1 Examples of relaxation enhancement

In these notes, we will discuss convergence to an equilibrium for solutions of partial differential equations. In this introductory part, we will focus on the situation when the equilibrium is simply $\phi(t, x) \equiv 0$, so the interest is in the decay rate of the solution to zero, and what can make this decay to be faster than naively expected. Generally, we will look at evolution problems of the form

$$\frac{\partial \phi}{\partial t} = L\phi - \Gamma\phi, \quad t \geq 0,$$

(1.1)

with an initial condition $\phi(0) = \phi_0 \in H$, where $H$ is some Hilbert space, $L$ is a skew-symmetric operator, and $\Gamma$ is a symmetric operator on $H$. Typically, we will take $H = L^2(\mathbb{R}^d)$ but sometimes we will think of more abstract settings. We will usually assume that $\Gamma$ is strictly dissipative (coercive) there exists $c_0 > 0$ so that

$$\langle \Gamma\phi, \phi \rangle \geq c_0 \|\phi\|^2.$$  

(1.2)

It is immediate to see from (1.2) that the solution to (1.1) satisfies the exponential decay estimate:

$$\frac{1}{2} \frac{d}{dt} (\|\phi(t)\|^2) = -\langle \Gamma\phi, \phi \rangle \leq -c_0 \|\phi\|^2,$$

(1.3)

so that

$$\|\phi(t)\| \leq \|\phi_0\| \exp(-c_0 t).$$

(1.4)

This estimate does not depend on $L$ at all, as long as $L$ is skew-symmetric, and holds, in particular, if $L = 0$.

On the other hand, since $L$ is skew-symmetric, solutions to the $\Gamma$-less equation

$$\frac{\partial \psi}{\partial t} = L\psi, \quad \psi(0) = \psi_0,$$

(1.5)

do not decay at all:

$$\frac{1}{2} \frac{d}{dt} (\|\psi(t)\|^2) = \langle L\psi, \psi \rangle = 0,$$

(1.6)
so that
\[ \|\psi(t)\| = \|\psi_0\|. \] (1.7)

The question we are interested in is if the presence of \( L \) can make the decay of the solutions in (1.1) much faster than the trivial bound (1.4) even though solutions to the ”purely \( L \)”-equation (1.6) have no decay whatsoever. This phenomenon is informally known as relaxation enhancement.

The basic mechanism behind relaxation enhancement is very simple. Let us assume that the operator \( \Gamma \) is self-adjoint and has a discrete spectrum with eigenvalues
\[ 0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots, \]
with
\[ \lambda_n \to +\infty \text{ as } n \to +\infty, \] (1.8)
with the corresponding eigenspaces
\[ X_k = \{ \phi \in H : \Gamma\phi = \lambda_k\phi \}. \]

Then solutions to the ”\( L \)-less” equation
\[ \frac{\partial \phi}{\partial t} = -\Gamma\phi, \] (1.9)
with an initial condition \( \phi_0 \in X_k \) will decay as
\[ \|\phi(t)\| = \|\phi_0\| \exp(-\lambda_k t). \] (1.10)

Therefore, the decay estimate
\[ \|\phi(t)\| \leq \|\phi_0\| \exp(-\lambda_0 t) \] (1.11)
is optimal, as \( \phi_0 \) generically has a non-trivial component in the eigenspace \( X_0 \).

Consider now what may happen if \( L \neq 0 \), and to emphasize the effect, let us put a large coefficient in front of \( L \):
\[ \frac{\partial \phi}{\partial t} = \frac{1}{\varepsilon} L\phi - \Gamma\phi, \] (1.12)
with \( \phi_0 \in X_k \), and with \( \varepsilon \ll 1 \). Then, for short times, solutions to (1.12) can be well approximated by
\[ \frac{\partial \phi_a}{\partial t} = \frac{1}{\varepsilon} L\phi_a, \] (1.13)
with \( \phi_a(0) = \phi_0 \). This is equivalent to a time-rescaling of
\[ \frac{\partial \psi}{\partial t} = L\psi, \quad \psi(0) = \phi_0, \] (1.14)
in the sense that \( \phi_a(t) = \psi(t/\varepsilon) \). If \( L \) and \( \Gamma \) do not commute, then there is no reason why \( \phi_a(t) \) would have a large component in the eigenspaces \( X_k \) corresponding to ”small” \( k \). We expect that \( \phi_a(t) \) will populate all eigenspaces of \( \Gamma \). Moreover, \( \phi_a(t) \) should not concentrate in the low eigenspaces of \( \Gamma \), and the bulk of the solution will be in the high eigenmodes of \( \Gamma \). But for such \( \phi_a(t) \) the operator \( \Gamma \) may dominate \( \varepsilon^{-1} L \) in (1.12) if, say, \( \lambda_k \gg \varepsilon^{-1} \) and solutions to the full problem (1.12) should decay rapidly because it has been moved into the high eigenmodes of \( \Gamma \) by the approximate evolution (1.14) that only involves \( L \). This is the basic scenario behind relaxation enhancement. Turning this into a theorem is not always simple but the mechanism is almost always the same.
A newborn toy example

Let us see how this works on the example of $2 \times 2$ matrices. We take

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(1.15)

with $\lambda \gg 1$, so that $L\Gamma \neq \Gamma L$, and $\Gamma$ has a small eigenvalue 1 and a large eigenvalue $\lambda$. The evolution by $\Gamma$ alone:

$$\frac{d\phi}{dt} = -\Gamma \phi,$$

(1.16)

only has the a priori decay

$$\|\phi(t)\| \leq \|\phi(0)\| e^{-t},$$

(1.17)

unless the initial condition is of the form $\phi(0) = (0, \alpha_0)$, so that the decay for the $L$-less equation (1.16) is governed by the small eigenvalue 1. On the other hand, the eigenvalues of the matrix

$$nL - \Gamma = \begin{pmatrix} -1 & n \\ -n & -\lambda \end{pmatrix}$$

are the solutions to

$$(\mu + 1)(\mu + \lambda) + n^2 = 0,$$

$$\mu_{1,2} = \frac{-1 - \lambda \pm \sqrt{(1 + \lambda)^2 - 4(n^2 + \lambda)}}{2}.$$

(1.18)

Then, for $n \gg \lambda$ we have

$$\text{Re}(\mu_{1,2}) = -\frac{1 + \lambda}{2}.$$

(1.19)

In other words, if $n$ is sufficiently large, then both eigenvalues $\mu_{1,2}$ of the matrix $nL - \Gamma$ have a very large negative real part and the "small" eigenvalue 1 of the matrix $\Gamma$ disappears. Therefore, solutions to

$$\frac{d\phi}{dt} = (nL - \Gamma)\phi, \quad \phi(0) = \phi_0,$$

(1.20)

obey the decay estimate

$$\|\phi(t)\| \leq C_0 e^{-(1+\lambda)t/2},$$

(1.21)

that is much better than (1.17) as $\lambda \gg 1$.

Let us look in a bit more detail at the solution to (1.20). Let $z(t)$ be the solution to

$$\frac{dz}{dt} = Lz, \quad z(0) = \phi_0 = (\phi_{10}, \phi_{20}),$$

(1.22)

so that

$$z(t) = \phi_{10} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \phi_{20} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

(1.23)

and write

$$\phi(t) = a(t)z(nt) + \frac{1}{n} \psi(t),$$

(1.24)
with a scalar function \(a(t)\) and a vector-valued function \(\psi(t)\) to be chosen. This gives
\[
\frac{da}{dt}z(nt) + \frac{1}{n} \frac{d\psi}{dt} = L\psi(t) - a(t)\Gamma z(nt) - \frac{1}{n} \Gamma \psi(t).
\] (1.24)

The function \(z(t)\) is periodic with a period \(2\pi\) and \(Lz(t)\) has mean zero because of (1.22). In addition, we have \(\|z(t)\| = \|\phi_0\|\) for all \(t \geq 0\). Taking the inner product with \(z(nt)\), averaging over the period \(T = 2\pi\) and recklessly dropping the terms we expect to be small, we arrive at
\[
\frac{da}{dt} = -\gamma a(t),
\] (1.25)
with
\[
\gamma = \frac{1}{T\|\phi_0\|^2} \int_0^T \langle \Gamma z(t), z(t) \rangle dt = \frac{1}{T\|\phi_0\|^2} \int_0^T (z_1^2(t) + \lambda z_2^2(t)) dt = \frac{1 + \lambda}{2}.
\] (1.26)

We used the explicit formula (1.23) above. This explains the precise expression \(\gamma = (+1\lambda)/2\) for the limit of the eigenvalues.

**Exercise 1.1** The above argument on the behavior of the solutions to (1.20) is a simple case of the general averaging theory for ODE with periodic oscillatory coefficients. Make it rigorous.

Note that one key feature of this very simple dynamics is that the solution to the "pure \(L\)" equation (1.22) spends a sufficient time in the eigenspace of the largest eigenvalue of \(\Gamma\) to allow for the decay to kick in. This is very important for the "combined \(\Gamma\) and \(L\)" decay mechanism.

**The Dirichlet eigenvalues for Laplacian with a drift**

We now investigate the same phenomenon for the Laplacian operator with a strong incompressible drift.

