R=\lambda_1, \tag{2.62}$$

and the proof of Poposition 2.4 is complete.

## Convergence of the solutions of the Cauchy problem

We now prove Theorem 2.2. Recall that we need to prove that if $\lambda_1<0$ then solutions of the Cauchy problem

$$\begin{aligned}&\frac{\partial u}{\partial t}=\Delta u+\mu(x)u-u^2,\quad t>0,\ x\in\mathbb{R}^n,\\&u(0,x)=u_0(x),\end{aligned} \tag{2.63}$$

with a bounded non-negative function $u_0(x)$ such that $u_0(x)\not\equiv 0$, have the long time limit

$$u(t,x)\to p(x)\text{ as }t\to+\infty, \tag{2.64}$$

uniformly on compact sets $K\subset\mathbb{R}^n$. Here, as before, $p(x)$ is the unique bounded positive solution of the steady problem

$$-\Delta p=\mu(x)p-p^2,\quad x\in\mathbb{R}^n.$$

Recall that we have already shown that if $\lambda_1\ge 0$ then

$$u(t,x)\to 0\text{ as }t\to+\infty, \tag{2.65}$$

uniformly in $\mathbb{R}^n$.

Let us first show that

$$\liminf_{t\to+\infty}u(t,x)\ge p(x). \tag{2.66}$$

To this end, we will use the sub-solution $\varepsilon\phi_R(x)$ we used in the proof of Lemma 2.3. First, we wait until time $t=1$ to make sure that $u(t=1,x)>0$ in $\mathbb{R}^n$. Then, we may find $\varepsilon>0$ sufficiently small, and $R$ sufficiently large, so that $\phi_\varepsilon(x)=\varepsilon\phi_R(x)$ is a sub-solution:

$$-\Delta\phi_\varepsilon\le\mu(x)\phi_\varepsilon-\phi_\varepsilon^2,$$

and $\phi_\varepsilon(x)<p(x)$ for all $x\in\mathbb{R}^n$ (we extend $\phi_\varepsilon(x)=0$ outside the ball $B(0,R)$). We also take $\varepsilon$ so small that $u(t=1,x)>\phi_\varepsilon(x)$ for all $x\in\mathbb{R}^n$. Let now $v(t,x)$ be the solution of the Cauchy problem

$$\begin{aligned}&\frac{\partial v}{\partial t}=\Delta v+\mu(x)v-v^2,\quad t>1,\ x\in\mathbb{R}^n,\\&v(t=1,x)=\phi_\varepsilon(x).\end{aligned} \tag{2.67}$$

The parabolic comparison principle implies immediately that $v(t,x)\le u(t,x)$ for all $t>1$.

**Exercise 2.5** Use the fact that $\phi_\varepsilon(x)$ (which is the initial data for $v(t,x)$), is a sub-solution, to show that $v(t,x)\ge\phi_\varepsilon(x)$ for all $t\ge 1$ and $x\in\mathbb{R}^n$.

**Exercise 2.6** Use the result of the previous exercise to show that $v(t,x)$ is strictly increasing in time. Hint: set, for all $h>0$,

$$v_h(t,x) = v(t+h,x) - v(t,x),$$

and verify that $v_h(t,x)$ satisfies

$$\frac{\partial v_h}{\partial t} = \Delta v_h + \mu(x)v_h - (v(t+h,x)+v(t,x))v_h,$$

with $v_h(t=1,x)\geq 0$ for all $x\in\mathbb{R}^n$. Use the parabolic comparison principle to deduce that $v_h(t,x)\geq 0$ for all $t\geq 1$, that is, the function $v(t,x)$ is monotonically increasing in $t$.

**Exercise 2.7** Use the fact that $p(x)$ is a solution, while $\phi_\varepsilon(x)$ is a sub-solution to show that $v(t,x)\leq p(x)$ for all $t\geq 1$, if $\varepsilon>0$ is sufficiently small.

A consequence of the above observations is that the limit

$$s(x) = \lim_{t\to+\infty} v(t,x)$$

exists and is a positive bounded steady solution:

$$-\Delta s = \mu(x)s - s^2.$$

Uniqueness of such solutions implies that $s(x)=p(x)$, and thus

$$\liminf_{t\to+\infty} u(t,x) \geq \lim_{t\to+\infty} v(t,x) = p(x), \tag{2.68}$$

as we have claimed. Moreover, if

$$u_0(x)\leq p(x) \text{ for all } x\in\mathbb{R}^n, \tag{2.69}$$

then by the same token we have $u(t,x)\leq p(x)$ for all $t\geq 0$, meaning that

$$\lim_{t\to+\infty} u(t,x) = p(x).$$

Let us finally see what happens if (2.69) does not hold. If we multiply $p(x)$ by a number $M>1$ and set $p_M(x)=Mp(x)$, we get a super-solution:

$$-\Delta p_M - \mu(x)p_M + p_M^2 = -M\Delta p - M\mu(x)p + M^2p^2 = -Mp^2 + M^2p^2 > 0, \tag{2.70}$$

as $M>1$. If we choose $M>1$ sufficiently large so that $u_0(x)\leq p_M(x)$ then $u(t,x)\leq w(t,x)$, solution of

$$\begin{aligned} &\frac{\partial w}{\partial t} = \Delta w + \mu(x)w - w^2, \quad t>0,\ x\in\mathbb{R}^n, \\ &w(t,x) = Mp(x). \end{aligned} \tag{2.71}$$

As $p_M(x)$ is a super-solution, the argument we used to show that $v(t,x)$ was increasing in time, shows that $w(t,x)$ is monotonically decreasing in time. In addition, as $M>1$, we know from the comparison principle that $w(t,x) \geq p(x)$ for all $t > 0$. Its point-wise limit (as $t \to +\infty$) is therefore a non-trivial steady solution of our problem and thus equals to $p(x)$:

$$\lim_{t \to +\infty} w(t,x) = p(x).$$

As a consequence, we obtain

$$\limsup_{t \to +\infty} u(t,x) \leq p(x). \tag{2.72}$$

This, together with (2.68) proves that

$$\lim_{t \to +\infty} u(t,x) = p(x),$$

and the proof of Theorem 2.2 is complete.

