Traveling fronts in porous media: existence and a singular limit

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Abstract

We consider a model for the propagation of a subsonic detonation wave through a porous medium introduced by Sivashinsky [8]. We show that it admits travelling wave solutions that converge in the limit of zero temperature diffusivity to the travelling fronts of a reduced system constructed in [6].

1 Introduction

The deflagration to detonation transition remains one of the most intriguing problems in combustion. A simple model for this phenomenon in a highly resistible porous medium has been recently proposed in [2]:

$$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= \varepsilon T_{xx} + Y\Omega(T), \\ P_t - T_t &= P_{xx}, \\ Y_t &= \varepsilon Le^{-1} Y_{xx} - \gamma Y\Omega(T). \end{aligned} \tag{1.1}$$

Here T, P and Y are the appropriately normalized temperature, pressure and concentration of the deficient reactant, $\gamma > 1$ is the specific heat ratio, $Y\Omega(T)$ is the normalized reaction rate, and ε is the ratio of thermal and pressure diffusivities. The first and the last equation in (1.1) represent the partially linearized conservation equations for energy and deficient reactant, while the second follows from the linearized continuity equation, and equations of state and momentum. We recall briefly the derivation of (1.1) in the appendix.

We are interested in the existence of travelling wave solutions of (1.1) of the form $T(x-ct)$, $P(x-ct)$, $Y(x-ct)$, where c is the a priori unknown front speed. Substituting this form of the solutions into (1.1) we obtain a reduced system of ODE's

$$-cT' + c(1-\gamma^{-1})P' = \varepsilon T'' + Y\Omega(T) \tag{1.2}$$

$$P'' = c(T' - P') \tag{1.3}$$

$$cY' + \varepsilon Y'' = \gamma Y\Omega(T) \tag{1.4}$$

with the front-like boundary conditions:

$$P(-\infty) = 1, \quad T(-\infty) = 1, \quad Y(-\infty) = 0 \tag{1.5}$$

$$T(+\infty) = 0, \quad P(+\infty) = 0, \quad Y(+\infty) = 1. \tag{1.6}$$

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We have set Le $= 1$ in (1.4) for simplicity. Note that unlike the situation in some other thermo-diffusive systems, a Lewis number not equal to one would not change the results of this paper. We assume that the function $\Omega(T)$ is of the Arrhenius type with an ignition cut-off, that is, $\Omega(T)$ vanishes on an interval $[0, \theta]$ and is positive for $T > \theta$:

$$\Omega(T) = 0 \quad \text{for} \quad 0 \le T < \theta < 1. \tag{1.7}$$

Moreover, $\Omega(T)$ is an increasing Lipschitz continuous function, except for a possible discontinuity at the ignition temperature $T = \theta$.

There has been a number of physical and mathematical studies of the system (1.2)-(1.6) as well as of its dynamical version (1.1): see a recent review [8] for references. Nevertheless, to the best of our knowledge, the rigorous results have only been obtained for the simplified version of the system (1.2)-(1.6) where ε is formally set equal to zero. This simplification is crucial as then T, Y and P are linearly dependent, which allows to reduce the original problem to a system of two ODE's. This degenerate case is well studied. In particular, it is known that the traveling wave solution exists and is unique [3],[6].

The most important case for the applications is when ε is small ($\varepsilon \sim 10^{-3} - 10^{-5}$). Thus setting $\varepsilon = 0$, that is, ignoring the thermal diffusivity, is very attractive and is believed to reflect the correct phenomena on the physical grounds. The goal of the present paper is to understand how singular the limit $\varepsilon \to 0$ actually is. We show that the full system (1.2)-(1.6) admits travelling wave solutions and that in the limit $\varepsilon \to 0$ they converge to that of (1.2)-(1.6) with $\varepsilon = 0$. Existence of the travelling waves with $\varepsilon > 0$ is established in Theorem 2.1 in Section 2 and the limit $\varepsilon \to 0$ is considered in Theorem 3.2 in Section 3. Finally, the appendix contains a sketch of the physical derivation of (1.1).

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2 Existence of the traveling waves

In this section we establish the existence of a traveling wave solution to the problem (1.2)-(1.6). Let us introduce λ as the positive solution of $\gamma(1-\lambda)(1+\varepsilon\lambda) = 1$ and decompose

$$T(x) = \lambda P(x) + (1-\lambda)R(x), \tag{2.1}$$

with an auxiliary function $R(x)$ defined by (2.1). Note that $0 < \lambda < 1$ and in the case when $\varepsilon < 1$, that is of the most interest for us,

$$\lambda = \frac{1-\varepsilon}{2\varepsilon}\left(\sqrt{1 + \frac{4\varepsilon(1-\gamma^{-1})}{(1-\varepsilon)^2}} - 1\right) = 1 - \gamma^{-1} + O(\varepsilon) \tag{2.2}$$

The following theorem holds.

Theorem 2.1 *The problem (1.2)-(1.6) has a travelling front solution (c, T, P, Y) with the following properties: $c > 0$, $R, P, T, Y \in W^{2,\infty}(\mathbb{R})$, $0 \le R, P, T, Y \le 1$ and*

$$-\frac{c}{\varepsilon\gamma(1-\lambda)} < R' < 0, \quad -c(1-\lambda) < P' < 0, \quad 0 < Y' < \frac{c}{\varepsilon}, \tag{2.3}$$

where $R(x)$ is defined in (2.1) and λ in (2.2).

Proof. The proof of Theorem 2.1 follows the general blueprint of [1] with necessary modifications and is based on the construction of a solution on a bounded interval and subsequent passage to the limit of the whole real line. Let us consider the system (1.2)-(1.4) on an interval $[-a, a]$

$$-cT' + c(1-\gamma^{-1})P' = \varepsilon T'' + Y\Omega(T) \tag{2.4}$$

$$P'' = c(T' - P') \tag{2.5}$$

$$cY' + \varepsilon Y'' = \gamma Y\Omega(T) \tag{2.6}$$

with the boundary conditions

$$P(-a) = 1, \quad T(-a) = 1, \quad Y(-a) = 0 \tag{2.7}$$

and

$$-cT(a) + c(1-\gamma^{-1})P(a) = \varepsilon T'(a), \quad P'(a) = cT(a) - cP(a), \quad cY(a) + \varepsilon Y'(a) = c. \tag{2.8}$$

We also impose the normalization condition:

$$T(0) = \theta, \tag{2.9}$$

where θ is the ignition temperature, as in (1.7).

In the variables (T, R, P, Y) the system (2.4)-(2.8) becomes

$$cR' + \varepsilon\gamma(1-\lambda)R'' = -\gamma Y\Omega(T) \tag{2.10}$$

$$P' = c(1-\lambda)(R - P) \tag{2.11}$$

$$cY' + \varepsilon Y'' = \gamma Y\Omega(T) \tag{2.12}$$

$$T(x) = \lambda P(x) + (1-\lambda)R(x). \tag{2.13}$$

The boundary conditions for (2.10)-(2.13) are

$$R(-a) = 1, \quad P(-a) = 1, \quad Y(-a) = 0 \tag{2.14}$$

and

$$cR(a) + \varepsilon\gamma(1-\lambda)R'(a) = 0, \quad cY(a) + \varepsilon Y'(a) = c. \tag{2.15}$$

Our goal now is to show that there exists $a_0 \geq 0$ so that solutions of (2.10)-(2.15) exist for all $a > a_0$ and converge as $a \to +\infty$ to solutions of (2.10)-(2.13) on the whole real line with the boundary conditions

$$R(-\infty) = P(-\infty) = T(-\infty) = 1, \; Y(-\infty) = 0, \; R(+\infty) = P(+\infty) = T(+\infty) = 0, \; Y(+\infty) = 1. \tag{2.16}$$

This immediately implies convergence of solutions of the system (2.4)-(2.8) to solutions of (1.2)-(1.6) as $a \to +\infty$ since the two systems (1.2)-(1.6) and (2.10)-(2.13) are related by a linear transformation.