**The eigenvalues of the Laplacian**

Before we explain how relaxation enhancement for the Laplacian operator with a drift comes about, let us first recall some very basic facts about the principal Dirichlet eigenvalues for the Laplacian on a bounded domain [6]. For any smooth bounded domain \(\Omega\) there exists an eigenvalue \(\lambda_1\) (called the principal eigenvalue) that corresponds to a positive eigenfunction \(\phi_1 > 0\) in \(\Omega\):
\[
-\Delta \phi_1 = \lambda_1 \phi_1, \quad x \in \Omega,
\] (1.27)
\[
\phi_1 = 0 \text{ on } \partial \Omega.
\]

Moreover, \(\lambda_1\) is the smallest of all eigenvalues of the Dirichlet Laplacian on \(\Omega\), \(\lambda_1\) is a simple eigenvalue and all other eigenfunctions of the Laplacian change sign in \(\Omega\). For example, if \(\Omega\) is an interval \((0, 1)\), the eigenvalues of the operator \(Lu = -u''\) with the Dirichlet boundary conditions \(u(0) = u(1) = 0\) are \(\lambda_n = n^2\pi^2\), and the corresponding eigenfunctions are
\[
u_n(x) = \sin(n\pi x).
\]
In this case, the principal eigenvalue is \( \lambda_1 = \pi^2 \).

In general, the principal eigenvalue of the Laplacian is given by the variational formula:

\[
\lambda_1 = \inf_{\psi \in H^1_0(\Omega) : \| \psi \|_2 = 1} \int_{\Omega} |\nabla \psi|^2 \, dx. \tag{1.28}
\]

The principal eigenvalue determines the long time decay of solutions of the parabolic initial value problem in the following way. Consider the initial value problem

\[
\begin{align*}
\psi_t &= \Delta \psi, \quad t > 0, x \in \Omega, \\
\psi(t, x) &= 0 \text{ on } \partial \Omega, \\
\psi(0, x) &= \psi_0(x).
\end{align*} \tag{1.29}
\]

As \( \phi_1(x) > 0 \) in \( \Omega \), and, as follows from the Hopf lemma, \( \partial \phi_1 / \partial \nu < 0 \) on \( \partial \Omega \), we can find a constant \( C > 0 \) so that \( |\psi(t = 1, x)| \leq C \phi_1(x) \) – we can not quite have such estimate at \( t = 0 \) since the initial condition \( \psi_0(x) \) may not satisfy the Dirichlet boundary conditions. The maximum principle implies that

\[
\psi(t, x) \leq Ce^{-\lambda_1(t-1)} \phi_1(x),
\]

for \( t > 1 \), and, similarly,

\[
-\psi(t, x) \leq Ce^{-\lambda_1(t-1)} \phi_1(x),
\]

so that

\[
|\psi(t, x)| \leq Ce^{-\lambda_1(t-1)} \phi_1(x), \quad t \geq 1. \tag{1.32}
\]

Therefore, all solutions of the Cauchy problem decay at the exponential rate determined by \( \lambda_1 \) as \( t \to +\infty \).

**The Dirichlet eigenvalues with a drift**

Let us now consider the Dirichlet principal eigenvalue problem in a smooth bounded domain \( \Omega \), for a diffusion with a strong incompressible flow:

\[
-\Delta \phi + \frac{1}{\varepsilon} u \cdot \nabla \phi = \lambda_1(\varepsilon) \phi, \quad \phi(x) > 0 \text{ in } \Omega, \phi = 0 \text{ on } \partial \Omega. \tag{1.33}
\]

This is an example of a problem like (1.12), with \( \Gamma = -\Delta \), and \( L = u \cdot \nabla \). We assume that \( u \) is an incompressible flow: \( \nabla \cdot u = 0 \), and that it does not penetrate the boundary:

\[
u(u \cdot \nu) = 0 \text{ on } \partial \Omega. \tag{1.34}
\]

This makes the operator \( L = u \cdot \nabla \) skew-symmetric:

\[
\langle L \psi, \psi \rangle = \int_{\Omega} (u \cdot \nabla \psi) \psi \, dx = 0, \tag{1.35}
\]

by the divergence theorem, as the boundary term vanishes, due to (1.34).
The operator in (1.33) is not self-adjoint (so that its eigenvalues are not necessarily real), and its eigenvalues do not obey an integral variational principle such as (1.28). Nevertheless, the Krein-Rutman theory for positive operators (see Chapter VIII of [5]) implies that it has a unique eigenvalue \( \lambda_1(\varepsilon) \) that corresponds to a positive eigenfunction \( \phi_1(x) \). This eigenvalue is real and simple, has the smallest real part of all eigenvalues, and is called the principal eigenvalue. As for the Laplacian, the maximum principle implies that the principal eigenvalue determines the long time decay of the solutions of the corresponding Cauchy problem:

\[
\psi_t + \frac{1}{\varepsilon} u \cdot \nabla \psi = \Delta \psi, \quad t > 0, x \in \Omega, \\
\psi(t, x) = 0 \text{ on } \partial \Omega, \\
\psi(0, x) = \psi_0(x),
\]

that is,

\[
\psi(t, x) \sim e^{-\lambda_1(\varepsilon)t} \phi_1(x), \quad \text{as } t \to +\infty.
\]

Note that when \( u = 0 \) (or, in our general terminology, \( L = 0 \)) the exponential rate of decay for the solutions of (1.36) is simply the principal eigenvalue of the Laplacian. On the other hand, solutions of the Laplacian-less problem

\[
\psi_t + \frac{1}{\varepsilon} u \cdot \nabla \psi = 0
\]

do not decay at all – their \( L^2 \) norm is preserved, as are all \( L^p \)-norms for \( p \geq 1 \). This is because the flow \( u \) is incompressible and parallel to \( \partial \Omega \) on the boundary.

Let us now understand whether it is possible that solutions of the ”combined” Cauchy problem (1.36) decay much faster in time than when \( u = 0 \) despite the fact that solutions of (1.38) have no decay whatsoever. To quantify this questions, let us ask if it is possible that

\[
\lambda_1(\varepsilon) \to +\infty \text{ as } \varepsilon \to 0.
\]

The above considerations make it clear that such phenomenon may only come from an interaction of the drift and the Laplacian.

Let us recall the probabilistic interpretation of the solutions of the Cauchy problem (1.36). Consider the stochastic differential equation

\[
dX_t = -\frac{1}{\varepsilon} u(X_t) dt + \sqrt{2} dW_t, \quad X_0 = x,
\]

starting at a point \( x \in \Omega \), and let \( \tau \) be the first time that the process \( X_t \) hits the boundary \( \partial \Omega \). Then solution of the Cauchy problem (1.36) can be expressed in terms of the diffusion \( X_t \) as

\[
\psi(t, x) = E_x[g(X_{\min(t,\tau)})]
\]

with the convention that

\[
g(X_\tau) = 0.
\]

When would we expect \( \psi(t, x) \) to be small as \( \varepsilon \to 0 \)? As one sees from (1.42), this would be true if, with a high probability we have \( \tau < t \) – the particle hits the boundary before a given
time \( t \). Intuitively, if the trajectories of the incompressible flow are “sufficiently mixing”, then, for any starting point \( x_0 \) in the interior of \( \Omega \), the trajectory of (1.40) that starts at \( x_0 \) eventually comes close to the boundary \( \partial \Omega \). Therefore, such flow, when sufficiently fast, will force solutions of (1.41) very quickly to pass very close to \( \partial \Omega \), and at that time diffusion term in (1.40) will force \( X_t \) to exit \( \Omega \) with a very high probability. Hence, when \( \varepsilon > 0 \) is sufficiently small, the exit time \( \tau \) of the solutions of (1.40) should be smaller than a given time \( t > 0 \) with a high probability. As we have mentioned, this makes \( \psi(t, x) \) given by (1.41) very small because of (1.42). Physically, this means that a sufficiently mixing flow, together with diffusion, should dramatically increase the cooling of the interior by the boundary. A natural questions is what ”mixing” means in this context, and how one can quantify such property. Usually, the mixing properties of a flow are defined in terms of the dynamic properties of the ODE

\[
\dot{X} = u(X),
\]

behave. Here, we are asking a PDE question – hence, the first problem is to define what “mixing” means for us. This is quantified by the following beautiful result due to Berestycki, Hamel and Nadirashvili [2]. We denote by \( I_0 \) the set of all first integrals of \( u \), solutions of

\[
u \cdot \nabla \phi = 0 \text{ a.e. in } \Omega, \tag{1.43}
\]

in the space \( H_0^1(\Omega) \).

**Theorem 1.2** The principal eigenvalue \( \lambda_1(\varepsilon) \) of (1.33) tends to \( +\infty \) as \( \varepsilon \to 0 \) if and only if the flow \( u \) has no first integral in \( H_0^1(\Omega) \). Moreover, if \( u \) has a first integral in \( H_0^1(\Omega) \), then

\[
\lambda_1(\varepsilon) \to \bar{\lambda} := \min_{w \in I_0} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega |w|^2 dx} \text{ as } \varepsilon \to 0, \tag{1.44}
\]

and the minimum in the right side is achieved.