# 3 The speed of invasion

We now turn to the heart of these lectures: finding the speed of invasion of the stable steady state $p(x)$ – in this section we assume that $\mu(x)$ is such that $\lambda_1 < 0$ so that the steady state does exist.

## 3.1 The homogeneous case

We first consider the uniform case $\mu(x) \equiv 1$, where the proof is much simpler, especially if we replace the nonlinearity $u - u^2$ by a function $f(u)$ which is linear close to zero:

$$f(u) = \begin{cases} u & \text{if } u \leq \theta, \\ u - u^2 & \text{if } u \text{ is close to } 1. \end{cases} \tag{3.1}$$

We also assume that $f(u)$ is smooth, and $f(u) \leq u$ for all $u \in [0,1]$ – this is the crucial Fisher-KPP assumption. Thus, we momentarily consider the problem

$$u_t = u_{xx} + f(u), \quad t > 0, x \in \mathbb{R}, \tag{3.2}$$

with a nonnegative initial condition $u(0,x) = u_0(x) \not\equiv 0$, and $f(u)$ as above. We assume that $u_0(x)$ is compactly supported – this is a very important assumption as a sufficiently slow decay at infinity may change the propagation speed, and even lead to fronts propagating super-linearly in time, an interesting subject outside of the scope of the present lectures. The unique stable steady state is $p(x) \equiv 1$, and we are interested in how fast it invades the areas where $u$ is small.

**An upper bound for the spreading speed**

The function $u(t,x)$ satisfies the inequality

$$u_t - u_{xx} \leq u. \tag{3.3}$$

Let us look for exponential super-solutions to (3.2) of the form

$$\bar{u}(t,x) = e^{-\lambda(x-ct)}.$$

Because of (3.3), the function $\bar{u}(t,x)$ is a super-solution if

$$\lambda^2 - c\lambda + 1 = 0. \tag{3.4}$$

As we need the supersolution $\bar{u}(t,x)$ to be real and positive, $\lambda$ has to be real, and (3.4) means that we have to take $c \geq 2$. In particular, for $c = 2$ we can take $\lambda = 1$. We conclude that if the initial data $u_0(x)$ satisfies

$$u_0(x) \leq Me^{-|x|}, \tag{3.5}$$

then $u(t,x)$ satisfies

$$u(t,x) \leq M \min\left(e^{-(x-2t)}, e^{x+2t}\right), \tag{3.6}$$

whence

$$\lim_{t\to+\infty} \sup_{|x|\geq ct} u(t,x) = 0, \tag{3.7}$$

for all $c > 2$. Therefore, the steady state $u \equiv 1$ can not invade with a speed larger than $c_* = 2$.

**A lower bound for the spreading speed**

Next, we show that the state $u \equiv 1$ invades with the speed at least equal to $c_* = 2$ (or, rather, faster than any speed smaller than $c_*$), matching the upper bound for the invasion speed. It is here that the assumption that $f(u) = u$ for small $u$ helps. It will be slightly easier to devise the lower bound in a moving frame. Let us take some $0 < c < 2$ and write $v(t,x) = u(t, x+ct)$, so that

$$v_t - cv_y = v_{yy} + f(v). \tag{3.8}$$

Because of the simplifying assumption (3.1) on the nonlinearity, any function $\underline{u}(t,x)$ such that

$$\underline{u}_t - c\underline{u}_y \leq \underline{u}_{yy} + \underline{u}, \tag{3.9}$$

and such that $\underline{u}(t,y) \leq \theta$ for all $t > 0$ and $x \in \mathbb{R}$ is a sub-solution to (3.8). We consider a time-independent exponential sub-solution

$$\underline{u}(y) = e^{-\lambda y},$$

but, as we take $c < 2$, the number $\lambda$, which satisfies (3.4), will have to be complex. In order to keep the sub-solution real, we set, for $t > 1$:

$$\underline{u}(y) = \begin{cases} m \exp\{-\mathrm{Re}\ \lambda x\} \cos(\mathrm{Im}\ \lambda y) \text{ if } |y| \leq \pi/(2\mathrm{Im}\ \lambda), \\ \qquad\qquad 0 \text{ otherwise.} \end{cases} \tag{3.10}$$

The constant $m > 0$ is chosen so that $\underline{u}(y) \leq \theta$, and, in addition, $\underline{u}(y) \leq u_0(y)$ – we assume here that $u_0(y) > 0$ on the interval $[-\pi/(2\mathrm{Im}\ \lambda), \pi/(2\mathrm{Im}\ \lambda)]$, otherwise we may simply wait until time $t = 1$, when $v(t = 1, y) > 0$ for all $y \in \mathbb{R}$, and put a small multiple of $\underline{u}(y)$ below $v(t = 1, y)$. We conclude that $v(t,y) \geq \underline{u}(y)$, for all $t > 0$. As a consequence, we immediately obtain that

$$\limsup_{t\to+\infty} u(t,ct) \geq \alpha_0 > 0, \text{ for all } 0 \leq c < 2, \tag{3.11}$$

with some $\alpha_0 > 0$.

**Exercise 3.1** Use the function $\underline{u}(y)$ as the initial data for the Cauchy problem in the moving frame to bootstrap the above argument to

$$\liminf_{t\to+\infty} u(t,ct) = 1, \tag{3.12}$$

for all $0 \le c < 2$. Hint: such solution will be monotonically increasing in time.