In order to obtain the convergence results as $a \to +\infty$ we have to obtain uniform bounds on the solution (c, T, R, P, Y) of (2.10)-(2.15), independent of a. We begin with the following proposition.

Proposition 2.2 *Any solution (c, T, P, R, Y) of the problem (2.9)-(2.15) with $c \geq 0$, has the following properties:*

$$c > 0 \tag{2.17}$$

$$0 \leq R \leq 1, \quad 0 \leq Y \leq 1, \quad 0 \leq P \leq 1, \quad 0 \leq T \leq 1 \quad \text{on} \quad [-a, a] \tag{2.18}$$

$$-\frac{c}{\varepsilon\gamma(1-\lambda)} \leq R' \leq 0, \quad 0 \leq Y' \leq \frac{c}{\varepsilon}, \quad -c(1-\lambda) \leq P' \leq 0 \quad \text{on} \quad [-a, a]. \tag{2.19}$$

Moreover, we have

$$R(x) \leq P(x). \tag{2.20}$$

Proof. **A.** Let us show first that $c = 0$ is impossible. Assume that $c = 0$, then the system (2.10)-(2.12) becomes:

$$\varepsilon R'' = -(1-\lambda)^{-1} Y \Omega(T) \quad (2.21)$$
$$P' = 0 \quad (2.22)$$
$$\varepsilon Y'' = \gamma Y \Omega(T) \quad (2.23)$$

and the boundary conditions (2.14), (2.15) become

$$R(-a) = 1, \quad P(-a) = 1, \quad Y(-a) = 0, \quad R'(a) = 0, \quad Y'(a) = 0. \quad (2.24)$$

Equation (2.22) and the boundary condition (2.24) imply $P(x) = 1$ on $(-a, a)$. Therefore, we have $T(x) = \lambda + (1-\lambda)R(x)$. Combining (2.21) and (2.23) we have:

$$(1-\lambda)\gamma R'' + Y'' = 0.$$

Integrating this equation between x and a, and taking into account the boundary conditions (2.24) we obtain:

$$(1-\lambda)\gamma R' + Y' = 0.$$

Integration from $-a$ to x together with (2.24) yields

$$Y = (1-\lambda)\gamma(1-R).$$

Thus, equation (2.21) takes the form:

$$\varepsilon R'' = -\gamma(1-R)\Omega(\lambda + (1-\lambda)R). \quad (2.25)$$

We claim that $R \leq 1$ on $[-a, a]$. Indeed, assume that this is false. Then either R attains an internal maximum at a point $x_0 \in (-a, a)$, or $R(a) > 1$. In the latter case (2.25) implies that $R(x)$ is convex near $x = a$ and hence can not attain its maximum at $x = a$ as $R'(a) = 0$, thus in both cases it has to have an internal maximum x_0 where $R(x_0) > 1$. Then $R'(x_0) = 0$, $R(x_0) = \max_{-a<x<a} R(x) > 1$ and $R'' \leq 0$. However, this contradicts (2.25) as

$$\Omega(\lambda + (1-\lambda)R(x_0)) \geq \Omega(1) > 0.$$

It follows that $R \leq 1$ everywhere on $[-a, a]$.

Next, integrating (2.25) between x and a we get

$$\varepsilon R' = \gamma \int_x^a (1 - R(\xi))\Omega(\lambda + (1-\lambda)R(\xi)) d\xi \geq 0.$$

Hence, R is a nondecreasing function on $(-a, a)$. In particular, $R(0) \geq R(-a) = 1$. Thus, we have $T(0) = \lambda + (1-\lambda)R(0) \geq 1$ which is in contradiction to (2.9). Therefore, $c = 0$ is impossible and (2.17) holds.

B. Y is positive. The proof is identical to the one presented in [1] for the thermo-diffusive system (Proposition 8.1.B).

C. Let us prove (2.18). First, we introduce a new variable $W = \varepsilon Y' + cY$. Then (2.12) becomes

$$W' = \gamma Y \Omega(T). \quad (2.26)$$

The boundary conditions (2.14), (2.15) imply $W(-a) = \varepsilon Y'(-a)$, $W(a) = c$. Note that $Y'(-a) \geq 0$ as $Y(x) \geq 0$ and $Y(-a) = 0$. Hence, W is a monotonic function on $(-a, a)$ that increases from

$W(-a) = \varepsilon Y'(-a) \geq 0$ to $W(a) = c$. Thus, in particular, $0 \leq W \leq c$. We may now rewrite the system (2.10)-(2.12) as a system of first order ODE's. Indeed, substituting (2.26) into (2.10)-(2.12) and integrating between x and a we obtain:

$$R' = \frac{c(1-R) - W}{\varepsilon\gamma(1-\lambda)} \tag{2.27}$$

$$P' = c(1-\lambda)(R-P) \tag{2.28}$$

$$Y' = \frac{W - cY}{\varepsilon}. \tag{2.29}$$

Integrating equations (2.27)-(2.29) between $-a$ and x we get

$$R(x) = e^{-\frac{c(a+x)}{\varepsilon\gamma(1-\lambda)}} + \int_{-a}^{x} \frac{c - W(\xi)}{\varepsilon\gamma(1-\lambda)} e^{\frac{c(\xi-x)}{\varepsilon\gamma(1-\lambda)}} d\xi \tag{2.30}$$

$$Y(x) = \int_{-a}^{x} \frac{W(\xi)}{\varepsilon} e^{\frac{c(\xi-x)}{\varepsilon}} d\xi \tag{2.31}$$

$$P(x) = e^{-c(1-\lambda)(a+x)} + \int_{-a}^{x} c(1-\lambda)R(\xi)e^{c(1-\lambda)(\xi-x)} d\xi. \tag{2.32}$$

Since $0 \leq W \leq c$ on $[-a, a]$, we conclude from (2.30), (2.31) that $0 \leq R(x) \leq 1$ and $0 \leq Y(x) \leq 1$. This estimate on R and (2.32) immediately imply that $0 \leq P(x) \leq 1$ and, as a consequence, $0 \leq T(x) \leq 1$. This proves (2.18).

D. It remains only to prove (2.19). The function R satisfies

$$cR' + \varepsilon\gamma(1-\lambda)R'' = -\gamma Y\Omega(T) \leq 0.$$

Therefore, we have $(R'e^{cx/\varepsilon\gamma(1-\lambda)})' \leq 0$. Integrating this expression between $-a$ and x we obtain

$$R'(x)e^{cx/\varepsilon\gamma(1-\lambda)} \leq R'(-a)e^{-ca/\varepsilon\gamma(1-\lambda)}. \tag{2.33}$$

Combining (2.10) and (2.12) and integrating between $-a$ and a we get

$$\gamma(1-\lambda)R'(-a) + Y'(-a) = 0.$$

Since $Y'(-a) \geq 0$, it follows that $R'(-a) \leq 0$ and so $R'(x) \leq 0$, as follows from (2.33). In a similar way, Y satisfies

$$cY' + \varepsilon Y'' = \gamma Y\Omega(T) \geq 0,$$

hence $(Y'e^{cx/\varepsilon})' \geq 0$. Integrating this expression between $-a$ and x leads to

$$Y'(x)e^{cx/\varepsilon} \geq Y'(-a)e^{-ca/\varepsilon}.$$

This observation together with the fact that $Y'(-a) \geq 0$ allows us to conclude that $Y'(x) \geq 0$. Next, differentiating (2.11) we see that

$$P'' + c(1-\lambda)P' = c(1-\lambda)R' \leq 0,$$

hence $(P'e^{c(1-\lambda)x})' \leq 0$. Integrating between $-a$ and x we obtain

$$P'(x)e^{c(1-\lambda)x} \leq P'(-a)e^{-c(1-\lambda)a}.$$

However, we also have $P'(-a) = 0$ due to (2.11) and boundary conditions (2.14). It follows that $P'(x) \leq 0$. Finally, the previous estimates together with (2.1) imply that $T'(x) \leq 0$. Furthermore, substituting the bounds on W, P, Y, R into (2.27)-(2.29), we immediately obtain the bounds on $|R'|$, $|P'|$ and $|Y'|$ in (2.19). We also observe that (2.20) follows immediately from (2.11) and the fact that $P' \leq 0$. $\square$