A couple of comments are in order. First, notice that the only information about the Laplacian operator in (1.33) that survives in the statement of the theorem is in the condition that the first integral lies in \( H_0^1(\Omega) \). This regularity requirement comes exactly from the presence of the Laplacian in (1.33), as irregular first integrals do not prevent strong decay of the solutions of the Cauchy problem. Second, the strong flow essentially forces the eigenfunction to be close to a first integral, and then the variational principle (1.29) for the Laplacian operator is replaced by essentially the same expression (1.44) except that the set of allowed test functions is restricted to the first integrals.

**Proof of Theorem 1.2**

The proof of this Theorem is nicely short. First, we claim that if \( u \) has a non-zero first integral \( w \) in \( H_0^1(\Omega) \), normalized so that

\[
\|w\|_{L^2} = 1,
\]

then we have

\[
0 \leq \lambda_1(\varepsilon) \leq \int_\Omega |\nabla w(x)|^2 dx, \tag{1.45}
\]
for any $\varepsilon \in \mathbb{R}$. In order to show that (1.45) holds, we take any $w \in I_0$, and multiply (1.33) by $w^2/(\phi + \delta)$ with $\delta > 0$ fixed:

$$-\int_{\Omega} \frac{w^2 \Delta \phi}{\phi + \delta} \, dx + \int_{\Omega} \frac{w^2}{\phi + \delta} (u \cdot \nabla \phi) \, dx = \lambda_1(\varepsilon) \int_{\Omega} \frac{w^2 \phi}{\phi + \delta} \, dx. \tag{1.46}$$

Integrating by parts in the first term gives

$$-\int_{\Omega} \frac{w^2 \Delta \phi}{\phi + \delta} \, dx = \int_{\Omega} \nabla \phi \cdot \nabla \left( \frac{w^2}{\phi + \delta} \right) \, dx = \int_{\Omega} \frac{2w(\phi + \delta) \nabla \phi \cdot \nabla w - w^2 |\nabla \phi|^2}{(\phi + \delta)^2} \, dx \leq \int_{\Omega} |\nabla w|^2 \, dx.$$

The second term in the left side of (1.46) vanishes because $\nabla \cdot u = 0$ and $w$ is a first integral:

$$\int_{\Omega} \frac{w^2}{\phi + \delta} (u \cdot \nabla \phi) \, dx = \int_{\Omega} w^2 (u \cdot \nabla (\log(\phi + \delta))) \, dx = -\int_{\Omega} 2w \log(\phi + \delta) (u \cdot \nabla w) \, dx = 0.$$

The boundary terms above vanish since $w \in H^1_0(\Omega)$ (it vanishes on the boundary). We conclude that

$$\lambda_1(\varepsilon) \int_{\Omega} \frac{w^2 \phi}{\phi + \delta} \, dx \leq \int_{\Omega} |\nabla w|^2 \, dx, \tag{1.47}$$

for any $w \in I_0$. Passing to the limit $\delta \to 0$ in the left side gives (1.45). Thus, existence of a first integral implies that $\lambda_1(\varepsilon)$ are uniformly bounded for all $\varepsilon \in \mathbb{R}$.

On the other hand, if there exists a sequence $\varepsilon_n \to 0$ such that $\lambda_1(\varepsilon_n)$ are bounded, then

$$\int_{\Omega} |\nabla \phi_n(x)|^2 \, dx = \lambda_1(\varepsilon_n) \int_{\Omega} |\phi_n(x)|^2 \, dx = \lambda_1(\varepsilon_n). \tag{1.48}$$

Here, $\phi_n(x)$ are the associated positive eigenfunctions $\phi_n(x)$ normalized so that $\|\phi_n\|_{L^2(\Omega)} = 1$. As $\lambda_1(\varepsilon_n)$ are uniformly bounded, it follows from (1.48) that there is a subsequence $\phi_{n_k}$ that converges weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$ to a function $\bar{w}(x) \in H^1_0(\Omega)$. Next, multiplying (1.33) by $\varepsilon_{n_k}$ and passing to the limit $k \to +\infty$ gives

$$\varepsilon_{n_k} \int_{\Omega} (-\Delta \phi_{n_k}) \eta \, dx = \varepsilon_{n_k} \int_{\Omega} (\nabla \phi_{n_k}) \cdot \nabla \eta \, dx \to 0, \tag{1.49}$$

for any test function $\eta \in H^1_0(\Omega)$, because of (1.48). We also have

$$\varepsilon_{n_k} \lambda_{n_k} \int_{\Omega} \phi_{n_k} \eta \, dx \to 0, \tag{1.50}$$

because $\|\phi_{n_k}\|_{L^2} = 1$ and $\lambda_1(\varepsilon_{n_k})$ is bounded. It follows that

$$u \cdot \nabla \bar{w} = 0, \quad \text{weakly in } H^1_0(\Omega),$$

which is the same as

$$u \cdot \nabla \bar{w} = 0, \quad \text{a.e. in } \Omega,$$
and
\[ \| \bar{w} \|_{L^2(\Omega)} = 1. \] (1.51)
Hence, \( \bar{w} \) is a first integral of \( u \) in \( H^1_0(\Omega) \). Thus, the non-existence of the first integral in \( H^1_0(\Omega) \) implies that
\[ \lim_{\varepsilon \to 0} \lambda_1(\varepsilon) = +\infty. \] (1.52)

Finally, to show that (1.44) holds if there is a first integral in \( H^1_0(\Omega) \), let us assume, once again, that there exists a sequence \( \varepsilon_n \to 0 \) such that \( \lambda_1(\varepsilon_n) \) are bounded. As the convergence of the subsequence \( \phi_{n_k} \) to the first integral \( \bar{w} \) is strong in \( L^2(\Omega) \) and weak in \( H^1_0(\Omega) \), it follows from (1.48), (1.51) and Fatou’s lemma that
\[ \liminf_{n \to +\infty} \lambda_1(\varepsilon_n) \geq \int_\Omega |\nabla \bar{w}(x)|^2 dx. \] (1.53)
It remains to notice that (1.53) and (1.45) together imply the Rayleigh quotient formula (1.44), and that the minimum is achieved at \( \bar{w}(x) \), finishing the proof of Theorem 1.2.

Directions were are not going to take

Let us finish this introductory section mentioning two directions that are important for the interaction of fast flow and diffusion but that we will not discuss. First, there is a large literature on estimating the effective diffusion in random and periodic flows, and its dependence on the fluid flow strength. Second, there is a very beautiful theory by Freidlin and Wentzell on weakly perturbed two-dimensional Hamiltonian flows.

2 Relaxation enhancement in time

As we have discussed, one interpretation of the eigenvalue enhancement estimate in Theorem 1.2 is in terms of the long time decay rate of the solution to the Cauchy problem
\[ \psi_t + \frac{1}{\varepsilon} u \cdot \nabla \psi = \Delta \psi, \quad t > 0, x \in \Omega, \]
\[ \psi(t, x) = 0 \text{ on } \partial \Omega, \]
\[ \psi(0, x) = g(x), \]
in \( \Omega \) with the Dirichlet boundary condition. Its solution has the long time asymptotics
\[ \psi(t, x) \sim e^{-\lambda_1(\varepsilon)t} \phi(x) \] (2.2)
for \( t \gg 1 \). Here, \( \phi(x) \) is the principal eigenfunction of the operator
\[ -\Delta \phi + \frac{1}{\varepsilon} u \cdot \nabla \phi = \lambda_1(\varepsilon) \phi, \] (2.3)
with the Dirichlet boundary conditions. We have seen in Theorem 1.2 that the principal eigenvalue, or the exponential rate of decay in (2.2), satisfies
\[ \lambda_1(\varepsilon) \to +\infty \text{ as } \varepsilon \to 0 \] (2.4)
if and only if the flow $u$ has no first integrals in $H^1_0(\Omega)$.

The "eigenvalue approach" to improved mixing by an interaction of a fluid flow and diffusion gets much more complicated if we pose the Neumann boundary conditions on $\partial \Omega$, or if $\Omega$ is a manifold without boundary, such as a torus. In that case, the principal eigenfunction is a constant, and the principal eigenvalue vanishes: $\lambda_0 = 0$, regardless of what the flow $u(x)$ is. One may instead study the second eigenvalue but that is not simple since we do not even know a priori that the second eigenvalue is real, or simple, and finding estimates for the real part of a complex eigenvalue that corresponds to an eigenfunction that also need not be real would not be an easy task. Moreover, even if the spectral gap estimate were available, generally it only provides a long time dynamical information, and how fast the long time limit is achieved may depend on $\varepsilon$, since the operator in the left side of (2.3) is neither self-adjoint nor normal: it does not commute with its formal adjoint operator

$$L^* \phi = -\Delta \phi - \frac{1}{\varepsilon} \nabla \cdot (u \phi).$$

This means that the long time behavior may depend not just on the spectrum of $L$ but also on its pseudo-spectrum: the set of $\lambda$ for which $(L - \lambda I)^{-1}$ exists but is large in an appropriate norm. It is a rather typical situation that the dynamical information is not quite easy to deduce from the spectrum alone.