The main point above is that if $c < 2$, we can find a compactly supported subsolution in the moving frame. This is the essence of the argument in the general periodic case as well – compactly supported subsolutions can be constructed in a moving frame that moves "not too fast". The physical reason for that is quite clear: consider the linear problem

$$u_t - ce \cdot \nabla u = \Delta u + u, \tag{3.13}$$

in a frame moving in a direction $e \in \mathbb{S}^{n-1}$ with a speed $c \ge 0$, with the Dirichlet boundary condition $u = 0$ on the boundary of an (also moving) ball $B(y_0, R)$. There is a competition between the linear growth term in the right side and the Dirichlet boundary conditions that promote decay. If $c = 0$ then the growth term always wins for a ball of a sufficiently large radius $R$. On the other hand, a large speed $c$ promotes a sweeping effect – the ball moves so fast that it spends too little time at any given point in the original frame for the growth to take place. In other words, for every $R > 0$ fixed, there exists $c_*(R)$ so that for all $c > c_*(R)$ solution of (3.13) tends to zero as $t \to +\infty$. Moreover, $c_*(R)$ is increasing in $R$, and the true propagation speed in the whole space may be guessed to be

$$\lim_{R\to+\infty} c_*(R).$$

We will leave the reader, for the moment, without the answer whether this guess is correct but nevertheless the above arguments, hopefully, convince that with a little bit of simplification, the speed of invasion in a homogeneous medium can be found very easily. In the remainder of this section we will drop the simplifying assumptions about the nonlinearity that we have used here, as well as consider a periodic reaction rate $\mu(x)$ – that will create some technical difficulties but not change the moral of the story.

## 3.2 The exponential solutions

As in the homogeneous case considered above, exponential solutions of the linearized problem play a crucial role in the general periodic case but their existence for a given speed $c \ge 0$ is a much more amusing problem than the simple quadratic equation (3.4). These are solutions of the equation

$$v_t = \Delta v + \mu(x)v, \quad x \in \mathbb{R}^n, \tag{3.14}$$

of the form

$$v(t,x) = e^{-\lambda(x\cdot e - ct)}\eta(x), \tag{3.15}$$

with a fixed unit vector $e \in \mathbb{R}^n$, $|e| = 1$, and a 1-periodic (in all directions) function $\eta(x)$. As we will use $v(t,x)$ as a super-solution, we will require that $\eta(x) > 0$ – this condition will produce a restriction on the range of speeds $c$ for which an exponential solution with a real

$\lambda$ can exist. It will be convenient to factor $\eta(x) = \phi(x)\Phi(x)$. Here, $\phi(x)$ is the principal (positive) periodic eigenfunction of the problem we have encountered before:

$$\begin{aligned} &-\Delta\phi - \mu(x)\phi = \lambda_1\phi \\ &\phi(x) \text{ is 1-periodic,} \\ &\phi(x) > 0 \text{ for all } x \in \mathbb{R}^n. \end{aligned} \tag{3.16}$$

Recall that our main assumption in this section is that $\lambda_1 < 0$. For $v$ to satisfy (3.14), the function $\Phi(x)$ has to be the solution of the eigenvalue problem

$$\begin{aligned} &\tilde{L}_\lambda\Phi = -(\lambda_1 + c\lambda)\Phi \\ &\Phi(x) \text{ is 1-periodic,} \\ &\Phi(x) > 0 \text{ for all } x \in \mathbb{R}^n, \end{aligned} \tag{3.17}$$

with the operator $\tilde{L}_\lambda$ (that depends parametrically on $\lambda$) given by

$$\tilde{L}_\lambda\Phi = -e^{\lambda x\cdot e}\left[\Delta(e^{-\lambda x\cdot e}\Phi) - 2\frac{\nabla\phi}{\phi}\cdot\nabla(e^{-\lambda x\cdot e}\Phi)\right]. \tag{3.18}$$

Therefore, the speed $c \in \mathbb{R}$ of an exponential solution and its decay rate $\lambda$ are related by the equation

$$c\lambda = -\lambda_1 - \mu_1^{per}(\tilde{L}_\lambda). \tag{3.19}$$

Here, $\mu_1^{per}(\tilde{L}_\lambda)$ is the principal periodic eigenvalue of the operator $\tilde{L}_\lambda$, and, as such, is a function of $\lambda$. Equation (3.19) is the relation between the speed $c$ and the exponential rate $\lambda$, which is the generalization to the periodic case of the quadratic equation (3.4) – the question is for which $c \geq 0$ can we find $\lambda > 0$ satisfying (3.19)?

**Theorem 3.2** *For every $e \in \mathbb{R}^n$, with $|e| = 1$ there exists $c_*(e) > 0$ so that (i) if $c < c_*(e)$, equation (3.19) has no solution $\lambda > 0$, (ii) if $c > c_*(e)$, equation (3.19) has two solutions $\lambda > 0$, and (iii) if $c = c_*(e)$, equation (3.19) has exactly one solution $\lambda > 0$.*

The key step in the proof of Theorem 3.2 is the next observation.

**Lemma 3.3** *The function $\mu_1^{per}(\tilde{L}_\lambda)$ is concave in $\lambda$.*

Let us step back and see what this result means in dimension $n = 1$ and when $\mu(x) \equiv 1$. Then $\lambda_1 = -1$, and both $\phi(x) \equiv 1$ and $\Phi(x) \equiv 1$, while $\mu_1(\tilde{L}_\lambda) = -\lambda^2$, so that (3.19) is simply

$$c\lambda = 1 + \lambda^2.$$

We see that in this special case the claim of Theorem 3.2 is true with $c_* = 2$, and that $\mu_1(\tilde{L}_\lambda) = -\lambda^2$ is, indeed, concave in $\lambda$. In the general case, the key to the proof of Lemma 3.3 is the following observation: set

$$E_\lambda = \{\psi \in C^2(\mathbb{R}^n) : \; e^{\lambda x\cdot e}\psi(x) \text{ is 1-periodic, } \psi(x) > 0 \text{ for all } x \in \mathbb{R}^n\}, \tag{3.20}$$

then $\mu_1^{per}(\tilde{L}_\lambda)$ has the min-max characterization

$$k(\lambda) := \mu_1^{per}(\tilde{L}_\lambda) = \max_{\psi\in E_\lambda}\inf_{x\in\mathbb{R}^n}\frac{\mathcal{L}\psi(x)}{\psi(x)}. \tag{3.21}$$