Remark 2.3 Since T is a monotone function on $[-a, a]$ and $T(0) = \theta$, the nonlinear term $Y\Omega(T)$ in (2.10)-(2.12) is equal identically to zero for all $x > 0$. Thus, (2.10)-(2.12) for $x > 0$ is a linear system of ODE's which can be solved analytically:

$$R(x) = R(0)e^{-\frac{cx}{\varepsilon\gamma(1-\lambda)}} \tag{2.34}$$

$$Y(x) = 1 - (1 - Y(0))e^{-\frac{cx}{\varepsilon}} \tag{2.35}$$

$$P(x) = P(0)e^{-c(1-\lambda)x} + \frac{\varepsilon\gamma(1-\lambda)^2 R(0)}{1 - \varepsilon\gamma(1-\lambda)^2}(e^{-c(1-\lambda)x} - e^{-\frac{cx}{\varepsilon\gamma(1-\lambda)}}). \tag{2.36}$$

Here $0 < R(0) \leq \theta$ – as follows from (2.9) and (2.20), $0 < Y(0) < 1$ and $0 < P(0) \leq \theta/\lambda$.

Proposition 2.4 *Any solution (c, T, R, P, Y) of the problem (2.9)-(2.12) on $[-a, a]$ with the boundary conditions (2.14), (2.15) satisfies the following bounds:*

$$|Y(x) + R(x) - 1| \leq (1 - \gamma(1-\lambda))(1 - R(x)) \tag{2.37}$$

$$|Y(x) + R(x) - 1| \leq \frac{1 - \gamma(1-\lambda)}{\gamma(1-\lambda)} Y(x) \tag{2.38}$$

$$|Y(x) + \gamma(1-\lambda)(R(x) - 1)| \leq (1 - \gamma(1-\lambda))(1 - R(x)) \tag{2.39}$$

$$|Y(x) + \gamma(1-\lambda)(R(x) - 1)| \leq (1 - \gamma(1-\lambda))Y(x). \tag{2.40}$$

In particular, we have

$$\gamma(1-\lambda)(1 - R(x)) \leq Y(x) \leq (1 - R(x)). \tag{2.41}$$

Proof. First we add (2.10) and (2.12) and integrate between x and a taking into account the boundary conditions (2.15). We obtain

$$\gamma(1-\lambda)R' + Y' + \frac{c}{\varepsilon}(R + Y - 1) = 0. \tag{2.42}$$

We introduce then new variable $z = R + Y - 1$. Due to the boundary conditions (2.14) we have $z(-a) = 0$. Equation (2.42) in terms of z can be re-written as follows:

$$z' + \frac{c}{\varepsilon}z = (1 - \gamma(1-\lambda))R'. \tag{2.43}$$

Integrating (2.43) we get

$$z(x)e^{cx/\varepsilon} = (1 - \gamma(1-\lambda)) \int_{-a}^{x} R'(\xi)e^{c\xi/\varepsilon}d\xi.$$

Since $R' \leq 0$ on $[-a, a]$ we have

$$|z(x)|e^{cx/\varepsilon} \leq (1 - \gamma(1-\lambda))e^{cx/\varepsilon} \int_{-a}^{x} (-R'(\xi))d\xi$$

which immediately implies (2.37).

In order to prove (2.38) we observe that z also satisfies

$$\gamma(1-\lambda)z' + \frac{c}{\epsilon}z = (\gamma(1-\lambda) - 1)Y'.$$

Integrating this equation we have

$$z(x)e^{cx/\varepsilon\gamma(1-\lambda)} = \frac{(\gamma(1-\lambda) - 1)}{\gamma(1-\lambda)} \int_{-a}^{x} Y'(\xi))e^{c\xi/\varepsilon\gamma(1-\lambda)}d\xi.$$

Using the fact that $Y' \geq 0$ on $[-a, a]$ we then obtain

$$|z(x)|e^{cx/\varepsilon\gamma(1-\lambda)} \leq \frac{(1-\gamma(1-\lambda))}{\gamma(1-\lambda)} e^{cx/\varepsilon\gamma(1-\lambda)} \int_{-a}^{x} Y'(\xi) d\xi$$

which implies (2.38).

In order to prove (2.39),(2.40) we observe that the variable $y = Y + \gamma(1-\lambda)(R-1)$ satisfies

$$y' + \frac{c}{\varepsilon} y = \frac{c}{\varepsilon}(1-\gamma(1-\lambda))(1-R) \tag{2.44}$$

$$\gamma(1-\lambda)y' + \frac{c}{\varepsilon} y = \frac{c}{\varepsilon}(1-\gamma(1-\lambda))Y \tag{2.45}$$

with $y(-a) = 0$. Integrating (2.44) between $-a$ and x we see that

$$y(x)e^{cx/\varepsilon} = \frac{c}{\varepsilon}(1-\gamma(1-\lambda)) \int_{-a}^{x} (1-R(\xi))e^{c\xi/\varepsilon} d\xi.$$

Since R is a non-increasing function we conclude that

$$|y(x)| \leq (1-\gamma(1-\lambda))(1-R(x))$$

which proves (2.39). Similarly, integrating (2.45) between $-a$ and x, and using the fact that Y is a nondecreasing function we obtain (2.40). Finally, we note that (2.41) follows from (2.37)-(2.40). □

Proposition 2.5 *Let* $G = \int_{\theta}^{1} (1-s)\Omega(s) ds < \infty$*, then for any solution* (c, R, Y, P) *of the problem (2.9)-(2.15) with* $c \geq 0$*, the speed* c *obeys a lower bound*

$$c \geq (2\varepsilon\gamma G)^{1/2}\gamma(1-\lambda) \tag{2.46}$$

Proof. First, let us prove the following estimate

$$\int_{-a}^{0} (R')^2 \leq \frac{c(1-R^2(0))}{2\varepsilon\gamma(1-\lambda)} \tag{2.47}$$

We have already proved that T is a monotone function, thus (2.9) implies that $T(x) \leq \theta$ for $x \in [0, a]$ and therefore $Y(x)\Omega(T(x)) = 0$ on $[0, a]$. It follows that

$$R'(0) = -\frac{c}{\varepsilon\gamma(1-\lambda)} R(0). \tag{2.48}$$

Next, we multiply (2.10) by R and integrate between $-a$ and 0. Taking into account the boundary condition (2.14) and (2.48) we get

$$\frac{c}{2}(1+R^2(0)) + \varepsilon\gamma(1-\lambda)R'(-a) + \varepsilon\gamma(1-\lambda) \int_{-a}^{0} (R')^2 = \gamma \int_{-a}^{0} Y\Omega(T)R. \tag{2.49}$$

The right side of (2.49) is bounded from above by $\gamma \int_{-a}^{0} Y\Omega(T)$, as $0 \leq R \leq 1$ and $Y\Omega(T) \geq 0$. The integration of (2.10) between $-a$ and 0 gives

$$\gamma \int_{-a}^{0} Y\Omega(T) = c + \varepsilon\gamma(1-\lambda)R'(-a).$$

This, together with (2.49) proves (2.47).