On the other hand, our general interest is in the speed of convergence of the solution to an equilibrium, the relaxation speed, and there are other ways to measure this, not in terms of the spectrum. Therefore, rather than try to address the spectral behavior, we will reformulate our questions purely in terms of the Cauchy problem. On the other hand, the information we will obtain will not translate into non-trivial quantitative properties of the spectrum in a straightforward way.

**Relaxation enhancement in shear flows and hypoellipticity**

A reasonable way to approach the relaxation speed for a parabolic equation of the form

$$\psi^\varepsilon_t + \frac{1}{\varepsilon} u \cdot \nabla \psi^\varepsilon = \Delta \psi^\varepsilon, \quad \psi^\varepsilon(0, x) = \psi_0(x), \quad (2.5)$$

posed in an unbounded domain is in terms of the $L^1 - L^\infty$ decay of the solutions. We will always assume that $u(x)$ is incompressible:

$$\nabla \cdot u = 0, \quad (2.6)$$

so that the flow map for the ODE

$$\frac{dX}{dt} = u(X), \quad X(0) = x, \quad (2.7)$$

is measure preserving, and the total mass is preserved by (2.5):

$$\int_{\mathbb{R}^n} \psi^\varepsilon(t, x) dx = \int_{\mathbb{R}^n} \psi_0(x) dx. \quad (2.8)$$
As the total mass of $\psi^\varepsilon(t,x)$ is conserved, we can measure the additional mixing by $u(x)$ in terms of the decay of the $L^\infty$-norm of $\psi^\varepsilon(t,x)$: the smaller $\|\psi^\varepsilon(t,\cdot)\|_{L^\infty}$ is, the more evenly the mass of $\psi^\varepsilon(t,x)$ is spread around. One can rephrase this in terms of the decay of any $L^p$-norm with $p > 1$ but the $L^\infty$-norm gives the most intuitive picture. Let us stress that we always talk about the decay of $\|\psi^\varepsilon(t,\cdot)\|_{L^\infty}$ at a fixed time $t > 0$ when $\varepsilon \ll 1$ is sufficiently small, and not as $t \to +\infty$. The reader may think simply of the $L^\infty$-norm of $\psi^\varepsilon(t,x)$ at $t = 1$.

One can get a simple estimate on the $L^1 - L^\infty$ decay multiplying (2.5) by $u$ and integrating by parts. Using incompressibility of $u$, gives

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\psi^\varepsilon|^2 dx = - \int_{\mathbb{R}^n} |\nabla \psi^\varepsilon|^2 dx. \quad (2.9)$$

**Exercise 2.1** Use the Nash inequality, conservation of the total mass, and (2.9) to show that there exists a constant $C > 0$ so that for all incompressible $u(x)$ and $\varepsilon > 0$ we have

$$|\psi^\varepsilon(t,x)| \leq \frac{C}{t^{n/2}} \|\psi_0\|_{L^1}. \quad (2.10)$$

The fact that the constant $C > 0$ does not depend on $u$ or $\varepsilon > 0$ in (2.9) shows that no incompressible flow can have too little mixing. The next exercise shows that this universal decay also incorporates the incompressibility of $u(x)$ so it does take into account some of the mixing properties.

**Exercise 2.2** Show that no such estimate (2.10) may hold with the same constant $C > 0$ for all $u(x)$ without the incompressibility assumption (2.6).

However, the estimate (2.10) does not show in any way an improvement of mixing by the flow $u(x)$. In general, this is quite difficult, so let us first look at a simple special case when everything can be done more or less explicitly: the shear flows considered in [3]. These are unidirectional flows of the form $u = (v(y),0)$, with a scalar-valued function $v(y)$. Here, we have introduced the coordinates $x = (x_1,y)$ on $\mathbb{R}^n$, with $x_1 \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Such flows automatically satisfy the incompressibility condition: $\nabla \cdot u = 0$. For simplicity, we will consider (2.5) in the cylinder $\Omega = \mathbb{R} \times \mathbb{T}^{n-1}$, so that both $v(y)$ and the solution to (2.5) are 1-periodic in the $y_k$-variables, $k = 1, \ldots, n-1$.

Let $\psi(t,x_1,y)$ be 1-periodic in $y \in \mathbb{T}^{n-1}$ and satisfy

$$\psi_t + \frac{1}{\varepsilon} v(y) \frac{\partial \psi}{\partial x_1} = \Delta_{x_1,y} \psi, \quad (2.11)$$

with the initial condition $\psi(0,x_1,y) = \phi_0(x_1,y)$. It is clear that if $v(y) \equiv \bar{v}$ is a constant flow, then the $L^1 - L^\infty$ decay of $\psi(t,x,y)$ is exactly the same as for the equation with $v(y) \equiv 0$, as $\bar{v}$ simply translates the solution in the $x_1$-direction. Another clear obstacle to a faster $L^1 - L^\infty$ is the existence of a plateau in the profile $v(y)$: if $v(y) \equiv \bar{v}$ for all $y \in D$, where $D$ is some open set. Indeed, in that case we may bound $\psi(t,x,y)$ from below by the solution to

$$\bar{\psi}_t + \frac{1}{\varepsilon} \bar{v} \frac{\partial \bar{\psi}}{\partial x_1} = \Delta_{x_1,y} \bar{\psi}, \quad (x_1,y) \in \mathbb{R} \times D, \quad (2.12)$$
with the Dirichlet boundary condition on $\partial D$. This is a translate of the solution to

$$\phi_t = \Delta_{x_1,y} \phi, \quad (x_1, y) \in \mathbb{R} \times D,$$

so that

$$\psi(t, x_1, y) \geq \tilde{\psi}(t, x_1, y) = \phi(t, x_1 - \bar{v} t, y), \quad (x_1, y) \in \mathbb{R} \times D. \quad (2.14)$$

This means that there is no speed-up of the $L^1 - L^\infty$ decay for $\psi(t, x, y)$ due to the flow $v(y)$ if $v(y)$ has a plateau – the rate of the decay, for a fixed $t > 0$, is constrained by the principal Dirichlet eigenvalue of the domain $D$.

Let us now assume that $v(y)$ does not have a plateau to see if "no plateau" is a sufficient condition for an improved $L^1 - L^\infty$ decay. A very simple observation is that $\psi(t, x, y)$ can be written as

$$\psi(t, x_1, y) = \int_{-\infty}^{\infty} G(t, x_1 - z) \Psi(t, z, y) dz \quad (2.15)$$

with the function $\Psi(t, x, y)$ satisfying the degenerate parabolic equation

$$\Psi_t + \frac{1}{\varepsilon v(y)} \frac{\partial \Psi}{\partial x_1} = \Delta_y \Psi, \quad (2.16)$$

with the initial condition $\Psi(0, x_1, y) = \psi_0(x_1, y)$ and the one-dimensional heat kernel

$$G(t, x_1) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x_1^2}{4t} \right).$$

Indeed, if $\Psi(t, x_1, y)$ is a solution to (2.16), then $\psi(t, x_1, y)$ defined by (2.15) satisfies

$$\psi_t + \frac{1}{\varepsilon v(y)} \frac{\partial \psi}{\partial x_1} = \int_{-\infty}^{\infty} [\Delta_{x_1} G(t, x_1 - z)] \Psi(t, z, y) dz$$

$$+ \int_{-\infty}^{\infty} G(t, x_1 - z) \left[ \Delta_y \Psi(t, z, y) - \frac{1}{\varepsilon v(y)} \frac{\partial \Psi(t, z, y)}{\partial z} \right] dz$$

$$+ \frac{1}{\varepsilon} \int_{-\infty}^{\infty} v(y) \frac{\partial G(t, x_1 - z)}{\partial x_1} \Psi(t, z, y) dz$$

$$= \int_{-\infty}^{\infty} [\Delta_{x_1} G(t, x_1 - z) \Psi(t, z, y) + G(t, x_1 - z) \Delta_y \Psi(t, z, y)] dz = \Delta_{x_1,y} \psi(t, x, y), \quad (2.17)$$

so that, indeed, $\psi(t, x_1, y)$ satisfies (2.11).

The operator

$$\mathcal{L} \psi = -\Delta_y \psi + \frac{1}{\varepsilon v(y)} \frac{\partial \psi}{\partial x_1} \quad (2.18)$$

that appears in (2.16) is not uniformly elliptic: it lacks the Laplacian in the $x_1$-direction. It is, however, hypoelliptic [11] if there is no point $y \in \mathbb{T}^{n-1}$, where all derivatives of $v(y)$ vanish. We will call this the $H$-condition. Indeed, the Lie algebra generated by the operators $\nabla_y$ and $v(y) \partial_x$ consists of vector fields of the form

$$\nabla_y, v(y) \frac{\partial}{\partial x}, \frac{\partial v(y)}{\partial y_k} \frac{\partial}{\partial x}, \frac{\partial^2 v(y)}{\partial y_k \partial y_m} \frac{\partial}{\partial x}, \ldots, v^{(n)}(y) \frac{\partial}{\partial x}, \ldots \quad (2.19)$$
which span $\mathbb{R}^n$ if $v(y)$ satisfies the H-condition. The study of existence of smooth fundamental solutions for such degenerate operators was initiated by Kolmogorov [15]. Kolmogorov’s work with $v(y) = y$ served in part as a motivation for the fundamental result on characterization of hypoelliptic operators of Hörmander [11]. The ”no plateau” condition for $v(y)$ is not equivalent to the H-condition but is reasonably close to it.