We have denoted

$$\mathcal{L}\psi = -\Delta\psi - 2\frac{\nabla\phi}{\phi}\cdot\nabla\psi.$$

With the above notation, we need to prove that for any $t \in [0,1]$ we have

$$tk(\lambda_1) + (1-t)k(\lambda_2) \le k(t\lambda_1 + (1-t)\lambda_2), \tag{3.22}$$

for all $\lambda_1, \lambda_2 > 0$. Let $\phi_1$ and $\phi_2$ be the principal eigenfunctions of the operators $\tilde{L}_{\lambda_1}$ and $\tilde{L}_{\lambda_2}$, respectively, and set

$$\psi_i(x) = e^{-\lambda_i x\cdot e}\phi_i(x), \quad i = 1,2, \quad \psi(x) = \psi_1^t(x)\psi_2^{1-t}(x).$$

Note that $\psi \in E_\lambda$, with $\lambda = t\lambda_1 + (1-t)\lambda_2$, and thus can be used as a test function in the max-min principle for $k(\lambda)$. We compute:

$$\frac{\nabla\psi}{\psi} = t\frac{\nabla\psi_1}{\psi_1} + (1-t)\frac{\nabla\psi_2}{\psi_2},$$

and

$$\frac{\Delta\psi}{\psi} = t\frac{\Delta\psi_1}{\psi_1} + (1-t)\frac{\Delta\psi_2}{\psi_2} + t(t-1)\left(\frac{\nabla\psi_1}{\psi_1} - \frac{\nabla\phi_2}{\phi_2}\right)^2.$$

It follows that

$$\begin{aligned}\frac{\mathcal{L}\psi(x)}{\psi(x)} &= -\frac{\Delta\psi(x)}{\psi(x)} - 2\frac{\nabla\phi}{\phi}\cdot\frac{\nabla\psi}{\psi} = t\frac{\mathcal{L}\psi_1(x)}{\psi_1(x)} + (1-t)\frac{\mathcal{L}\psi_2(x)}{\psi_2(x)} - t(t-1)\left(\frac{\nabla\psi_1}{\psi_1} - \frac{\nabla\phi_2}{\phi_2}\right)^2\\ &\ge t\frac{\mathcal{L}\psi_1(x)}{\psi_1(x)} + (1-t)\frac{\mathcal{L}\psi_2(x)}{\psi_2(x)},\end{aligned}$$

and thus

$$\inf_{x\in\mathbb{R}^n}\frac{\mathcal{L}\psi(x)}{\psi(x)} \ge t\inf_{x\in\mathbb{R}^n}\frac{\mathcal{L}\psi_1(x)}{\psi_1(x)} + (1-t)\inf_{x\in\mathbb{R}^n}\frac{\mathcal{L}\psi_2(x)}{\psi_2(x)},$$

We deduce that

$$\sup_{\psi\in E_\lambda}\inf_{x\in\mathbb{R}^n}\frac{\mathcal{L}\psi(x)}{\psi(x)} \ge t\sup_{\psi_1\in E_{\lambda_1}}\inf_{x\in\mathbb{R}^n}\frac{\mathcal{L}\psi_1(x)}{\psi_1(x)} + (1-t)\sup_{\psi_2\in E_{\lambda_2}}\inf_{x\in\mathbb{R}^n}\frac{\mathcal{L}\psi_2(x)}{\psi_2(x)},$$

which is nothing but (3.22). Hence, the function $\mu_1^{per}(\lambda)$ is, indeed, concave in $\lambda$.

Now, we can prove Theorem 3.2. Let us first summarize some basic properties of the function

$$s(\lambda) = -\lambda_1 - \mu_1^{per}(\tilde{L}_\lambda).$$

We have just shown that it is convex and, in addition, by assumption we have $s(0) = -\lambda_1 > 0$.

**Exercise 3.4** Show that

$$\lim_{\lambda\to+\infty}\frac{\mu_1^{per}(\tilde{L}_\lambda)}{\lambda^2} = -1$$

This exercise implies that the function $s(\lambda)$ is super-linear at infinity:

$$\lim_{\lambda\to+\infty}\frac{s(\lambda)}{\lambda}=+\infty. \tag{3.23}$$

**Exercise 3.5** Use finite differences to show that the function $k(\lambda)=\mu_1^{per}(\tilde{L}_\lambda)$ and the corresponding eigenfunction $\phi_\lambda$ of $\tilde{\mathcal{L}}_\lambda$ are differentiable in $\lambda$ (in fact, analytic).

The last property of $s(\lambda)$ that we will need is

$$s'(0)=k'(0)=0. \tag{3.24}$$

To see this, recall that

$$-\Delta\phi_\lambda+2\lambda(e\cdot\nabla\phi_\lambda)-(\lambda^2+\frac{2\lambda}{\phi}(e\cdot\nabla\phi))\phi_\lambda+\frac{2\nabla\phi}{\phi}\cdot\nabla\phi_\lambda=k(\lambda)\phi_\lambda, \tag{3.25}$$

hence (with $\phi_0=\phi_{\lambda=0}$),

$$-\Delta\phi_0+\frac{2\nabla\phi}{\phi}\cdot\nabla\phi_0=k(0)\phi_0. \tag{3.26}$$