Next, we multiply (2.10) by R' and integrate it between $-a$ and 0. This leads to

$$c\int_{-a}^{0}(R')^2+\frac{\varepsilon\gamma(1-\lambda)}{2}(R'^2(0)-R'^2(-a))=-\gamma\int_{-a}^{0}Y\Omega(T)R'. \tag{2.50}$$

Since R is a monotone function, we make a change of variables and consider Y and T as a function of R. Moreover, we have $P\geq R$ (see (2.11) and $P'<0$ (see (2.19)). As a result, $T\geq R$, and in particular $T(0)=\theta\geq R(0)$. In addition we have $Y(x)\geq\gamma(1-\lambda)(1-R)$ (see (2.41)). This allows us to write

$$-\int_{-a}^{0}Y\Omega(T)R'dx=\int_{R(0)}^{1}Y\Omega(T)dR\geq\gamma(1-\lambda)\int_{\theta}^{1}(1-R)\Omega(R)dR=\gamma(1-\lambda)G. \tag{2.51}$$

Combining (2.48), (2.50) and (2.51) we then have

$$c\int_{-a}^{0}(R')^2+\frac{c^2R^2(0)}{2\varepsilon\gamma(1-\lambda)}\geq\gamma^2(1-\lambda)G.$$

This expression together with the estimate (2.47) implies (2.46). □

Proposition 2.6 *There exist $a_0>0$ and $D_0(\varepsilon)$ so that given $a>a_0$, every solution (c,T,R,P,Y) of the problem (2.9)-(2.15) with $c\geq 0$ obeys an upper bound on the speed c:*

$$c\leq D_0(\varepsilon). \tag{2.52}$$

Proof. The proof is based on the comparison principle [4] and a construction of a super-solution for (2.4). First, using the fact that T is a linear combination of P and R we rewrite (2.4) as

$$-c\kappa T'-\varepsilon T''=Y\Omega(T)+c(1-\gamma^{-1})(\lambda^{-1}-1)R'.$$

It follows from Proposition 2.2 that $R'(x)\leq 0$ for $x\in[-a,a]$ and $0\leq Y\leq 1$. Therefore, we have a differential inequality

$$-c\kappa T'-\varepsilon T''\leq MT,$$

where $M=\sup_{0\leq s\leq 1}\Omega(s)/s$. Moreover, at the left end we have $T(-a)=1$. In order to estimate $T(a)$ we use (2.20), (2.34) and (2.36) to obtain:

$$\begin{aligned}T(x)&\leq P(x)\leq P(0)e^{-c(1-\lambda)x}+\varepsilon\gamma(1-\lambda)^2R(0)e^{-c(1-\lambda)x}\leq P(0)\left(1+\varepsilon\gamma(1-\lambda)^2\right)e^{-c(1-\lambda)x}\\&\leq\frac{\theta}{\lambda}\left(1+\frac{\varepsilon}{\gamma}\right)e^{-c(1-\lambda)x}\end{aligned}$$

for $x\geq 0$. In particular, we have

$$T(a)\leq Be^{-ca(1-\lambda)},\quad B=\frac{\theta}{\lambda}\left(1+\frac{\varepsilon}{\gamma}\right).$$

Consider now a function $\bar{T}_A=A\exp(-\alpha(x+a))$ with $\alpha>0$ chosen so that

$$c\geq\frac{M}{\kappa\alpha}+\frac{\varepsilon\alpha}{\kappa},\quad c\geq\frac{2\alpha}{1-\lambda}. \tag{2.53}$$

Then a direct calculation using the first condition in (2.53) shows that given any $A \geq 0$, the function $\bar{T}_A$ satisfies a differential inequality

$$-c\kappa \bar{T}_A' - \varepsilon \bar{T}_A'' \geq M\bar{T}_A.$$

We claim that

$$T(x) \leq \bar{T}_A(x) \text{ for all } A \geq 1. \tag{2.54}$$

Indeed, this is clearly true if A is so large that $T_A(x) \geq 2$ for all $x \in [-a, a]$. Let us now assume that there exists $A_1 > 1$ so that $\bar{T}_{A_1}(x_0) \leq T(x_0)$ for some $x_0 \in [-a, a]$ and define

$$A_0 = \sup\{A : \text{there exists } x \in [-a, a] \text{ so that } \bar{T}_A(x) \leq T(x).\}$$

Then there exists $x_0 \in [-a, a]$ so that $\bar{T}_{A_0}(x_0) = T(x_0)$ and, moreover, $\bar{T}_{A_0}(x) \geq T(x)$ for all $x \in [-a, a]$. However, the function $\phi = \bar{T}_{A_0}(x) - T(x)$ satisfies

$$-c\kappa\phi' - \phi'' \geq M\phi.$$

Hence, the maximum principle implies that it can not attain its minimum equal to zero inside the open interval $(-a, a)$. However, we have at the end points:

$$\bar{T}_{A_0}(-a) = A_0 > A_1 > 1 = T(-a)$$

and

$$\bar{T}_{A_0}(a) = A_0 e^{-2\alpha a} \geq e^{-2\alpha a} \geq B e^{-c(1-\lambda)a}$$

if a is sufficiently large and α satisfies the second inequality in (2.53). Therefore, the function ϕ may not be zero at the endpoints of the interval either. This contradiction shows that $A_0 = 1$ and (2.54) holds.

On the other hand, (2.54) implies that for $\alpha > \log(\theta^{-1})/a$ we have $T(0) \leq \bar{T}_1(0) < \theta$. This, however, contradicts the normalization condition (2.9). Thus, no α satisfying (2.53) may exist, and therefore c is uniformly bounded from above as in (2.52) with a constant D_0 that may depend on ε but not on a. □

Proposition 2.7 *There exists a constant $a_0 > 0$ so that for any $a > a_0$ there exists a solution (c, T, R, P, Y) of (2.9)-(2.15) on $[-a, a]$.*

Proof. Given the a priori bounds in Propositions 2.2 and 2.5 and 2.6, the proof is standard [1]. Consider the space $\mathcal{C} = [C^{1,\alpha}([-a, a])]^3 \times \mathbb{R}$. For each $\tau \in [0, 1]$ we define a map $\mathcal{K}_\tau : \mathcal{C} \to \mathcal{C}$, $\mathcal{K}_\tau(\bar{R}, \bar{Y}, \bar{P}, c) = (R, Y, P, \theta^\tau)$ as follows. Let $(\bar{R}, \bar{Y}, \bar{P}) \in \mathcal{C}$ and let $c \in \mathbb{R}$. Then the functions (R, Y, P) are the solutions of the linear forced system

$$cR' + \varepsilon\gamma(1-\lambda)R'' = -\tau\gamma\bar{Y}\Omega(\bar{T}), \quad \bar{T} = \lambda\bar{P} + (1-\lambda)\bar{R} \tag{2.55}$$
$$P' + c(1-\lambda)P = c\tau(1-\lambda)\bar{R} \tag{2.56}$$
$$cY' + \varepsilon Y'' = \tau\gamma\bar{Y}\Omega(\bar{T}) \tag{2.57}$$

with the boundary conditions

$$R(-a) = 1, \quad P(-a) = 1, \quad Y(-a) = 0 \tag{2.58}$$
$$cR(a) + \varepsilon\gamma(1-\lambda)R'(a) = 0, \quad cY(a) + \varepsilon Y'(a) = c. \tag{2.59}$$