If $v(y)$ satisfies the H-condition, then the theory of Hörmander [11], and the results of Ichihara and Kunita [12] imply that there exists a smooth transition probability density $p_\varepsilon(t, x_1, y, y')$ such that

$$\Psi(t, x_1, y) = \int_{\mathbb{R}} \int_{0}^{H} p_\varepsilon(t, x_1 - x', y, y') \psi_0(x', y')dy'dx. \quad (2.20)$$

In particular, the function $p_\varepsilon(t)$ is uniformly bounded from above for any $t > 0$ [12]. Then we have

$$\|\psi(t)\|_{L^\infty_{x_1,y}} \leq \|\Psi(t)\|_{L^\infty_{x_1,y}} \leq \|p_\varepsilon(t)\|_{L^\infty_{x_1,y}} \|\psi_0\|_{L^1_{x_1,y}}. \quad (2.21)$$

It is straightforward to observe that $p_\varepsilon$ has a simple scaling property

$$p_\varepsilon(t, x_1, y, y') = \varepsilon p_0(t, \varepsilon x, y, y') \quad (2.22)$$

with $p_0$ being the transition probability density for (2.16) with $\varepsilon = 1$. That is, $p_0$ satisfies

$$\frac{\partial p_0}{\partial t} + v(y) \frac{\partial p_0}{\partial x_1} = \Delta_y p_0, \quad (2.23)$$

with the initial condition $p_0(0, x, y, y') = \delta(x)\delta(y - y')$. Therefore, we obtain

$$\|\psi(t)\|_{L^\infty} \leq \varepsilon \|p_0(t)\|_{L^\infty_{x,y}} \|\psi_0\|_{L^1_{x,y}} \leq C\varepsilon \|\psi_0\|_{L^1_{x,y}}. \quad (2.24)$$

This is a version of the relaxation enhancement in the whole space that we were looking for: the $L^1 - L^\infty$ decay at a fixed time $t > 0$ is faster as $\varepsilon \to 0$.

As a side remark, we note that this very simple example also shows a connection between relaxation enhancement and hypoellipticity.

**Relaxation enhancing flows on a torus**

Let us now consider relaxation enhancement on the $n$-dimensional torus. The discussion below applies verbatim to the case of a smooth compact $n$-dimensional Riemannian manifold $\Omega$, and generalizations are very straightforward, so we do not discuss them – see [4] for some of the full cornucopia. We consider solutions to the passive scalar equation

$$\phi^\varepsilon_t + \frac{1}{\varepsilon}u(x) \cdot \nabla \phi^\varepsilon - \Delta \phi^\varepsilon = 0, \quad \phi^\varepsilon(0, x) = \phi_0(x), \quad (2.25)$$

on $\Omega = \mathbb{T}^n$, supplemented by periodic boundary conditions. As always, we assume that $u$ is incompressible: $\nabla \cdot u = 0$. The solution $\phi^\varepsilon(t, x)$ tends to its average,

$$\bar{\phi}(t) = \frac{1}{|\Omega|} \int_{\Omega} \phi^\varepsilon(t, x) \, d\mu = \frac{1}{|\Omega|} \int_{\Omega} \phi_0(x) \, dx, \quad (2.26)$$
as $t \to +\infty$. Here $|\Omega|$ is the volume of $\Omega$. To see that, first, integrating (2.25) over $M$ and using incompressibility of $u(x)$ gives

$$
\frac{d}{dt} \int_{\Omega} \phi(t, x) dx = 0,
$$

hence $\bar{\phi}(t) = \bar{\phi}(0)$ is preserved in time. Next, multiplying (2.25) by $\phi^\varepsilon(t, x) - \bar{\phi}$, and again using incompressibility of $u(x)$, we have

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\phi(t, x) - \bar{\phi}|^2 dx = -\int_{\Omega} |\nabla \phi^\varepsilon(t, x)|^2 dx. \tag{2.27}
$$

The Poincaré inequality implies that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\phi(t, x) - \bar{\phi}|^2 dx \leq -C_p \int_{\Omega} |\phi^\varepsilon(t, x) - \bar{\phi}|^2 dx, \tag{2.28}
$$

whence

$$
\|\phi(t, \cdot) - \bar{\phi}\|_{L^2(\Omega)} \leq e^{-C_p t} \|\phi_0 - \bar{\phi}\|_{L^2(\Omega)}. \tag{2.29}
$$

**Exercise 2.3** Strengthen this result to show that

$$
\|\phi(t, \cdot) - \bar{\phi}\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to +\infty. \tag{2.30}
$$

Note that the decay rate in (2.29) holds for all incompressible flows $u(x)$, with the same constant $C_p$ — this is the analog of the universal $L^1 - L^\infty$ decay estimate (2.10) that holds in the whole space for all incompressible flows $u(x)$. The same is true for (2.30) — the rate of convergence is uniform in incompressible flows.

We would like to understand how the speed of convergence to the average in (2.30) depends on the properties of the flow and determine which flows are particularly efficient in enhancing the relaxation process. We will use the following “fixed time” (no long time limit!) definition as a measure of the flow efficiency in improving the relaxation of the solution to a uniform state.

**Definition 2.4** An incompressible flow $u$ is relaxation enhancing if for all $\tau > 0$ and $\delta > 0$, there exists $\varepsilon(\tau, \delta)$ such that for any $\varepsilon < \varepsilon(\tau, \delta)$ and any $\phi_0 \in L^2(\Omega)$, with $\|\phi_0\|_{L^2(\Omega)} = 1$, we have

$$
\|\phi^\varepsilon(\tau, \cdot) - \bar{\phi}\|_{L^2(\Omega)} < \delta, \tag{2.31}
$$

where $\phi^\varepsilon(t, x)$ is the solution of (2.25) and $\bar{\phi}$ the average of $\phi_0$.

**Exercise 2.5** Show that the choice of the $L^2$ norm in the definition is not essential and can be replaced by any $L^p$-norm with $1 \leq p \leq \infty$, without changing the class of relaxation enhancing flows.
Let us mention that there are various results on Gaussian and other estimates on the heat kernel corresponding to the incompressible drift and diffusion on manifolds such as in the work of Norris [16] and Franke [10], but these estimates lead to upper bounds on the convergence rate to the equilibrium which essentially do not improve as \( \varepsilon \to 0 \), and thus do not quite address the effect of a strong flow. Such general estimates often deteriorate as the flow gets stronger, which is exactly the opposite of what interests us. Surprisingly, there seems to be no general method to incorporate the "helpful" affects of the advection into the proofs of the heat kernel estimates.

The original motivation for this definition came from the work of Fannjiang, Nonnemacher and Wolowski [7, 8, 9], where relaxation enhancement was studied in the discrete setting (see also [14] for related earlier references). In these papers, a unitary evolution step (a certain measure preserving map on the torus) alternates with a dissipation step, which, for example, acts simply by multiplying the Fourier coefficients by damping factors. The absence of sufficiently regular eigenfunctions appears as a key for the enhanced relaxation in this particular class of dynamical systems. In [7, 8, 9], the authors also provide finer estimates of the dissipation time for particular classes of toral automorphisms – they estimate how many steps are needed to reduce the \( L^2 \) norm of the solution by a factor of two if the dissipation strength is \( \varepsilon \).

To understand why and when we expect relaxation enhancement, let us first look at the time-splitting approximation for (2.25), in the spirit of [7, 8, 9]. Assume that \( \psi(t, x) \) solves the advection equation

\[
\psi_t + \frac{2}{\varepsilon} u \cdot \nabla \psi = 0, \quad n\tau \leq t \leq (n + 1/2)\tau, \tag{2.32}
\]

followed by the heat equation

\[
\psi_t = 2\Delta \psi, \quad (n + 1/2)\tau \leq n\tau, \tag{2.33}
\]

and then again (2.32) followed by (2.33), and so on. As the time step \( \tau \to 0 \), the solution of this time-splitting scheme converges to the solution of (2.32). However, the smallness of \( \tau \) that is required to make the error small depends on \( \varepsilon \) in a way that is very difficult to control efficiently. If we, in a cavalier fashion, instead fix the size of \( \tau \) that is independent of \( \varepsilon \), then solution of the very first step is

\[
\psi(\tau/2, x) = \phi_0(X(\tau/\varepsilon, x)), \tag{2.34}
\]

where \( X(t, x) \) is the trajectory

\[
\dot{X}(t) = -u(X), \quad X(0) = x. \tag{2.35}
\]

If the flow of (2.35) is sufficiently complex and \( \varepsilon \) is sufficiently small, the points \( X(\tau/\varepsilon, x) \) and \( X(\tau/\varepsilon, x') \) may be very far apart, even if \( x \) and \( x' \) are very close. This would make the difference \( \psi(\tau/2, x) - \psi(\tau/2, x') \) large, so that the function \( \psi(\tau/2, x) \) given by (2.34) would have a large gradient. This means that the initial condition for the second step in the time-splitting scheme

\[
\psi_t = 2\Delta \psi, \quad \tau/2 \leq \tau, \tag{2.36}
\]
has a very large gradient. On the other hand, the dissipation identity for (2.36)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\psi - \bar{\psi}|^2 = -2 \int_\Omega |\nabla \psi|^2 dx
\]

(2.37)
tells us that solutions with a large gradient and zero average decay very fast. Therefore, we would deduce that for "sufficiently mixing" flows \(u(x)\) solutions of this time splitting scheme converge to their average very fast if \(\varepsilon\) is small. The problem with making this argument rigorous is that, as we have mentioned, for the convergence of the time-splitting scheme to the true solution we would need to take \(\tau\) not fixed but \(\tau \ll \varepsilon\), making the interaction of advection and diffusion non-trivial and very difficult to account for carefully. Nevertheless, this intuition is correct. Here is the main result of this section.

