Lecture notes for Math 256B, Version 2024

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Abstract

Nothing found here is original except for a few mistakes and misprints here and there. These notes are simply a record of what I cover in class, to spare the students the necessity of taking the lecture notes. The readers should consult the original books for a better presentation and context. We plan to follow the material from the following books: J. Bedrossian and V. Vicol "The Mathematical Analysis of the Incompressible Euler and Navier-Stokes Equations" C. Doering and J. Gibbon "Applied Analysis of the Navier-Stokes Equations", A. Majda and A. Bertozzi "Vorticity and Incompressible Flow", P. Constantin and C. Foias "The Navier-Stokes Equations", as well as lecture notes by Vladimir Sverak on the mathematical fluid dynamics that can be found on his website.

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1 The derivation of the Navier-Stokes and Euler equations

The state of a fluid is characterized by its density $\rho(t,x)$ and fluid velocity $u(t,x)$, and our first task is to derive the partial differential equations that govern their evolution. They will come from the conservation of mass, Newton's second law and, finally, an assumption on the material properties of the fluid.

1.1 The continuity equation

Each fluid particle is following a trajectory governed by the fluid velocity $u(t,x)$:

$$\frac{dX(t,\alpha)}{dt} = u(t, X(\alpha,t)), \quad X(0,\alpha) = \alpha. \tag{1.1}$$

Here, α is the starting position of the particle, and is sometimes called "the label", and the inverse map $A_t : X(t, \alpha) \to \alpha$ is called the "back-to-the-labels" map. If the flow $u(t, x)$ is sufficiently smooth so that the particles can not coalesce, and the particles are never removed, the forward map $\alpha \to X(t, \alpha)$ should preserve the total mass.

Let us first assume that the fluid density $\rho(t, x) = \rho_0$ is a constant, and see what can be deduced from mass preservation – the fluid is neither created nor destroyed. In the constant density case, mass preservation is equivalent to the conservation of the volume. That is, if $V_0 \subset \mathbb{R}^d$, $(d = 2, 3)$ is an initial volume of a parcel of the fluid, then the set

$$V(t) = \{X(t, \alpha) : \ \alpha \in V_0\}$$

of where the particles that started in V_0 at $t = 0$ ended up at a later time $t > 0$, should have the same volume as V_0. In order to quantify this property, let us define the Jacobian

$$J(t, \alpha) = \det(\frac{\partial X_i(t, \alpha)}{\partial \alpha_j}).$$

The change of variables formula, for the coordinate transformation $\alpha \to X(t, \alpha)$, implies that volume preservation means that $J(t, \alpha) \equiv 1$. As $J(0, \alpha) \equiv 1$, this condition is equivalent to

$$\frac{dJ}{dt} \equiv 0. \tag{1.2}$$

Thus, our first task is to compute the time derivative dJ/dt for a general velocity field $u(t, x)$. It follows from (1.1) that the full derivative matrix

$$H_{ij}(t, \alpha) = \frac{\partial X_i(t, \alpha)}{\partial \alpha_j}$$

obeys the evolution equation

$$\frac{dH_{ij}}{dt} = \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k} \frac{\partial X_k}{\partial \alpha_j}, \tag{1.3}$$

which, in the matrix form, is

$$\frac{dH}{dt} = (\nabla u)H, \quad (\nabla u)_{ik} = \frac{\partial u_i}{\partial x_k}. \tag{1.4}$$

The matrix H_{ij} is also known as the deformation tensor. For example, if $u = \bar{u}$ is a constant vector, so that

$$X(t, \alpha) = \alpha + \bar{u}t,$$

then $H = \mathrm{Id}$ is the identity matrix. In order to find dJ/dt, with $J(t, \alpha) = \det H(t, \alpha)$, we consider a general $n \times n$ time-dependent matrix $A_{ij}(t)$ and decompose, for each $i = 1, \ldots, n$ fixed:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} M_{ij} A_{ij}.$$