The number θ^τ is then defined by

$$\theta^\tau = \theta - \max_{x\geq 0} T(x) + c,$$

where $T = \lambda P + (1-\lambda)R$ and θ is the ignition temperature. The operator $\mathcal{K}_\tau$ is a mapping of the Banach space $\mathcal{C}$, equipped with the norm

$$\|(R,Y,P,c)\|_{\mathcal{C}} = \max(\|R\|_{C^{1,\alpha}([-a,a])}, \|Y\|_{C^{1,\alpha}([-a,a])}), \|P\|_{C^{1,\alpha}([-a,a])}, |c|)$$

onto itself. A solution $\mathbf{q} = (R,Y,P,c)$ of (2.9)-(2.15) is a fixed point of $\mathcal{K}_1$ and satisfies $\mathcal{K}_1\mathbf{q} = \mathbf{q}$, and vice versa: a fixed point of $\mathcal{K}_1$ provides a solution to (2.9)-(2.15). Hence, in order to show that (2.10)-(2.12) has a traveling front solution it suffices to show that the kernel of the operator $\mathcal{F}_1 = \mathrm{Id} - \mathcal{K}_1$ is not trivial. The standard elliptic regularity results, as well as the explicit formulas for the solutions of (2.55)-(2.59) imply that the operators $\mathcal{K}_\tau$ are compact and depend continuously on the parameter $\tau \in [0,1]$. Thus the Leray-Schauder topological degree theory can be applied. Let us introduce the set $B_M = \{\|(R,Y,P,c)\|_{\mathcal{C}} \leq M\} \cap \{c > M^{-1}\}$. Then Propositions 2.2, 2.5 and 2.6 show that the operator $\mathcal{F}_\tau$ does not vanish on the boundary ∂B_M with M sufficiently large for any $\tau \in [0,1]$. It remains only to show that the degree $\deg(\mathcal{F}_1, B_M, 0)$ in $\bar{B}_M$ is not zero. However, the homotopy invariance property of the degree implies that $\deg(\mathcal{F}_\tau, B_M, 0) = \deg(\mathcal{F}_0, B_M, 0)$ for all $\tau \in [0,1]$. Moreover, the degree at $\tau = 0$ can be computed explicitly as the operator $\mathcal{F}_0$ is given by

$$\mathcal{F}_0(R,P,Y,c) = (R - R_0^c, P - P_0^c, Y - Y_0^c, \max_{x\geq 0}[\lambda P + (1-\lambda)R - \theta_0]).$$

Here the functions $R_0^c(x)$, $P_0^c(x)$ and $Y_0^c(x)$ solve

$$\begin{aligned}
&c\frac{dR_0^c}{dx} + \varepsilon\gamma(1-\lambda)\frac{R_0^c}{dx^2} = 0, \quad R_0(-a) = 1, \quad cR_0^c(a) + \varepsilon\gamma(1-\lambda)\frac{dR_0^c}{dx}(a) = 0\\
&\frac{dP_0^c}{dx} + c(1-\lambda)P_0^c = 0, \quad P_0^c(-a) = 1\\
&c\frac{Y_0^c}{dx} + \varepsilon\frac{d^2Y_0^c}{dx^2} = 0, \quad Y_0^c(-a) = 0, \quad cY_0^c(a) + \varepsilon\frac{Y_0^c}{dx}(a) = c,
\end{aligned}$$

and are given by

$$R_0^c(x) = \frac{e^{-cx/(\varepsilon\gamma(1-\lambda))}}{e^{ca/(\varepsilon\gamma(1-\lambda))}}, \quad P_0^c = \frac{e^{-c(1-\lambda)x}}{e^{ca(1-\lambda)}}, \quad Y_0^c = 1 - \frac{e^{-cx/\varepsilon}}{e^{ca/\varepsilon}}.$$

The mapping $\mathcal{F}_0$ is homotopic to

$$\Phi(R,P,Y,c) = (R - R_0^c, P - P_0^c, Y - Y_0^c, \max_{x\geq 0}[\lambda P_0^c(x) + (1-\lambda)R_0^c(x)] - \theta)$$

that in turn is homotopic to

$$\tilde{\Phi}(R,P,Y,c) = (R - R_0^{c_*}, P - P_0^{c_*}, Y - Y_0^{c_*}, \lambda P_0^c(0) + (1-\lambda)R_0^c(0) - \theta),$$

where c_*^0 is the unique number so that

$$T_0^{c_*}(0) = \lambda P_0^{c_*}(0) + (1-\lambda)R_0^{c_*}(0) = \theta.$$

The degree of the mapping $\tilde{\Phi}$ is the product of the degrees of each component. The first three have degree equal to one, and the last to -1, as the function $T_0^c(0)$ is decreasing in c. Thus $\deg\mathcal{F}_0 = -1$ and hence $\deg\mathcal{F}_1 = -1$ so that the kernel of $\mathrm{Id} - \mathcal{K}_1$ is not empty. This finishes the proof of Proposition 2.7. □

The last step in the proof of Theorem 2.1 is the passage to the limit $a \to \infty$.

Proposition 2.8 *There exists an increasing subsequence* $\{a_n\}_{n\in N}$ *with* $a_n > a_0$, $\lim_{n\to\infty} a_n = \infty$ *such that solution* $(R_{a_n}, P_{a_n}, T_{a_n}, Y_{a_n}, c_{a_n})$ *of (2.10)- (2.15) converges in the topology of* $C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}) \times \mathbb{R}$ *to the solution of (2.10)-(2.13) on the whole real line with the boundary conditions (2.16). Moreover:*

$$0 \le R, P, T, Y \le 1 \quad on \quad \mathbb{R} \tag{2.60}$$

$$R, P, T, Y \in W^{2,\infty}(\mathbb{R}) \tag{2.61}$$

$$-c/\varepsilon\gamma(1-\lambda) < R' < 0, \quad -c(1-\lambda) < P' < 0, \quad T' = \lambda P' + (1-\lambda)R', \quad 0 < Y' < c/\varepsilon \tag{2.62}$$

$$0 < \underline{c} \le c \le \bar{c} < \infty. \tag{2.63}$$

Proof. Consider solutions $(R_a, P_a, T_a, Y_a, c_a)$ of (2.10)-(2.15). By Propositions 2.5 and 2.6 there exist two constants $0 < \underline{c} < \bar{c} < \infty$ independent of a such that $\underline{c} \le c_a \le \bar{c}$. Using Proposition 2.2 we have $0 \le R_a, P_a, T_a, Y_a \le 1$ and

$$-c/\varepsilon\gamma(1-\lambda) < R'_a < 0, \quad -c(1-\lambda) < P'_a < 0,$$

and

$$T'_a = \lambda P'_a + (1-\lambda)R'_a, \quad 0 < Y' < c/\varepsilon.$$

Moreover $0 \le \Omega(s) \le M$ for all $0 \le s \le 1$. We then deduce that

$$R''_a = (-c_a R'_a - \gamma Y_a \Omega(T_a))/\varepsilon\gamma(1-\lambda)$$

and

$$P''_a = c_a(1-\lambda)(R'_a - P'_a), \quad Y''_a = (-c_a Y'_a + \gamma Y_a \Omega(T_a))/\varepsilon$$

are bounded independently of a, hence so is T''_a which is a linear combination of R''_a and P''_a. Therefore, R_a, P_a, T_a, Y_a are bounded independently of a in $W^{2,\infty}(-a, a)$. As a consequence we obtain the convergence in the topology of $C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}) \times \mathbb{R}$ of a sub-sequence $(R_{a_n}, P_{a_n}, T_{a_n}, Y_{a_n}, c_{a_n})$ to a limit (R, P, T, Y, c). The latter satisfies the system

$$cR' + \varepsilon\gamma(1-\lambda)R'' = -\gamma Y \Omega(T) \tag{2.64}$$

$$P' = c(1-\lambda)(R-P) \tag{2.65}$$

$$cY' + \varepsilon Y'' = \gamma Y \Omega(T) \tag{2.66}$$

$$T(x) = \lambda P(x) + (1-\lambda)R(x) \tag{2.67}$$

on the whole real line. Properties (2.60)-(2.63) are clearly satisfied as well. Remark 2.3 implies that we have $R(+\infty) = P(+\infty) = T(+\infty) = 0$, $Y(+\infty) = 1$. Moreover, $T(0) = \theta$ since $T_a(0) = \theta$ for all $a > 0$. Monotonicity and boundedness of the functions T, R, P and Y imply that the limits $(T^-, R^-, P^-, Y^-) = \lim_{x\to-\infty}(T, R, P, Y)(x)$ exist. Moreover, $Y'(-\infty) = 0$ and similarly $Y''(-\infty) = 0$. The function Y also satisfies (2.66) and thus $Y^-\Omega(T^-)(-\infty) = 0$, but $T^- > \theta$ since T is strictly decreasing and therefore $\Omega(T^-)(-\infty) \neq 0$. Thus $Y^- = 0$. This fact together with inequality (2.41) of Proposition 2.2 implies $R^- = 1$. Finally, as $P'(-\infty) = 0$, we use (2.65) to conclude that $P^- = 1$, and as a consequence $T^- = 1$. □