**Theorem 2.6** ([4]) A Lipschitz continuous incompressible flow \(u \in \text{Lip}(\Omega)\) is relaxation enhancing if and only if the operator \(u \cdot \nabla\) has no eigenfunctions in \(H^1(\Omega)\), other than the constant function.

As in the Berestycki-Hamel-Nadiarshvili theorem, the only information about the Laplacian is in the requirement that the eigenfunction lies in the space \(H^1(\Omega)\): rough eigenfunctions do not preclude relaxation enhancement. We will explain below how flows with such rough eigenfunctions can be constructed, more or less explicitly.

The "sufficiently mixing" property of \(u\) is encoded in this theorem in the same requirement that it does not have an eigenfunction in \(H^1(\Omega)\). The reason for that condition can also be seen from the approximation by the time-splitting scheme (2.34)-(2.36). The operator \(u \cdot \nabla\) is skew-symmetric when \(u\) is divergence free:

\[
\int_\Omega (u \cdot \nabla \eta(x)) \eta(x) dx = 0,
\]

(2.38)
for all \(\eta \in H^1(\Omega)\). Therefore, all eigenvalues \(\lambda = i\omega\) of \(u \cdot \nabla\) are purely imaginary, and if \(\phi \in H^1(\Omega)\) is an eigenfunction:

\[
u \cdot \nabla \phi = i\omega \phi,
\]

(2.39)
then solution of (2.34)

\[
\psi_t + \frac{2}{\varepsilon} u \cdot \nabla \psi = 0, \quad 0 \leq t \leq \tau/2
\]

(2.40)
with the initial condition \(\psi(0, x) = \phi(x)\) satisfies

\[
\psi(t, x) = e^{2i\omega t/\varepsilon} \phi(x).
\]

Therefore, the \(H^1\)-norm of \(\psi(t, x)\) does not increase:

\[
\|\psi(\tau/2, x)\|_{H^1(\Omega)} = \|\phi\|_{H^1(\Omega)},
\]

hence the advection step does not prepare an irregular initial condition for the heat equation in the second step of the time-splitting scheme, and there is no intuitive reason to expect relaxation enhancement when \(\varepsilon \to 0\).

The discrepancy between Theorems 1.2 and 2.6 may seem surprising – after all, on the physical level, the conditions for the relaxation enhancement and eigenvalue enhancement need
not be very different but the eigenvalue enhancement (with the Dirichlet boundary conditions) requires that the operator \( u \cdot \nabla \) does not have first integrals while relaxation enhancement (with the periodic or Neumann boundary conditions) requires that this operator does not have eigenfunctions in \( H^1(\Omega) \) with any eigenvalue (the first integral corresponds to a zero eigenvalue). This issue is resolved by the following

**Proposition 2.7** Let \( u \in \text{Lip}(\Omega) \). If \( \phi \in H^1(\Omega) \) is an eigenfunction of the operator \( u \cdot \nabla \) corresponding to an eigenvalue \( i\omega, \omega \in \mathbb{R} \), then \( |\phi| \in H^1(\Omega) \) and it is the first integral of \( u \), that is, \( u \cdot \nabla|\phi| = 0 \).

**Proof.** The fact that \( |\phi| \in H^1 \) follows from the well-known properties of Sobolev functions (see, for example, [6]). If \( \phi(x) \) satisfies
\[
u \cdot \nabla \phi = i\omega \phi
\]
then
\[
u \cdot \nabla|\phi|^2 = \nu \cdot \nabla(\phi \bar{\phi}) = \phi(\nu \cdot \nabla \bar{\phi}) + \bar{\phi}(\nu \cdot \nabla \phi) = -i\omega \phi \bar{\phi} + i\omega \phi \bar{\phi} = 0,
\]
hence \( \nu \cdot \nabla|\phi| = 0 \). □

Therefore, in the case of the Dirichlet boundary conditions, if \( \phi \in H^1_0(\Omega) \) is an eigenfunction of the operator \( \nu \cdot \nabla \) then \( |\phi| \) is its first integral. Naturally, \( |\phi| \) can not be equal identically to a constant since \( \phi \) satisfies the Dirichlet boundary conditions, as it lies in \( H^1_0(\Omega) \), and \( \phi \neq 0 \). Moreover, if \( \phi \in H^1_0(\Omega) \) is a first integral: \( \nu \cdot \nabla \phi = 0 \) then it is an eigenfunction corresponding to eigenvalue \( \lambda = 0 \). Hence, for the Dirichlet boundary conditions the requirement that \( \nu \cdot \nabla \) does not have a first integral in \( H^1_0(\Omega) \) is equivalent to the condition that it does not have eigenfunctions in \( H^1_0(\Omega) \).

On the other hand, existence of mean zero \( H^1(\Omega) \) eigenfunctions, without imposing the Dirichlet boundary condition, need not guarantee the existence of a mean zero first integral, as can be seen from the following well-known example. Let \( \alpha \in \mathbb{R}^n \) be a constant vector generating an irrational rotation on the \( n \)-dimensional torus \( \Omega \), in the sense that the components of \( \alpha \) are independent over the rationals. The operator \( \alpha \cdot \nabla \) has eigenvalues \( 2\pi i(\alpha \cdot k) \), with any \( k \in \mathbb{Z}^n \). The corresponding eigenfunctions are
\[
w_k(x) = e^{2\pi ik \cdot x}.
\]
Their absolute value is 1, which is a first integral of \( \alpha \cdot \nabla \) but there are no non-constant first integrals since \( \alpha \) is irrational. Indeed, if there exists a function \( \psi \in L^1(\Omega) \) such that
\[
\psi(x + \alpha t) = \psi(x), \quad \text{for all } x \in \Omega \text{ and all } t \in \mathbb{R},
\]
then the Fourier coefficients of the function \( \psi \), defined by
\[
\hat{\psi}(k) = \int_{\Omega} e^{-2\pi ik \cdot y} \psi(y) dy,
\]
should satisfy
\[
\hat{\psi}(k) = e^{2\pi i k \cdot \alpha t} \hat{\psi}(k), \quad \text{for all } k \in \mathbb{Z}^n \text{, and all } t \in \mathbb{R}.
\]
Therefore, either all $\hat{\psi}_k = 0$ for $k \neq 0$, or there exists $k \neq 0$ such that
\[ k \cdot \alpha = 0. \]
The latter, however, is impossible since $\alpha$ is irrational. Hence, $\hat{\psi}_k = 0$ for all $k \neq 0$, and the only first integrals of $\alpha \cdot \nabla$ for an irrational $\alpha$ are constant functions. Thus, this flow is not relaxation enhancing, since it has eigenfunctions in $H^1(\Omega)$, even though it has no first integrals other than a constant function.

**Examples of relaxation enhancing flows**

We now present some examples of relaxation enhancing flows on a torus, to assure the reader that this class is not empty. We first describe flows with very rough eigenfunctions, none of which lie in $H^1(\Omega)$, and then flows that have no eigenfunctions – they are weakly mixing. In both cases, the construction is based on a simple modification of a shear flow.

**Flows with rough eigenfunctions**

Here, we describe a smooth incompressible flow $u(x, y)$, $\nabla \cdot u = 0$, on a torus $\mathbb{T}^2$ that has a purely discrete spectrum but none of the eigenfunctions are in $H^1(\mathbb{T}^2)$. The idea of the construction goes back to Kolmogorov [15]. We present some but not all of the full technical details of the construction [1, 13]. We denote by $U^t$ the flow on $L^2(\mathbb{T}^2)$ generated by $u$:
\[ U^t f(x) = f(X(t; x)), \]
where $X(t, x)$ is the trajectory of
\[ \frac{dX}{dt} = -u(X), \quad X(0; x) = x. \]

Here is the key result of this section.

**Proposition 2.8** There exists a smooth incompressible (with respect to the Lebesgue measure) flow $u(x, y)$ on a two-dimensional torus $\mathbb{T}^2$ so that the corresponding unitary evolution $U^t$ has a discrete spectrum on $L^2(\mathbb{T}^2)$ but none of the eigenfunctions of $U^t$ are in $H^1(\mathbb{T}^2)$.