Note that the minors M_{ij}, for all $1 \le j \le n$, do not depend on the matrix element A_{ij}, hence

$$\frac{\partial}{\partial A_{ij}} (\det A) = (-1)^{i+j} M_{ij}.$$

We conclude that

$$\frac{d}{dt}(\det A) = \sum_{i,j=1}^{n} \frac{\partial(\det A)}{\partial A_{ij}} \frac{dA_{ij}}{dt} = \sum_{i,j=1}^{n} (-1)^{i+j} M_{ij} \frac{dA_{ij}}{dt}. \tag{1.5}$$

Recall also that

$$(A^{-1})_{ij} = (1/\det A)(-1)^{i+j} M_{ji},$$

meaning that

$$(\det A)\delta_{ik} = (\det A) \sum_{j=1}^{n} A_{kj}(A^{-1})_{ji} = \sum_{j=1}^{n} (-1)^{j+i} M_{ij} A_{kj}. \tag{1.6}$$

We apply now (1.5)-(1.6) to the matrix H_{ij}:

$$\frac{dJ}{dt} = \sum_{i,j=1}^{n} (-1)^{i+j} M_{ij} \frac{dH_{ij}}{dt}, \tag{1.7}$$

and

$$J\delta_{ik} = \sum_{j=1}^{n} (-1)^{j+i} M_{ij} H_{kj} \tag{1.8}$$

Here, M_{ij} are the minors of the matrix H_{ij}. Using (1.3) and (1.8) in (1.7) gives

$$\frac{dJ}{dt} = \sum_{i,j,k=1}^{n} (-1)^{i+j} M_{ij} \frac{\partial u_i}{\partial x_k} H_{kj} = \sum_{i,k=1}^{n} \frac{\partial u_i}{\partial x_k} J\delta_{ik} = J(\nabla \cdot u). \tag{1.9}$$

This is the equation for dJ/dt that we sought. Preservation of the volume means that $J \equiv 1$. As $H(0) = \mathrm{Id}$ and $J(0) = 1$, this is equivalent to the incompressibility condition:

$$\nabla \cdot u = 0. \tag{1.10}$$

Here, we use the notation

$$\nabla \cdot u = \mathrm{div} u = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k}.$$

More generally, if the density is not constant, mass conservation would require that for any initial volume V_0 we would have (recall that $\rho(t,x)$ is the fluid density)

$$\frac{d}{dt} \int_{V(t)} \rho(t,x)dx = 0, \tag{1.11}$$

where

$$V(t) = \{X(t,\alpha) : \ \alpha \in V_0\}.$$

Using the change of variables $\alpha \to X(t,\alpha)$ and writing

$$\int_{V(t)} \rho(t,x)dx = \int_{V_0} \rho(t, X(t,\alpha)) J(t,\alpha) d\alpha, \tag{1.12}$$

we see that mass conservation is equivalent to the condition

$$\frac{d}{dt}(\rho(t, X(t,\alpha))J(t,\alpha)) = 0. \tag{1.13}$$

Using (1.1) and (1.9) leads to

$$\frac{\partial \rho}{\partial t}J + (u \cdot \nabla \rho)J + \rho(\nabla \cdot u)J = 0. \tag{1.14}$$

Dividing by J we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \tag{1.15}$$

We note briefly some basic properties of (1.15). First, the total mass over the whole space is conserved:

$$\int_{\mathbb{R}^d} \rho(t,x)dx = \int_{\mathbb{R}^d} \rho(0,x)dx. \tag{1.16}$$

This follows both from (1.15) after integration over $\mathbb{R}^d$ (assuming an appropriate decay at infinity), and, independently, from our derivation of the continuity equation. If (1.15) is posed in a bounded domain Ω then, in order to ensure mass preservation, one may assume that the flow does not penetrate the boundary $\partial\Omega$:

$$u \cdot \nu = 0 \text{ on } \partial\Omega. \tag{1.17}$$

Here, ν is the outward normal to $\partial\Omega$. Under this condition, we have

$$\int_{\Omega} \rho(t,x)dx = \int_{\Omega} \rho(0,x)dx. \tag{1.18}$$

This may be verified directly from (1.15) but it also follows from our derivation of the continuity equation since (1.17) implies that Ω is an invariant region for the flow u: if $\alpha \in \Omega$ then $X(t,\alpha) \in \Omega$ for all $t > 0$.