3 The singular limit

In this section we show that solutions of the problem (1.2)-(1.6) that we have constructed in the previous section converge to the unique travelling front solution of the limiting problem (that is the problem (1.2)-(1.6) with $\varepsilon = 0$) as $\varepsilon \to 0$: see Theorem 3.2.

As in the previous section we will re-write the system (2.10)-(2.15) in an equivalent form

$$\omega = \frac{\varepsilon}{c} Y' + Y \tag{3.1}$$

$$\omega' = c^{-1}\gamma Y \Omega(T), \quad T = \lambda P + (1-\lambda) R, \tag{3.2}$$

$$R' = c\frac{1 - R - \omega}{\varepsilon\gamma(1-\lambda)} \tag{3.3}$$

$$P' = c(1-\lambda)(R - P) \tag{3.4}$$

$$Y' = c\frac{\omega - Y}{\varepsilon}. \tag{3.5}$$

The boundary conditions are:

$$\begin{aligned} R(+\infty) = P(+\infty) = T(+\infty) = 0, \quad Y(+\infty) = \omega(+\infty) = 1, \\ R(-\infty) = P(-\infty) = T(-\infty) = 1, \quad Y(-\infty) = \omega(-\infty) = 0. \end{aligned} \tag{3.6}$$

In the sequel we will work with the system (3.1)-(3.6).

As we have mentioned, our goal is to show that for small ε solutions of the system converges to the solutions of the limiting problem

$$\omega_0' = c_0^{-1}\gamma\omega_0\Omega(T_0) \tag{3.7}$$

$$P_0' = c_0\gamma^{-1}(1 - \omega_0 - P_0) \tag{3.8}$$

$$Y_0 = \omega_0, \quad R_0 = 1 - \omega_0, \quad T_0 = (1-\gamma^{-1})P_0 + \gamma^{-1}(1-\omega_0) \tag{3.9}$$

with the boundary conditions

$$P_0(+\infty) = 0, \quad \omega_0(+\infty) = 1, \qquad P_0(-\infty) = 1, \quad \omega_0(-\infty) = 0 \tag{3.10}$$

This problem is obtained from (3.1)-(3.6) by setting $\varepsilon = 0$. The system (3.7)-(3.10) is well understood. In particular, the following result has been established in [6].

Theorem 3.1 *[6] A travelling front solution (c_0, ω_0, P_0) of (3.7)-(3.10) exists if and only if*

$$\theta < 1 - \gamma^{-1}. \tag{3.11}$$

Moreover, in that case the travelling front solution is unique and satisfies

$$\frac{d\omega_0}{dP_0} = \frac{\gamma^2\omega_0\Omega((1-\gamma^{-1})P_0 + \gamma^{-1}(1-\omega_0))}{c_0^2(1-\omega_0-P_0)}, \quad \omega_0(1) = 0, \quad \omega_0\left(\frac{\theta}{1-\gamma^{-1}}\right) = 1 \text{ for } x < 0, \tag{3.12}$$

$$\omega_0(x) = 1, \quad P_0(x) = \frac{\theta}{1-\gamma^{-1}} e^{-c_0\gamma^{-1}x} \text{ for } x > 0, \tag{3.13}$$

and

$$T_0(x) = (1-\gamma^{-1})P_0(x) + \gamma^{-1}(1-\omega_0(x)), \quad R_0(x) = 1 - \omega_0(x), \quad Y_0(x) = \omega_0(x). \tag{3.14}$$

for all $x \in \mathbb{R}$.

In accordance to this theorem we will assume below that (3.11) holds. Note that this implies, in particular, that

$$\theta < \lambda_\varepsilon \tag{3.15}$$

for a sufficiently small ε, as follows from (2.2). We have the following result.

Theorem 3.2 *Solutions of the problem (3.1)- (3.6) converge as $\varepsilon \to 0$ uniformly to the unique travelling front solution of the limiting problem (3.12)-(3.14).*

Proof. Most of the estimates on the travelling front solutions for $\varepsilon > 0$, that we have obtained in the previous section, were sufficient to establish the existence of a travelling front for $\varepsilon > 0$ but diverge as $\varepsilon \to 0$. On the other hand, as a first step in the passage to the limit $\varepsilon \to 0$ we need uniform estimates on the travelling front speed c_ε. Therefore in order to investigate the limit $\varepsilon \to 0$ we need to obtain better estimates for c_ε. The following two propositions show that c_ε is bounded from above and below independent of $\varepsilon \in (0, 1)$.

Proposition 3.3 *Assume that ε is sufficiently small, so that (3.15) holds, then*

$$c \geq \frac{\gamma}{\sqrt{\lambda}}\sqrt{\int_\theta^\lambda \Omega(s)ds} \tag{3.16}$$

Proof. Consider (3.2) and (3.4) with the boundary conditions (3.6). Since all functions T, P, R and ω are monotonic we can map the system (3.2), (3.4) onto the phase plane

$$\frac{d\omega}{dP} = -\frac{\gamma Y \Omega(T)}{c^2(1-\lambda)(P-R)}, \quad \omega(1) = 0, \quad \omega(P_0) = \omega_0, \tag{3.17}$$

where $\omega = \omega(P)$ and $P_0 = P(0)$, $\omega_0 = \omega(0)$. Therefore, we have

$$c^2 = \frac{\gamma}{\omega_0(1-\lambda)}\int_{P_0}^1 \frac{Y\Omega(T)}{(P-R)}dP. \tag{3.18}$$

As we know from Proposition 2.2 $P > R$, and $P < 1$. It follows that $P - R < 1 - R$. We also have $T \geq \lambda P$, and, furthermore, $Y \geq \gamma(1-\lambda)(1-R)$ – see (2.41). Moreover, we have $\omega_0 < 1$ and $P_0 \leq \theta/\lambda$. Thus, we get a lower bound for c:

$$c^2 \geq \gamma^2 \int_{\theta/\lambda}^1 \Omega(\lambda P)dP = \frac{\gamma^2}{\lambda}\int_\theta^\lambda \Omega(s)ds. \tag{3.19}$$

This proves (3.16). □

Proposition 3.4 *The speed c obeys the following upper bound:*

$$c \leq \sqrt{\frac{2\gamma(1+\varepsilon\gamma(1-\lambda)^2)(1+\varepsilon)}{(1+\varepsilon)(1-\lambda)\theta^2}\int_\theta^1 \Omega(s)ds}. \tag{3.20}$$