**Proof.** The basic idea behind the construction is quite simple: we want to create a unidirectional flow such that the speed with which the particles move along various lines is sufficiently mismatched to create large gradients. If the speed were constant along each straight line trajectory, that would be a shear flow. We will see that it is impossible so we will need the speed to vary along the trajectory. This is incompatible with incompressibility but the flow will preserve another measure that has a non-constant density with respect to the Lebesgue measure. An appropriate mapping of a flow constructed this way will lead to an incompressible flow. On a slightly more technical level, we will look for a flow that can be mapped to a constant flow $\bar{u} = (\alpha, 1)$ by a measure preserving map $S$ with very low regularity properties. Since the eigenfunctions of the constant flow are explicitly computable, we can compute the eigenfunctions of the original flow. Due to the roughness of $S$, these will be highly irregular.
As outlined above, we will look at a time change of the constant linear translation flow, of the form
\[ \frac{dx}{dt} = \frac{\alpha}{F(x, y)}, \quad \frac{dy}{dt} = \frac{1}{F(x, y)}, \quad x(0) = x_0, \quad y(0) = y_0, \] (2.41)
with an appropriately chosen \( \alpha \in \mathbb{R} \) and \( F(x, y) \). The trajectories of (2.41) are straight lines:
\[ x(t) - \alpha y(t) = x_0 - \alpha y_0, \quad \text{for all } t \geq 0, \] (2.42)
and the time it takes for the trajectory to go from a point \((x, 0)\) at height \(y = 0\) to the point \((x + \alpha y, y)\), when the trajectory reaches the height \(y\) is
\[ T(x, y) = \int_0^y F(x + \alpha z, z)dz. \] (2.43)

Hence, the function \( F(x, y) \) is simply the local time change of the flow.

It would be very convenient to take \( F(x, y) \) in the form
\[ F(x, y) = Q(x - \alpha y), \] (2.44)
so that \( F(x, y) \) would be constant on each trajectory of the flow (2.41), and (2.41) would really be a shear flow in the direction \((\alpha, 1)\). However, for \( F(x, y) \) as in (2.44) to be 1-periodic both in \(x\) and \(y\), the function \( Q(x) \) has to be both 1-periodic and \( \alpha \)-periodic. If \( \alpha \) is irrational, this is impossible unless \( Q(x) \equiv 1 \).

Thus, instead of trying (2.44), we use cut-offs to modify (2.44), setting
\[ F(x, y) = m + \psi(y)(Q(x - \alpha y) - m), \quad 0 \leq x, y \leq 1. \] (2.45)
Here, \( Q(x, y) > 0 \) is a 1-periodic function \( Q(x) > 0 \) such that
\[ \int_0^1 Q(\xi) d\xi = 1. \] (2.46)
A smooth cut-off function \( \psi(y) \geq 0 \) in (2.45) is such that
\[ \int_0^1 \psi(y) dy = 1, \] (2.47)
and
\[ \psi(y) = 0 \quad \text{for } 0 \leq y \leq y_0 \quad \text{and} \quad y_1 \leq y \leq 1 \quad \text{with} \quad y_0 \text{ close to zero and } y_1 \text{ close to one.} \] (2.48)

The constant \( m \) in (2.45) is such that \( 0 < m < \min Q(s) \). The choice of \( m \) ensures that the function \( F(x, y) > 0 \) - this is needed both to interpret \( F(x, y) \) as a local time change, and to be able to divide by \( F(x, y) \) in (2.41). Note that the function \( F(x, y) \) is already 1-periodic in \( x \) because \( Q(x) \) is 1-periodic. As, in addition, \( F(x, y) \equiv m \) near \( y = 0 \) and \( y = 1 \), we can extend \( F(x, y) \) to the whole plane so that it is also periodic in \( y \). The smoothness of \( Q(x) \) and (2.48) imply that the extension is smooth. In addition, because of (2.46) and (2.47), the total mass of \( F(x, y) \) is
\[ \int_0^1 \int_0^1 F(x, y) dxdy = 1. \] (2.49)
In order to map the flow (2.41) to a constant speed flow \((\alpha, 1)\) moving along the same straight lines, it is natural to attempt to define the transformation \(S : (x, y) \to (X, Y)\) as

\[
X(x, y) = x + \alpha(Y(x, y) - y), \quad Y(x, y) = T(x - \alpha y, y), \quad (2.50)
\]

with \(T(x - \alpha y, y)\) as in (2.43) – the time it takes for the particle starting at \(t = 0\) at the point \((x - \alpha y, 0)\) at height \(y = 0\) to reach the point \((x, y)\) at the height \(y\). In addition, the transformation (2.50) satisfies \(x - \alpha y = X - \alpha Y\), thus it preserves the flow trajectories. Hence, the particle would move with a constant speed along the straight lines in the new variables, the speed in the \(Y\)-direction would equal to one, and in the \(X\)-direction it would equal to \(\alpha\).

However, the map (2.50) is not well-defined on the torus \(T^2\): it is easy to see that \(Y(x, y + 1) = T(x - \alpha y - \alpha, y + 1) \neq T(x - \alpha y, y)\) mod 1. (2.51)

Thus, we modify (2.50) as [15, 18]

\[
X(x, y) = x + \alpha(Y(x, y) - y), \quad Y(x, y) = T(x - \alpha y, y) + R(x - \alpha y), \quad (2.52)
\]

adding a compensatory shift \(R(x - \alpha y)\) that is constant on each trajectory.

We claim that if we choose the 1-periodic function \(R(x)\) that satisfies the homology equation [1]

\[
R(\xi + \alpha) - R(\xi) = Q(\xi) - 1, \quad \xi \in S^1, \quad (2.53)
\]

then the map (2.52) is well-defined on \(T^2\). Note that for (2.53) to have a measurable solution the function \(Q(\xi)\) should satisfy the normalization (2.46). For the moment, we will not worry about the existence of \(R(x)\) and its properties but will come back to this soon.

Let us now check that, indeed, if \(R\) is a solution to the homology equation, then (2.52) defines a mapping of the torus onto itself. The shift in \(x\) is simple to understand: the function \(T(x, y)\) is clearly 1-periodic in \(x\) since \(F(x, y)\) is periodic in \(x\), thus

\[
Y(x + 1, y) = Y(x, y), \quad (2.54)
\]

while

\[
X(x + 1, y) = 1 + X(x, y) = X(x, y) \mod 1. \quad (2.55)
\]

To verify what happens under the shift \(y \to y + 1\), we first make some preliminary observations. The normalization (2.47) implies that

\[
T(x, 1) = \int_0^1 F(x + \alpha z, z)dz = \int_0^1 [m + \psi(z)(Q(x) - m)]dz = Q(x). \quad (2.56)
\]

Now, it follows that

\[
T(x, y + 1) = \int_0^{y+1} F(x + \alpha z, z)dz = \int_0^1 F(x + \alpha z, z)dz + \int_1^{y+1} F(x + \alpha z, z)dz \\
= Q(x) + \int_0^y F(x + \alpha + \alpha z, z + 1)dz = Q(x) + \int_0^y F(x + \alpha + \alpha z, z)dz \quad (2.57)
\]

\[
= Q(x) + T(x + \alpha, y).
\]
Using this identity, and the homology equation (2.53) for the function \( R \) gives

\[
Y(x, y + 1) = T(x - \alpha y - \alpha, y + 1) + R(x - \alpha y - \alpha) \\
= T(x - \alpha y, y) + Q(x - \alpha y - \alpha) + R(x - \alpha y) - Q(x - \alpha y - \alpha) + 1 \\
= T(x - \alpha y, y) + R(x - \alpha y) + 1 = Y(x, y) + 1 = Y(x, y) \mod 1.
\]

This is really the reason why we have chosen \( R(x) \) as the solution to the homology equation.

Finally, for \( X(x, y) \) we have

\[
X(x, y + 1) = x + \alpha(Y(x, y + 1) - y - 1) = x + \alpha(Y(x, y) + 1 - y - 1) = x + \alpha(Y(x, y) - y) = X(x, y).
\]

(2.59)

We conclude from (2.54), (2.55), (2.58) and (2.59) that \( S \) is a well-defined mapping of \( \mathbb{T}^2 \) to itself.

A key observation is that solutions \( R(x) \) of the homology equation (2.53) can be very rough even if the function \( Q \in C^\infty(S^1) \) is smooth. To see that, let us go back to (2.53):

\[
R(\xi + \alpha) - R(\xi) = Q(\xi) - 1, \quad \xi \in S^1.
\]

(2.60)

Note that it can be solved explicitly using the Fourier transform:

\[
R(\xi) = \sum_{n \in \mathbb{Z}} \hat{R}_n e^{2\pi i n \xi},
\]

(2.61)

with the Fourier coefficients

\[
\hat{R}_n = \frac{\hat{Q}_n}{\exp(2\pi i \alpha n) - 1}.
\]

(2.62)

The denominators in (2.62) can be dangerously small if \( \alpha n \) can be very close to an integer, that is, if \( \alpha \) is a Liouvillean irrational number. Recall that an irrational number \( \alpha \in \mathbb{R} \) is called \( \beta \)-Diophantine if there exists a constant \( C \) such that for each \( k \in \mathbb{Z} \setminus \{0\} \) we have

\[
\inf_{p \in \mathbb{Z}} |\alpha k + p| \geq \frac{C}{|k|^\beta + 1}.
\]

The vector \( \alpha \) is Liouvillean if it is not Diophantine for any \( \beta > 0 \). The Liouvillean numbers (and vectors) are the ones which can be very well approximated by rationals. The following Proposition is a particular case of Theorem 4.5 of [13].