Furthermore, (1.15) preserves the positivity of the solution: if $\rho(0,x) \geq 0$ then $\rho(t,x) \geq 0$ for all $t > 0$ and x – this also follows from common sense: density can not become negative.

1.2 Newton's second law in an inviscid fluid

The continuity equation for the evolution of the density $\rho(t,x)$ should be supplemented by an evolution equation for the fluid velocity $u(t,x)$. This will come from Newton's second law of motion. Consider a fluid volume V. If the fluid is inviscid, so that there is no "internal friction" in the fluid, the only force acting on this volume is due to the pressure:

$$F = -\int_{\partial V} p\nu dS = -\int_V \nabla p dx, \tag{1.19}$$

where ∂V is the boundary of V, and ν is the outside normal to ∂V. Taking V to be an infinitesimal volume around a point $X(t)$, which moves with the fluid, Newton's second law of motion leads to the balance

$$\rho(t, X(t))\ddot{X}(t) = -\nabla p(t, X(t)). \tag{1.20}$$

We may compute $\ddot{X}(t)$ from (1.1):

$$\ddot{X}_j(t) = \frac{d}{dt}(u_j(t,X(t)) = \frac{\partial u_j(t,X(t))}{\partial t} + \sum_k \dot{X}_k(t)\frac{\partial u_j(t,X(t))}{\partial x_k} \tag{1.21}$$
$$= \frac{\partial u_j(t,X(t))}{\partial t} + u(t,X(t))\cdot\nabla u_j(t,X(t)).$$

Therefore, we have the following equation of motion:

$$\rho\Big(\frac{\partial u}{\partial t} + u\cdot\nabla u\Big) + \nabla p = 0. \tag{1.22}$$

Equations (1.15) and (1.22) do not form a closed system of equations by themselves – they involve $n+1$ equations for $n+2$ unknowns (the density $\rho(t,x)$, the pressure $p(t,x)$ and the fluid velocity $u(t,x)$). The missing equation should provide the connection between the density and the pressure, and this comes from the physics of the problem, that goes into the assumptions on the material properties of the fluid. In gas dynamics, it often takes the form of a constitutive relation $p = F(\rho)$, where $F(\rho)$ is a given function, such as $F(\rho) = C\rho^\gamma$ with some constant $\gamma > 0$. Then, the full system becomes

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho u) = 0\\ &u_t + u\cdot\nabla u + \frac{1}{\rho}\nabla p = 0,\\ &p = F(\rho). \end{aligned} \tag{1.23}$$

The pressure may also depend on the temperature, and then the evolution of the local temperature has to be included as well but we will not discuss this at the moment.

1.2.1 The linearized equations

The simplest solution of (1.23) is the constant density and pressure, zero fluid velocity state:

$$\rho = \rho_0,\ p = p_0 = F(\rho_0) \text{ and } u = 0. \tag{1.24}$$

Let us consider a small perturbation around this state:

$$\begin{aligned} &\rho = \rho_0 + \varepsilon\eta + O(\varepsilon^2),\\ &p = p_0 + \varepsilon F'(\rho_0)\eta + O(\varepsilon^2)\\ &u = \varepsilon v + O(\varepsilon^2), \end{aligned} \tag{1.25}$$

with $\varepsilon \ll 1$. Inserting these expansions into (1.23) gives, in the (leading) order $O(\varepsilon)$:

$$\begin{aligned} &\eta_t + \rho_0\nabla\cdot v = 0\\ &v_t + \frac{F'(\rho_0)}{\rho_0}\nabla\eta = 0. \end{aligned} \tag{1.26}$$

It is common to write this system in terms of v and the pressure perturbation $\tilde{p} = F'(\rho_0)\eta$. After dropping the tilde it becomes the linearized acoustic system

$$\kappa_0 p_t + \nabla\cdot v = 0 \tag{1.27}$$
$$\rho_0 v_t + \nabla p = 0. \tag{1.28}$$

Here, $\kappa_0 = 1/(F'(\rho_0)\rho_0)$ is the compressibility constant. Equations (1.27)-(1.28) form what is known as the linearized acoustics system. Differentiating (1.27) in time and using (1.28) leads to the wave equation for pressure:

$$\frac{1}{c_0^2} p_{tt} - \Delta p = 0, \tag{1.29}$$

with the sound speed

$$c_0 = \frac{1}{\sqrt{\rho_0 \kappa_0}} = \sqrt{F'(\rho_0)}. \tag{1.30}$$

The linearized acoustics is what governs most of the "real-world" applications at "bearable" sound levels.