It follows that

$$k(0)=0\text{ and }\phi_0=1. \tag{3.27}$$

Differentiating (3.25) in $\lambda$, we obtain, at $\lambda=0$:

$$-\Delta\psi_0+\frac{2\nabla\phi}{\phi}\cdot\nabla\psi_0-\frac{2}{\phi}(e\cdot\nabla\phi)\phi_0+2(e\cdot\nabla\phi_0)=k(0)\psi_0+k'(0)\phi_0,$$

with the function

$$\psi_0=\frac{d\phi_\lambda}{d\lambda}|_{\lambda=0}.$$

Taking (3.27) into account, this simplifies to

$$-\Delta\psi_0+\frac{2\nabla\phi}{\phi}\cdot\nabla\psi_0-\frac{2}{\phi}(e\cdot\nabla\phi)=k'(0). \tag{3.28}$$

The adjoint equation to (3.26) is

$$-\Delta\phi_0^*-2\nabla\cdot\left(\frac{\nabla\phi}{\phi}\phi_0^*\right)=0, \tag{3.29}$$

or

$$-\nabla\cdot\left(\nabla\phi_0^*+\frac{2\nabla\phi}{\phi}\phi_0^*\right)=0.$$

It is satisfied by $\phi_0^*(x)=1/\phi^2(x)$. Multiplying then (3.28) by $\phi^{-2}(x)$ and integrating over the period cell gives

$$k'(0)\int_{\mathbb{T}^n}\frac{dx}{\phi_1^2(x)}=-2\int_{\mathbb{T}^n}\frac{e\cdot\nabla\phi}{\phi^3}dx=0,$$

hence $k'(0)=0$.

Let us summarize the above observations about the function $s(\lambda)$: we know that $s(\lambda)$ is convex, super-linear at infinity, $s(0) > 0$ and $s'(0) = 0$. It follows that there exists a threshold $c_*(e)$ so that the equation

$$s(\lambda) = c\lambda, \tag{3.30}$$

has no solutions for $0 < c < c_*(e)$, one solution for $c = c_*(e)$ and two solutions for $c > c_*(e)$ – this proves Theorem 3.2.

We will denote below by $\lambda_*(e)$ the unique solution of (3.30) at $c = c_*(e)$, and by $\lambda_e(c)$ the smaller of the two positive solutions for $c > c_*(e)$.

## 3.3 The Freidlin-Gärtner formula

With the exponential solutions in hand, we look at the invasion speed of the solutions of the Cauchy problem. As we have discussed above, in the introduction to this chapter, the speed of invasion in a direction $e \in \mathbb{S}^{n-1}$ is given not by $c_*(e)$ (as may be naively expected) but by (1.8):

$$w_*(e) = \inf_{|e'|=1,(e\cdot e')>0} \frac{c_*(e')}{(e \cdot e')}. \tag{3.31}$$

The geometric reason for this modification is that we have to account for the interaction between propagation in various directions – going fast or slow in a different direction $e'$ has implications for the propagation in the direction $e$, thus the invasion speed is $w_*(e)$ and not $c_*(e)$. We will make a (slightly) simplifying assumption that

$$\mu(x) > 0, \text{ for all } x \in \mathbb{T}^n. \tag{3.32}$$

This assumption is not necessary – our usual hypothesis $\lambda_1 < 0$ (implied by (3.32)) is sufficient, and we will point out specifically when we use it.

**Theorem 3.6** *Let $u(t, x)$ be the solution of the Cauchy problem*

$$u_t = \Delta u + \mu(x)u - u^2, \tag{3.33}$$

*with bounded and non-negative compactly supported initial data $u_0(x) \not\equiv 0$. Then for every $e \in \mathbb{S}^{n-1}$ and all $x \in \mathbb{R}^n$ we have*

$$\lim_{t\to+\infty} u(t, x + cte) = 0 \text{ if } c > w_*(e), \tag{3.34}$$

*and*

$$\lim_{t\to+\infty} |u(t, x + cte) - p(x + cte)| = 0 \text{ if } 0 \le c < w_*(e). \tag{3.35}$$

As in the homogeneous case, the asymptotics of $u(t, x)$ close to $x = w_*(e)t$ is a more delicate question, and we will not discuss it here [12]. We should also stress again that if the initial data $u_0(x)$ is not compactly supported, and decays sufficiently slowly at infinity, the propagation speed is faster than that given by the Freidlin-Gärtner formula, and for initial data decaying algebraically at infinity, solutions may propagate super-linearly in time.

In order to understand quantitatively why the propagation speed is given by (3.31) and not by $c_*(e)$, recall that for any $e \in \mathbb{S}^{n-1}$ the exponential solution

$$v_e(t,x) = e^{-\lambda_*(e)(x\cdot e - c_*(e)t)}\phi_e(x),$$

with $\phi_e(x) = \phi_{\lambda_*(e)}(x)$, is a super-solution to the Cauchy problem:

$$v_t \geq \Delta v + \mu(x)v - v^2. \tag{3.36}$$

Therefore, the function

$$\bar{v}(t,x) = \inf_{|e|=1} e^{-\lambda_*(e)(x\cdot e - c_*(e)t)}\phi_e(x)$$

is also a super-solution. Hence, any solution of the Cauchy problem

$$u_t = \Delta u + \mu(x)u - u^2, \tag{3.37}$$

with a compactly supported function $u_0(x)$, lies below a large multiple of $\bar{v}(t,x)$. In order to see how small $\bar{v}(t,x)$ is far away in a fixed direction $e \in \mathbb{S}^{n-1}$, we look at when

$$\bar{v}(t,cte) = \inf_{|e'|=1} e^{-\lambda_*(e')(ct(e\cdot e') - c_*(e')t)}\phi_{e'}(cte)$$

is exponentially small. Note that

$$\sup_{e'\in\mathbb{S}^{n-1}, x\in\mathbb{T}^n} \phi_{e'}(x) < +\infty,$$

thus $\bar{v}(t,cte)$ is small as $t \to +\infty$, if there exists some $e' \in \mathbb{S}^{n-1}$ such that