Proof. It is convenient now to use directly the system (1.2)-(1.4). First, we multiply (1.3) by P' and integrate:

$$\int_{-\infty}^\infty T'P' = \int_{-\infty}^\infty P'^2. \tag{3.21}$$

Next, we multiply (1.2) by P' and integrate between $-\infty$ and $+\infty$:

$$-c\int_{-\infty}^\infty T'P' + c(1-\gamma^{-1})\int_{-\infty}^\infty P'^2 = \varepsilon\int_{-\infty}^\infty T''P' + \int_{-\infty}^\infty Y\Omega(T)P'. \tag{3.22}$$

Using (3.21) and the fact that $P' = c(T - P)$, we have

$$-c\gamma^{-1}\int_{-\infty}^{\infty} P'^2 = c\varepsilon\int_{-\infty}^{\infty} T''(T-P) + \int_{-\infty}^{\infty} Y\Omega(T)P' = -c\varepsilon\int_{-\infty}^{\infty} T'^2 + c\varepsilon\int_{-\infty}^{\infty} T'P' + \int_{-\infty}^{\infty} Y\Omega(T)P'.$$

Since P is a monotonic function we also have

$$\int_{-\infty}^{\infty} Y\Omega(T)P' = -\int_{P_0}^{1} Y\Omega(T)dP$$

so that, using (3.21) again, we get

$$-c\gamma^{-1}\int_{-\infty}^{\infty} P'^2 = -c\varepsilon\int_{-\infty}^{\infty} T'^2 + c\varepsilon\int_{-\infty}^{\infty} P'^2 - \int_{0}^{1} Y\Omega(T)dP \tag{3.23}$$

and, finally,

$$c(\gamma^{-1}+\varepsilon)\int_{-\infty}^{\infty} P'^2 = \int_{0}^{1} Y\Omega(T)dP + c\varepsilon\int_{-\infty}^{\infty} T'^2. \tag{3.24}$$

Similarly, we multiply (1.2) by T' and integrate to obtain

$$-c\int_{-\infty}^{\infty} T'^2 + c(1-\gamma^{-1})\int_{-\infty}^{\infty} P'T' = \frac{\varepsilon}{2}\int_{-\infty}^{\infty} (T'^2)' + \int_{-\infty}^{\infty} Y\Omega(T)T'. \tag{3.25}$$

Again, we use (3.21) and the fact that T is monotonic:

$$c\int_{-\infty}^{\infty} T'^2 = c(1-\gamma^{-1})\int_{-\infty}^{\infty} P'^2 + \int_{0}^{1} Y\Omega(T)dT. \tag{3.26}$$

Combining (3.24) and (3.26) we obtain

$$c\gamma^{-1}(1+\varepsilon)\int_{-\infty}^{\infty} P'^2 = \int_{0}^{1} Y\Omega(T)dP + \varepsilon\int_{0}^{1} Y\Omega(T)dT \le (1+\varepsilon)\int_{0}^{1} \Omega(s)ds, \tag{3.27}$$

as (2.20) implies that $T \le P$ and hence $\Omega(T) \le \Omega(P)$. Now, we multiply the equation

$$P'' = c(1-\lambda)(R' - P')$$

by P' and integrate to get

$$\int_{-\infty}^{\infty} P'^2 = \int_{-\infty}^{\infty} P'R'. \tag{3.28}$$

Then, since $P' = c(1-\lambda)(R-P)$, we have

$$\int_{-\infty}^{\infty} P'^2 \ge \int_{0}^{\infty} P'^2 = c(1-\lambda)\int_{0}^{\infty} (RP' - PP') = \frac{c(1-\lambda)}{2}P^2(0) + c(1-\lambda)\int_{0}^{\infty} RP'. \tag{3.29}$$

On the other hand, R is monotonic and the nonlinearity $Y\Omega(T) = 0$ for all $x > 0$. Thus, for $x > 0$ we have, using (2.10), $cR = -\varepsilon\gamma(1-\lambda)R'$. Moreover, $P(0) \ge \theta$ so that

$$\int_{-\infty}^{\infty} P'^2 \ge \frac{c(1-\lambda)}{2}\theta^2 - \varepsilon\gamma(1-\lambda)^2\int_{0}^{\infty} R'P'. \tag{3.30}$$

However, we have

$$\int_{0}^{\infty} R'P' \le \int_{-\infty}^{\infty} R'P' = \int_{-\infty}^{\infty} P'^2.$$

Thus, we obtain

$$(1+\varepsilon\gamma(1-\lambda)^2)\int_{-\infty}^{\infty}P'^2 \geq \frac{c(1-\lambda)}{2}\theta^2 \tag{3.31}$$

and therefore

$$\int_{-\infty}^{\infty}P'^2 \geq \frac{c(1-\lambda)}{2(1+\varepsilon\gamma(1-\lambda)^2)}\theta^2. \tag{3.32}$$

Now, we combine (3.27) and (3.32):

$$c^2 \leq \frac{2\gamma(1+\varepsilon\gamma(1-\lambda)^2)}{(1-\lambda)\theta^2}\int_{\theta}^{1}\Omega(T)dT. \tag{3.33}$$

This proves (3.20). □

Proposition 3.5 *If* (c, ω, R, P, Y) *is a solution of (3.2)-(3.6) then*

$$||1-R-\omega||_{L^p} \leq C\varepsilon, \quad ||\omega - Y||_{L^p} \leq C\varepsilon, \quad \textit{for all} \quad 1 \leq p \leq \infty. \tag{3.34}$$

Proof. Let us consider (3.3) and (3.5). First, due to Proposition 2.2 we have $Y' \geq 0$ and $R' \leq 0$ for all x. Therefore, we have $\omega \geq Y$ and

$$\int_{-\infty}^{\infty}\omega - Y = \int_{-\infty}^{\infty}|\omega - Y| = \varepsilon/c \tag{3.35}$$

and similarly

$$\int_{-\infty}^{\infty}|1-R-\omega| = \varepsilon\gamma(1-\lambda)/c. \tag{3.36}$$

As we have already proved that c is bounded above and below by two positive constants that are independent of ε, it follows that

$$||\omega - Y||_{L^1} \leq C\varepsilon, \quad ||1-R-\omega||_{L^1} \leq C\varepsilon. \tag{3.37}$$

Next, we note that (3.5) can be rewritten as:

$$Y(x) = \frac{c}{\varepsilon}\int_{-\infty}^{x}\omega(\xi)e^{c(\xi-x)/\varepsilon}d\xi. \tag{3.38}$$

Therefore, we have

$$|Y(x)-\omega(x)| \leq \frac{c}{\varepsilon}\int_{-\infty}^{x}|\omega(\xi)-\omega(x)|e^{c(\xi-x)/\varepsilon}d\xi \leq \|\omega'\|_{L^\infty}\frac{c}{\varepsilon}\int_{-\infty}^{x}(x-\xi)e^{c(\xi-x)/\varepsilon}d\xi \leq \frac{B\varepsilon\|\omega'\|_{L^\infty}}{c}.$$