1.2.2 Euler's equations in incompressible fluids

A common approximation in the fluid dynamics is to assume that the fluid is incompressible, that is, its density is constant: $\rho(t, x) = \rho_0$, as the fluid can not be compressed. Using this condition in (1.15), leads to another form of the incompressibility condition:

$$\nabla \cdot u = 0, \tag{1.31}$$

that we have already seen before in (1.10) as the volume preservation condition for the flow. That is natural: conservation of density means exactly that the volume of a fluid is preserved.

Equations (1.22) and (1.31) together form Euler's equations for an incompressible fluid:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho_0} \nabla p = 0, \tag{1.32}$$

$$\nabla \cdot u = 0. \tag{1.33}$$

Unlike in the acoustics system, the pressure $p(t, x)$ is not prescribed but is rather determined by the fluid incompressibility condition. In other words, $p(t, x)$ has to be chosen is such a way that the solution to (1.32) remains divergence free. In order to find the pressure, we may take the divergence of (1.32), leading to the Poisson equation for the pressure in terms of the velocity field:

$$\Delta p = -\rho_0 \nabla \cdot (u_t + u \cdot \nabla u) = -\rho_0 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \Big(u_k \frac{\partial u_j}{\partial x_k} \Big) = -\rho_0 \sum_{i,j=1}^{n} \frac{\partial u_k}{\partial x_j} \frac{\partial u_j}{\partial x_k}. \tag{1.34}$$

We used the incompressibility condition (1.33) in the second and third equalities above. Equations (1.32)-(1.34) together may be thought of as a closed system of equations for the velocity $u(t, x)$ alone since $p(t, x)$ is determined by $u(t, x)$ via (1.34). An extremely important point is that the Poisson equation (1.34) for the pressure means that $p(t, x)$ is a non-local function of the velocity. Hence the Euler equations are a non-local system of equations for the fluid velocity – the pressure field at a given point depends on the velocity distribution in the whole space.

When the problem is posed in a bounded domain, we need to prescribe the boundary conditions for the fluid velocity and pressure. If the physical domain Ω is fixed and the fluid does not penetrate through its boundary, a natural physical condition for the fluid velocity is that the normal component of the velocity vanishes at the boundary:

$$\nu \cdot u = 0 \text{ on } \partial\Omega, \tag{1.35}$$

where ν is the outward normal to the boundary. It follows that

$$\nu \cdot \frac{\partial u}{\partial t} = 0 \text{ on } \partial\Omega, \tag{1.36}$$

thus the pressure satisfies the Neumann boundary conditions

$$\frac{\partial p}{\partial \nu} = -\rho_0 \nu \cdot (u \cdot \nabla u) \text{ on } \partial\Omega. \tag{1.37}$$

Often, as a simplification we will consider the Euler equations either in the whole space, with the decaying boundary conditions at infinity, or with the periodic boundary conditions on a two- or three-dimensional torus, as the boundaries bring extra (and, admittedly, very interesting) difficulties into an already difficult problem.