$$c(e\cdot e') \gg c_*(e'),$$

that is, for $c > w_*(e)$. This is where the formula (3.31) for $w_*(e)$ comes from. More precisely, the above argument shows that if we take any $c > w_*(e)$, then we have, for all $x \in \mathbb{R}^n$ fixed:

$$\lim_{t\to+\infty} u(t, x + cte) = 0. \tag{3.38}$$

The next (and harder) step is to prove that for each $c \in (0, w_*(e))$ we have

$$\lim_{t\to+\infty} |u(t, x + cte) - p(x + cte)| = 0. \tag{3.39}$$

Here, $p(x)$ is the unique positive bounded steady solution to (3.33). We take $c < w_*(e)$, and go into the moving frame: set $v(t,y) = u(t, y + cte)$:

$$v_t - ce\cdot\nabla v = \Delta v + \mu(y + cte)v - v^2. \tag{3.40}$$

As in the homogeneous case, the proof of (3.39) boils down to finding a compactly supported sub-solution to (3.40) that does not vanish as $t \to +\infty$. The first (and the most difficult) step is to show that solution is strictly positive at distances of the order $ct$ in the direction $e$, with $c < w_*(e)$.

**Proposition 3.7** *Given $e \in \mathbb{S}^{n-1}$, and $R > 0$, solution of the Cauchy problem in the moving frame:*

$$\frac{\partial v}{\partial t} - ce \cdot \nabla v = \Delta v + \mu(y + cte)v - v^2, \tag{3.41}$$

*with a compactly supported initial data $v_0(x)$, satisfies*

$$\liminf_{t \to +\infty} \inf_{|y| \le R} v(t, y) > 0. \tag{3.42}$$

We leave the second step as an exercise.

**Exercise 3.8** Show that the Freidlin-Gärtner formula follows from Proposition 3.7.

In order to prove Proposition 3.7 we will establish the following result for rational angles, that is, vectors $e \in \mathbb{S}^{n-1}$ that have all rational components.

**Proposition 3.9** *Let $e \in \mathbb{S}^{n-1}$ be rational, and $0 \le c < w_*(e)$. There exists $R_0 > 0$ sufficiently large, $\gamma > 0$, and a positive bounded function $s_e(t, y)$ that satisfies*

$$\frac{\partial s_e}{\partial t} - ce \cdot \nabla s_e = \Delta s_e + \mu(y + cte)s_e - \gamma s_e, \quad t \in \mathbb{R}, \quad |y| \le R_0, \tag{3.43}$$

*with the Dirichlet boundary condition $s_e(t, y) = 0$ for $|y| = R_0$, and such that*

$$\liminf_{t \to +\infty} \inf_{|x| \le R_0/2} s_e(t, y) > 0. \tag{3.44}$$

Note that when $c = 0$, we can take $\gamma = -\lambda_R$, the principal eigenvalue of the Dirichlet problem in a large ball, and Proposition 2.4 implies $\gamma > 0$ under the assumption $\lambda_1 < 0$. Proposition 3.9 shows that this negativity extends up to the speed $w_*(e)$.

Let us explain how the conclusion of Proposition 3.7 for rational directions follows. If $e \in \mathbb{S}^{n-1}$ is rational, the function $s_\varepsilon(t, x) = \varepsilon s_e(t, x)$ with $s_e(t, x)$ as is in Proposition 3.9, is a sub-solution to (3.41), provided that $\varepsilon > 0$ is sufficiently small:

$$\frac{\partial s_\varepsilon}{\partial t} - ce \cdot \nabla s - \Delta s_\varepsilon - \mu(y + cte)s_\varepsilon + s_\varepsilon^2 = -\gamma \varepsilon s_e + \varepsilon^2 s_e^2 < 0.$$

This implies that there exists $\beta > 0$ sufficiently small so that $v(t, x) > \beta s_\varepsilon(t, x)$ (choosing $\beta$ so that this inequality holds at $t = 1$). Now, (3.42) follows from (3.44) if $e \in \mathbb{S}^{n-1}$ is a rational direction by putting a small multiple of $s_0(0, x)$ under $v_0(x)$ at $t = 0$ and using the maximum principle.

The proof of Proposition 3.7 for irrational directions is by a density argument. As we will use various directions, it is easier to work in the original frame:

$$u_t = \Delta u + \mu(x)u - u^2, \tag{3.45}$$

and we need to show that

$$\liminf_{t \to +\infty} \inf_{|y| \le R} u(t, cte + y) > 0. \tag{3.46}$$

The Harnack inequality implies that it actually suffices to show that

$$\liminf_{t \to +\infty} u(t, cte) > 0. \tag{3.47}$$

Here (and only here), we will use the simplifying assumption (3.32).

**Exercise 3.10** It follows from (3.32) that the solution propagates in all directions at least at the speed $2\sqrt{m} := \underline{c}$. That is, for any $0 \le c < \underline{c}$ we have the following: for any $\delta > 0$ there exists $\alpha > 0$ such that if $u(\tau, y) > \delta$ for all $y \in B(x, 1)$ and some $\tau > 0$, then for all $t > \tau + 1$ we have

$$\inf_{y \in B(x, c(t-\tau))} u(t, y) > \alpha. \tag{3.48}$$

Given $c < w_*(e)$, we take $\varepsilon > 0$ sufficiently small, and consider a rational direction $e_\varepsilon$ at distance at most $\varepsilon^2$ from $e$ (rational points are dense on the unit sphere):

$$|e - e_\varepsilon| \le \varepsilon^2.$$

In addition, we require that $e_\varepsilon$ is so close to $e$ that

$$|w_*(e) - w_*(e_\varepsilon)| \le \varepsilon^2,$$

and

$$c_\varepsilon = c(1 + \varepsilon) < w_*(e_\varepsilon).$$

Consider now a large time $T > 0$ and the corresponding positions along the two rays:

$$X = c_\varepsilon T e \text{ and } X_\varepsilon = c_\varepsilon T e_\varepsilon.$$