Observing that c and ω' are bounded for all ε we conclude that

$$||Y(x)-\omega(x)||_{L^\infty} \leq C\varepsilon. \tag{3.39}$$

Similar manipulations with (3.3) imply that

$$||1-R-\omega||_{L^\infty} \leq C\varepsilon. \tag{3.40}$$

Inequalities (3.37), (3.39) and (3.40) imply (3.34). □

Propositions 3.3, 3.4 and 3.5 together with (3.2)-(3.6) and Proposition 2.2 imply that the functions P_ε, ω_ε, Y_ε, T_ε and R_ε are all uniformly bounded together with the first derivatives, independent of $\varepsilon > 0$. It also follows from (3.2)-(3.6) that the same estimates hold for the second derivatives ω''_ε and P''_ε, except possibly at the point $x = 0$ where $\Omega(T)$ may have a jump if the function Ω is discontinuous at $T = \theta$. Therefore, the functions ω_ε, T_ε, R_ε, P_ε and Y_ε converge point-wise, along a subsequence $\varepsilon_k \to 0$, to the respective limits $\bar{\omega}$, $\bar{T}$, $\bar{R}$, $\bar{P}$ and $\bar{Y}$. Moreover, the derivatives of ω_ε and P_ε also converge to the corresponding limits: $\omega'_\varepsilon \to \bar{\omega}'$ and $P'_\varepsilon \to \bar{P}'$, and the limits satisfy the algebraic relations:

$$\bar{R} + \bar{\omega} = 1, \quad \bar{Y} = \bar{\omega}, \quad \bar{T} = \lambda_0 \bar{P} + (1 - \lambda_0)(1 - \bar{\omega}) \tag{3.41}$$

with $\lambda_0 = 1 - \gamma^{-1}$. After passing once again to a subsequence, the speed c_{ε_k} converges to a limit $\bar{c}$. The above arguments imply that the limits satisfy the system

$$\bar{c}\bar{\omega}' = \gamma\bar{\omega}\Omega((1 - \gamma^{-1})\bar{P} + \gamma^{-1}(1 - \bar{\omega})) \tag{3.42}$$

$$\bar{P}' = \bar{c}\gamma^{-1}(1 - \bar{\omega} - \bar{P}) \tag{3.43}$$

for $x < 0$ and

$$\bar{\omega}' = 0$$
$$\bar{P}' = \bar{c}\gamma^{-1}(1 - \bar{\omega} - \bar{P})$$

for $x > 0$. Moreover, it follows from Remark 2.3 that $\bar{R} = 0$ and $\bar{Y} = 1$ for $x > 0$. Therefore, (3.41) implies that $\bar{\omega} = \bar{Y} = 1$ for $x > 0$. The continuity of $\bar{\omega}$ at $x = 0$ implies that $\bar{\omega}(0) = 1$, hence $\bar{R}(0) = 0$ and

$$\bar{P}(0) = \frac{\bar{T}(0)}{\lambda_0} = \frac{\theta}{1 - \gamma^{-1}}.$$

The monotonicity of P_ε and Y_ε imply that the limits $\bar{P}$ and $\bar{Y}$ are also monotonic and hence so is $\bar{\omega} = \bar{Y}$. Therefore, as $0 \le \bar{\omega}, \bar{P} \le 1$, the limits

$$P_- = \lim_{x \to -\infty} \bar{P}(x), \quad \omega_- = \lim_{x \to -\infty} \bar{\omega}(x)$$

exist. It follows from (3.42) and the fact that $\Omega(\bar{T}) > 0$ for $x < 0$ that $\omega_- = 0$. Then (3.43) implies that $P_- = 1$. This shows that $(\bar{c}, \bar{P}, \bar{\omega})$ satisfy the limiting problem (3.12)-(3.14) with the correct boundary conditions. As such travelling front is unique, the conclusion of Theorem 3.2 follows. □

Remark 3.6 It has been pointed out to us by J.-M. Roquejoffre that Theorem 3.2 can apparently be proved using geometric singular perturbation theory similar to one in [5].

Remark 3.7 It is important to note that Theorem 4.1 does not provide any information about uniqueness of the solution even for small ε. There is still a possibility of non-uniqueness even in the neighborhood of $\varepsilon = 0$. It would be interesting to perform a bifurcation analysis around this point.

A Derivation of the model

In this section, following [2], we briefly sketch a derivation of the model (1.2)- (1.6). Consider a porous medium filled with combustible gas. Assume that the relaxation time of pressure is much larger than that of temperature. Then the system can be described in the framework of the single temperature model. We will also assume that the solid phase (skeleton) has a small specific heat and low volumetric fraction. Thus the effective features of the reactive gas-porous medium system are controlled exclusively by its gaseous phase subjected to the resistance of the porous media matrix. In this setting the system of governing equations reads as follows

$$\begin{aligned} c_p\rho(\Theta_\tau + u\Theta_\xi) - (\Pi_\tau + u\Pi_\xi) = qW + (c_p\rho D_{th}\Theta_\xi)_\xi \quad \text{Energy} \\ \rho(C_\tau + uC_\xi) = -W + (T^{-1}D_{mol}(\rho\Theta C)_\xi)_\xi \quad \text{Concentration} \\ W = Z\rho C\exp(-E/RT) \quad \text{Chemical kinetics} \\ \rho_\tau + (\rho u)_\xi = 0 \quad \text{Continuity} \\ \rho u = -K\nu^{-1}\Pi_\xi \quad \text{Momentum (Darcy's law)} \\ \rho = \Pi/(c_p - c_v)\Theta \quad \text{State} \end{aligned} \tag{A.1}$$

The system is written in the frame of reference attached to the skeleton. Here u is the gas velocity relative to the skeleton, C is the concentration of the deficient reactant, ρ, Π, and Θ are density, pressure and temperature of the gas-solid system, W is the chemical reaction rate, ν is the kinematic viscosity, D_{th},D_{mol} are thermal and molecular diffusivities, K is the permeability of the porous medium, Z is a frequency factor, E is the activation energy, R is the universal gas constant, and q is the heat release.

In order to simplify the system we will adopt the small heat release approximation [7] where the variations of pressure temperature density and gas velocity are regarded as small and hence the nonlinear effects are ignored everywhere but in the reaction term that is generally highly sensitive to temperature changes. Under this assumption the system (A.1) after some simple manipulations can be reduced to the following

$$\begin{aligned} c_p\rho_0\Theta_\tau - \Pi_\tau = c_p\rho_0 D_t h\Theta_{\xi\xi} + qW, \\ \frac{1}{\Theta_0}\Pi_\tau - \frac{\Pi_0}{\Theta_0^2}\Theta_\tau = \frac{K(c_p - c_v)}{\nu}\Pi_{\xi\xi}, \\ \rho_0 C_t = \rho_0 D_{mol}C_{\xi\xi} - W. \end{aligned} \tag{A.2}$$

The first and third equations are the partially linearized equations for conservation of energy and deficient reactant, the second equation is the linearized continuity equation taking into account equations of the state and momentum.

We set

$$T = \frac{\Theta - \Theta_0}{\Theta_\infty - \Theta_0}, \quad P = \frac{\Pi - \Pi_0}{\Pi_\infty - \Pi_0}, \quad Y = C/C_0 \tag{A.3}$$

where Θ_0, Π_0 and C_0 are temperature pressure and concentration at $\tau = 0$ while $\Theta_\infty = \Theta_0(1 + qC_o/c_vT_0)$, $\Pi_\infty = \Pi_0(1 + qC_0/c_vT_0)$ and $C_\infty = 0$ are temperature pressure and concentration at $\tau \to \infty$ in case of the homogeneous explosion. Introducing appropriate scaling of space and time coordinates $t = \tau/\bar{\tau}$, $x = \xi/\bar{\xi}$ with $\bar{\tau} = Z^{-1}\beta\exp(E/RT_\infty)$ and $\bar{\xi} = \sqrt{D_b\bar{\tau}}$ we have

$$\begin{aligned} T_t - (1 - \gamma^{-1})P_t = \varepsilon T_{xx} + Y\Omega(T), \\ P_t - T_t = P_{xx}, \\ Y_t = \varepsilon Le^{-1}Y_{xx} - Y\Omega(T) \\ \Omega(T) = \beta\gamma^{-1}\exp(\beta(T-1)/(\sigma + (1-\sigma)T)), \end{aligned} \tag{A.4}$$

where $D_b = KT_\infty c_v(\gamma - 1)/\nu$, $\beta = (1-\sigma)E/RT_\infty$, $Le = D_{th}/D_{mol}$ are (pressure) barodiffusivity, Zeldovich and Lewis numbers and $\sigma = T_0/T_\infty$, $\gamma = c_p/c_v$, $\varepsilon = D_{th}/D_b$.

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