**Proposition 2.9** Let \( \alpha \) be a Liouvillean irrational number. There exists a \( C^\infty(S^1) \) function \( Q(\xi) \) so that the homology equation (2.53) has a unique (up to an additive constant) measurable solution \( R(\xi) : S^1 \rightarrow \mathbb{R} \) such that for any \( \lambda \in \mathbb{R} \setminus \{0\} \), the function \( R_\lambda(\xi) = e^{i\lambda R(\xi)} \) is discontinuous everywhere.

We will not prove this proposition here.

Without loss of generality we may assume that \( Q(\xi) \) given by Proposition 2.9 is positive: otherwise, we choose \( M \) so that \( Q(\xi) + M > 1 \) and consider a rescaled function

\[
Q_M(\xi) = (M + Q(\xi))/(M + 1).
\]
Then, the function

\[ R_M(\xi) = \frac{R(\xi)}{M+1} \]

is the solution to (2.53) with \( Q_M \) in the right side and, of course, \( R_M(\xi) \) has the same set discontinuities as \( R(\xi) \).

Let us see what happens to the flow (2.41) under the map (2.52):

\[
\frac{dx}{dt} = \frac{\alpha}{F(x,y)}, \quad \frac{dy}{dt} = \frac{1}{F(x,y)}, \quad x(0) = x_0, \quad y(0) = y_0.
\]  

(2.63)

Note that

\[ x(t) - y(t) = x_0 - \alpha y_0, \]

hence \( Y(t) \) is given by

\[
Y(t) = T(x(t) - \alpha y(t), y(t)) + R(x(t) - \alpha y(t)) = T(x_0 - \alpha y_0, y(t)) + R(x_0 - \alpha y_0),
\]  

(2.64)

so that

\[
\frac{dY}{dt} = \frac{\partial T(x_0 - \alpha y_0, y(t))}{\partial y} \frac{\dot{y}(t)}{1} = F(x_0 - \alpha y_0 + \alpha y(t), y(t)) \frac{1}{F(x(t), y(t))} = 1.
\]  

(2.65)

On the other hand, for \( X(t) \) we have

\[
\frac{dX}{dt} = \dot{x}(t) + \alpha(\dot{Y}(t) - \dot{y}(t)) = \frac{\alpha}{F(x(t), y(t))} + \alpha - \frac{\alpha}{F(x(t), y(t))} = \alpha.
\]  

(2.66)

Therefore, the image of the flow (2.41) under \( S \) is simply the uniform flow:

\[
\frac{dX}{dt} = \alpha, \quad \frac{dY}{dt} = 1,
\]  

(2.67)

as we desired. We will denote \( \bar{u} = (\alpha, 1) \).

Note that the map \( S \) is invertible with a measurable inverse. Indeed, we have

\[
X - \alpha Y = x - \alpha y,
\]  

(2.68)

so that

\[
Y = T(X - \alpha Y, y) + R(X - \alpha Y).
\]  

(2.69)

As the function \( F \) is strictly positive, the function \( T(x, y) \) is strictly increasing in \( y \), hence (2.69) has a unique solution \( y(X, Y) \), and then (2.68) defines \( x(X, Y) \) uniquely.

In addition, \( S \) is measure preserving in the following sense:

\[
\int [S^*f](x, y) F(x, y) dxdy = \int f(S(x, y)) F(x, y) dxdy = \int f(X, Y) dXdY
\]  

(2.70)
for any function \( f \in C(T^2) \). In order to see that, let us introduce intermediate changes of variables: \( S = S_3 \circ S_2 \circ S_1 \), with \( S_1: (x, y) \rightarrow (z, y_1) \) with

\[
z = x - \alpha y, \quad y_1 = y, \]

followed by \( S_2: (z, y_1) \rightarrow (Z, y_2) \)

\[
Z = z, \quad y_2 = T(z, y_1) + R(z),
\]

and finally \( S_3: (Z, y_2) \rightarrow (X, Y) \), with

\[
X = Z + \alpha y_2, \quad Y = y_2.
\]

The corresponding Jacobians are:

\[
J_1 = J_3 = 1, \quad J_2 = \frac{\partial T}{\partial y_1}(z, y_1) = F(z + \alpha y_1, y_1) = F(x, y).
\]

Therefore, the Jacobian of \( S \) is, indeed,

\[
J = J_1 J_2 J_3 = F(x, y),
\]

hence (2.70) holds and \( S \) is measure-preserving.

Hence, \( S^* \) may be extended as an operator \( L^2(dx dy) \rightarrow L^2(d\mu) \) with the preservation of the corresponding norms. It follows that the unitary evolutions \( U^t_w \) and \( U^t_{unif} \) generated by the flow \( w \) given by (2.63) and the uniform flow \( \bar{u} \), respectively, are conjugated by means of the unitary transformation

\[
S^*: L^2(T^2, dX dY) \rightarrow L^2(T^2, d\mu),
\]

that is, we have

\[
U^t_{unif} = [S^*]^{-1} U^t_w S^*.
\]

Therefore, \( U^t_w \) and \( U^t_{unif} \) have the same spectrum:

\[
\lambda_{nl} = 2\pi i n \alpha + 2\pi i l, \quad l, n \in \mathbb{Z}.
\]

It also follows that the eigenfunctions of the operator \( U_w \) may be written as

\[
\psi_{nl}^w(x, y) = e^{2\pi i n X(x, y) + 2\pi i Y(x, y)} = e^{2\pi i n (x - \alpha y + \alpha Y(x, y)) + 2\pi i Y(x, y)}
\]

\[
= e^{2\pi i n (x - \alpha y)} e^{(2\pi i n + 2\pi i l) T(x - \alpha y) + R(x - \alpha y)} = \zeta(x, y) e^{(2\pi i n + 2\pi i l) R(x - \alpha y)}
\]

with a smooth function \( \zeta(x, y) \in C^\infty([0, 1]^2) \). Note that the function

\[
\zeta(x, y) = e^{2\pi i n (x - \alpha y)} e^{(2\pi i n + 2\pi i l) T(x - \alpha y) + R(x - \alpha y)}
\]

is not periodic in \( y \), even though the function \( \psi_{nl}^w(x, y) \) is periodic, but that plays no role. In order to verify that \( \psi_{nl}^w \) are not in \( H^1(T^2) \) it suffices to check that the function

\[
\Theta_\lambda(x, y) = e^{i \lambda R(x - \alpha y)} = R_\lambda(x - \alpha y)
\]

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is not in $H^1([0,1]^2)$ for any real $\lambda \neq 0$. Here, $R_{\lambda}(s)$ is as defined in Proposition 2.9. Since the function $\Theta_{\lambda}(x,y)$ is constant on the lines 

$$x - \alpha y = \text{const},$$

if it were in $H^1([0,1]^2)$, it would force the function $R_{\lambda}(s)$ to be in $H^1(S^1)$ and hence continuous. However, $R_{\lambda}$ is discontinuous everywhere according to Proposition 2.9. Therefore, the eigenfunctions $\psi_{nl}^w$ cannot be in $H^1(T^2)$ unless $n = l = 0$.

Finally, to obtain an incompressible flow (with respect to the standard Lebesgue measure) with rough eigenfunctions, we introduce a smooth transformation of the torus

$$\bar{S}: (x,y) \rightarrow (p,q)$$

by setting

$$p = \int_0^x F(s)ds, \quad q = \frac{1}{F(x)} \int_0^y F(x,z)dz, \text{ where } F(x) = \int_0^1 F(x,z)dz.$$

Note that $\bar{F}(x)$ is periodic, and

$$p(x+1,y) = \int_0^{x+1} \bar{F}(s)ds = p(x,y) + \int_0^1 \bar{F}(s)ds = p(x,y) + 1.$$

We also have $q(x+1,y) = q(x,y)$ and

$$q(x,y+1) = \frac{1}{F(x)} \int_0^{y+1} F(x,z)dz = q(x,y) + 1.$$

Therefore, indeed, $\bar{S}$ is a mapping of $T^2$ to itself. Since $F(x,y)$ is positive, $\bar{S}$ is one-to-one. It is immediate to verify that it maps the measure $d\mu$ onto the Lebesgue measure $dpdq$ – the Jacobian of $\bar{S}$ is $F(x,y)$. Hence, the evolution group generated by the image $u(p,q)$ of the flow $w(x,y)$ will have the same discrete spectrum as $U_w$. In addition, the eigenfunctions $\psi_{nl}^w$ of $U_w$ are the images of the eigenfunctions $\psi_{nl}^u$ of $u$ under $\bar{S}^*:

$$\psi_{nl}^w(x,y) = (\bar{S}^* \psi_{nl}^u)(x,y) = \psi_{nl}^u(\bar{S}(x,y)).$$

As the functions $\psi_{nl}^w$ are not in $H^1(T^2)$ and the map $\bar{S}$ is smooth, it follows that all the eigenfunctions of the incompressible flow $u(p,q)$ are not in $H^1(T^2)$. This finishes the proof of Proposition 2.8. □

References


[12] Ichihara, K. snd Kunita, H., A classification of the second order degenerate elliptic opera-


[15] A.N. Kolmogorov, On dynamical systems with an integral invariant on the torus (Rus-