As $c_\varepsilon < w_*(e_\varepsilon)$, if $T$ is sufficiently large (possibly depending on $\varepsilon$), then, by what we have shown for the propagation in a rational direction, the function $u(T, x)$ is larger than some $\delta > 0$ in a ball centered at $X_\varepsilon$. Therefore, as follows from Exercise 3.10, at the time $T' = T + \varepsilon T$, $u$ will be larger than $\alpha > 0$ in a ball of radius at least $\underline{c}\varepsilon T$, centered at $X_\varepsilon$. However, for small $\varepsilon$, the point $X$ is in this ball:

$$|X - X_\varepsilon| \le c_\varepsilon \varepsilon^2 T \le \underline{c}\varepsilon T.$$

It follows that for all $T$ sufficiently large we have

$$u(T(1 + \varepsilon), c(1 + \varepsilon)Te) \ge \delta,$$

which implies that

$$\liminf_{t \to +\infty} u(t, cte) > 0.$$

Thus, the proof of the Freidlin-Gärtner formula hinges on Proposition 3.9.

### The proof of Proposition 3.9

The proof relies on an alternative characterization of $w_*(e)$ in terms of compactly supported sub-solutions. Thus, while $w_*(e)$ was originally defined in terms of the exponential solutions which are super-solutions to the Fisher-KPP problem, it can also be characterized in terms of sub-solutions, which leads to the tight propagation bounds in the Freidlin-Gärtner formula.

We assume that $e \in \mathbb{Q}^n$ is a "rational direction", that is, all components of $e$ are rationally dependent. Then the coefficient

$$a(t, y) = \mu(y + cte)$$

is 1-periodic in $y$ and is also periodic in time, with the period $T_c = M/c$. Here, $M$ is the smallest number so that all $Me_j$ are integers. A key role in the characterization of $w_*(e)$ in terms of sub-solutions is played by the principal Dirichlet eigenfunction for the problem

$$\begin{aligned}
&z_t - \Delta z - ce \cdot \nabla z - a(t,y)z = \lambda_1(c,R)z, \quad t \in \mathbb{R},\ y \in B_R = \{|y| \le R\}, \qquad (3.49)\\
&z(t,y) > 0 \text{ is } T_c\text{-periodic in } t,\\
&z(t,y) = 0 \text{ for } |y| = R.
\end{aligned}$$

This is the problem we have discussed before – the drift term $ce \cdot \nabla z$ is the sweeping effect that enhances the effect of the boundary and "wants" to make $\lambda_1(c,R)$ positive, while the growth term $a(t,y)z$ on the left tries to make $\lambda_1(c,R)$ negative. We will be interested in the balance between these two effects. To simplify slightly the notation, we do not show explicitly the dependence of $\lambda_1(c,R)$ on $e$.

**Lemma 3.11** *There exists $R_1$ so that for all $R > R_1$ we can find $c_*(R)$ such that*

$$\lambda_1(c_*(R), R) = 0.$$

The principal periodic eigenvalue $\lambda_1$ of the operator

$$-\Delta - \mu(x)$$

is negative when $c = 0$ – this is our main assumption. Proposition 2.4 tells us that then the principal Dirichlet eigenvalue $\lambda_1(R)$ on the ball $B_R$ of the same operator is also negative – in other words, in our curent notation, $\lambda_1(0,R) < 0$ for $R$ sufficiently large – this sets $R_1$. The function $\lambda_1(c,R)$ is analytic in $c$, thus $\lambda_1(c,R) < 0$ for all $c > 0$ sufficiently small and $R$ large enough. On the other hand, for all $c > 0$ sufficiently large we have $\lambda_1(c,R) > 0$. To see that, set

$$z(t,x) = e^{-c(x\cdot e)/2}\tilde{z}(t,x).$$

The function $\tilde{z}(t,x)$ satisfies

$$\tilde{z}_t - \Delta\tilde{z} + \frac{c^2}{4}\tilde{z} - a(t,y)\tilde{z} = \lambda_1(c,R)\tilde{z}, \qquad (3.50)$$

with the periodic boundary conditions in $T$ and the Dirichlet boundary conditions on $\partial B_R$.

**Exercise 3.12** Show that $\lambda_1(c,R) > 0$ if

$$c > \sqrt{1 + 4\|a\|_\infty}. \qquad (3.51)$$

Thus, there exist $c(R) > 0$ so that $\lambda_1(c(R),R) = 0$ and we will denote by $c_*(R)$ the smallest such $c > 0$ (once again, $c_*(R)$ depens also on $e$ but we do not indicate this dependence explicitly in our notation). Note that $c_*(R)$ is bounded from above because of (3.51).

**Exercise 3.13** Show that $c_*(R)$ is uniformly bounded from below as $R \to +\infty$ (also uniformly in $e \in \mathbb{S}^{n-1}$).

Here is the key lemma, connecting sub-solutions to the invasion speed $w_*(e)$.

**Lemma 3.14** *We have, for all $e \in \mathbb{R}^n$ with $|e| = 1$,*

$$\liminf_{R \to +\infty} c_*(R) \geq w_*(e). \tag{3.52}$$

Let $z_R(t,x)$ be the Dirichlet eigenfunction at $c = c_*(R)$:

$$\begin{aligned}
&\frac{\partial z_R}{\partial t} - \Delta z_R - c_*(R) e \cdot \nabla z_R - a(t,y) z_R = 0, \quad t \in \mathbb{R},\ y \in B_R \\
&z_R(t,y) > 0 \text{ is } T_*(R)\text{-periodic in } t, \\
&z_R(t,y) = 0 \text{ for } |y| = R,
\end{aligned} \tag{3.53}$$

normalized so that $z_R(0,0) = 1$. We denote

$$T_*(R) = T_{c_*(R)} = M/c_*(R).$$

Because of the uniform bounds on $c_*(R)$, we can extract a sub-sequence $R_n \to +\infty$ so that $c_*(R_n) \to \bar{c}$ and the periods $T_*(R_n) = M/c_*(R) \to M/\bar{c}$, and, moreover the functions $z_{R_n}(t,x)$ converge (after possibly extracting another subsequence) locally uniformly to a positive $T$-periodic function $q(t,x)$ that solves

$$q_t - \Delta q - \bar{c} e \cdot \nabla q - a(t,y) q = 0, \quad t \in \mathbb{R},\ y \in \mathbb{R}^n, \tag{3.54}$$

and satisfies $q(0,0) = 1$. If $q(t,y)$ were an exponential solution we would immediately conclude that $\bar{c} > c_*(e) \geq w_*(e)$. However, we do not know that, and instead of showing that $q(t,y)$ is itself an exponential solution, we will use $q(t,y)$ to construct an exponential solution to (3.54) that will possibly move in a different direction $e' \in \mathbb{S}^{n-1}$. This will lead to the lower bound (3.52).

The construction proceeds as follows. Let $\hat{e}_1$ be the first coordinate vector. The Harnack inequality implies that there exists a constant $m$ so that

$$m q(t, y + \hat{e}_1) \leq q(t + T, y), \tag{3.55}$$

for all $y \in \mathbb{R}$ and $t \in \mathbb{R}$. As the function $q(t,y)$ is $T$-periodic, we conclude that there exist $m, M > 0$ so that

$$m q(t, y + \hat{e}_1) \leq q(t,y) \leq M q(t, y + \hat{e}_1), \tag{3.56}$$

Let $M_1$ be the smallest $M$ so that this inequality holds. If there exists $t_0, y_0$ so that

$$q(t_0, y_0) = M_1 q(t_0, y_0 + \hat{e}_1),$$

and $q(t,y) \leq M_1 q(t, y + \hat{e}_1)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$, 1-periodicity of $a(t,y)$ in $y_1$, and the maximum principle would imply that

$$q(t,y) = M q(t, y + \hat{e}_1), \text{ for all } t \in \mathbb{R} \text{ and } y \in \mathbb{R}^n. \tag{3.57}$$

On the other hand, if there exists a sequence of points $t_n, y_n$ such that

$$q(t_n, y_n) \geq (M_1 - \frac{1}{n}) q(t_n, y_n + \hat{e}_1),$$

then by considering the shifted functions $q_m(t,y) = q(t+t_m, y+[y_m])$ and passing to the limit $n \to +\infty$ we would construct a solution $\bar{q}(t,x)$ of (3.54) such that

$$\bar{q}(0,\bar{y}) = M_1\bar{q}(0,\bar{y}+\hat{e}_1),$$

with some $\bar{y} \in [0,1]^n$, and $\bar{q}(t,y) \le M_1\bar{q}(t,y+\hat{e}_1)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$. The maximum principle would, once again, imply that $\bar{q}(t,x)$ satisfies (3.57). In other words, in either case $\bar{q}(t,y)$ is a solution of

$$\bar{q}_t - \Delta\bar{q} - \bar{c}e\cdot\nabla\bar{q} - a(t,y)\bar{q} = 0, \tag{3.58}$$

which, in addition, satisfies (with $\lambda_1 = \log M_1$)

$$\bar{q}(t,y) = e^{-\lambda_1 y_1}\Psi_1(t,y), \tag{3.59}$$

with a function $\Psi_1(t,y)$ which is 1-periodic in $y_1$ and $T$-periodic in $t$. Iterating this process we will construct a solution of (3.58) that is of the form

$$\bar{q}(t,y) = e^{-\sum_{i=1}^n \lambda_i y_i}\Psi(t,y). \tag{3.60}$$

Here, the function $\Psi(t,y)$ is $T$-periodic in $t$ and 1-periodic in all $y_i$. In the original variables this corresponds to a solution of

$$r_t = \Delta r + \mu(x)r \tag{3.61}$$

of the form

$$r(t,x) = e^{-\sum_{i=1}^n \lambda_i(x_i - \bar{c}e_i t)}\Phi(t,x). \tag{3.62}$$

As $T = M/\bar{c}$, the function $\Phi(t,x) = \Psi(t,x-\bar{c}te)$ is $T$-periodic in time, 1-periodic in all $x_i$, and satisfies an autonomous equation

$$\Phi_t + \bar{c}(e\cdot\lambda)\Phi = \Delta\Phi - 2\lambda\cdot\nabla\Phi + |\lambda|^2\Phi + \mu(x)\Phi.$$

**Exercise 3.15** Use the Krein-Rutman theorem to show that the function $\Phi(t,x)$ does not depend on $t$.

Finally, we set $e'_i = \lambda_i/|\lambda|$ and write

$$\sum_{i=1}^n \lambda_i(x_i - \bar{c}e_i t) = |\lambda|\sum_{i=1}^n (x_i e'_i - \bar{c}te'_i e_i) = |\lambda|[(x\cdot e') - \bar{c}'t],$$

with $\bar{c}' = \bar{c}(e\cdot e')$. Therefore, $r(t,x)$ is an exponential solution in the direction $e'$ moving with the speed $\bar{c}'$. It follows that $\bar{c}' \ge c_*(e')$, hence

$$\bar{c} \ge \frac{c_*(e')}{(e\cdot e')} \ge w_*(e),$$

and the proof of Lemma 3.14 is complete.

Returning to the proof of Proposition 3.9, we conclude that for any $c < w_*(e)$, we can find $R$ sufficiently large so that $\lambda_1(R) < 0$. Taking the corresponding eigenfunction for (3.49) on $B_R$ as the function $s_e(t,x)$ in (3.43), and $\gamma$ as the corresponding eigenvalue (with the minus sign), we deduce the claim of that proposition.

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