1.3 The viscous stress and the Navier-Stokes equations

The previous discussion did not take into account the viscosity of a fluid, which comes from the forces that resist the shearing motions because of the microscopic friction. The forces normal to a given area element are associated to the pressure (which we did take into account), while those acting in the plane of the area element are associated to the shear stress. In order to derive the fluid motion equations, as a generalization of the force on a volume element V coming from the pressure field:

$$F = -\int_{\partial V} p\nu dS = -\int_V \nabla p dx, \tag{1.38}$$

we may write, for the force that acts on an infinitesimal surface area dS of a volume element V:

$$dF_j = \sum_{k=1}^{n} \nu_k \tau_{kj} dS, \tag{1.39}$$

where ν is the outward normal to dS, and τ is the total stress tensor that includes both the pressure and the shear stress. We will soon start making assumptions on the stress tensor but for moment, we simply assume that the surface force has the form (1.39) with some tensor τ_{kj}. Integrating this expression over the boundary ∂V leads to the total force acting on the volume V:

$$F_j = \sum_{k=1}^{n} \int_{\partial V} \nu_k \tau_{kj} dS = \sum_{k=1}^{n} \int_V \frac{\partial \tau_{kj}}{\partial x_k} dx. \tag{1.40}$$

We will use the notation $\nabla \cdot \tau$ for the vector with the components

$$(\nabla \cdot \tau)_j = \sum_{k=1}^{n} \frac{\partial \tau_{kj}}{\partial x_k}, \tag{1.41}$$

as well as denote

$$(\nu \cdot \tau)_j = \sum_{k=1}^{n} \nu_k \tau_{kj}. \tag{1.42}$$

In addition to the surface forces, there may internal forces that act inside the volume V, that need to be balanced with the surface forces. Let us assume for the moment that the fluid is in equilibrium, and let f be the internal forces, τ be the stress tensor, and V be an arbitrary volume element. Then the balance of forces says that

$$\int_V f dx + \int_V (\nabla \cdot \tau) dx = 0, \tag{1.43}$$

which means that in an equilibrium we have

$$f + \nabla \cdot \tau = 0. \tag{1.44}$$

The total angular momentum of the force should also vanish, meaning that (in three dimensions)

$$\int_V (f \times x) dx + \int_{\partial V} ((\nu \cdot \tau) \times x) dS = 0, \tag{1.45}$$

for each volume element V. The surface integral above can be re-written as[1]

$$\int_{\partial V} \varepsilon_{ijk} \nu_l \tau_{lj} x_k dS = \int_V \varepsilon_{ijk} \frac{\partial}{\partial x_l} (\tau_{lj} x_k) dx = \int_V \varepsilon_{ijk} \Big(\frac{\partial \tau_{lj}}{\partial x_l} x_k + \tau_{kj} \Big) dx, \quad \text{for each } i = 1, 2, 3. \tag{1.46}$$

Here, ε_{ink} is the totally anti-symmetric tensor: $(v \times w)_i = \varepsilon_{ijk} v_j w_k$, and $\varepsilon_{ijk} = 0$ if any pair of the indices i, j, k coincide, while if all i, j, k are different, then $\varepsilon_{ijk} = (-1)^{p+1}$, where $p = 1$ if (ijk) is an even permutation, and $p = 0$ if it is odd. Using (1.44) in (1.46), we get

$$\int_{\partial V} \varepsilon_{ijk} \nu_l \tau_{lj} x_k dS = \int_V \varepsilon_{ijk} \Big(- f_j x_k + \tau_{kj} \Big) dx, \quad \text{for each } i = 1, 2, 3. \tag{1.47}$$

Returning to (1.45), and combing it with (1.47), we obtain

$$0 = \int_V \varepsilon_{ijk} f_j x_k dx + \int_V \varepsilon_{ijk} \Big(- f_j x_k + \tau_{kj} \Big) dx = \int_V \varepsilon_{ijk} \tau_{kj} dx, \text{ for each } i = 1, 2, 3. \tag{1.48}$$

As a consequence,

$$\varepsilon_{ijk} \tau_{jk} = 0, \text{ for each } i = 1, 2, 3, \tag{1.49}$$

which means that the tensor τ_{ij} has to be symmetric.

Exercise. Modify the above computation to show that the stress tensor is symmetric even if the fluid is not in an equilibrium.

We may now go back to the derivation of the Euler equations and proceed as before, the difference being that the force term in the Newton second law is not $-\nabla p$ but $\nabla \cdot \tau$. This will lead to the equation of motion

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \frac{1}{\rho} \nabla \cdot \tau. \tag{1.50}$$

[1]From now we will use the convention that the repeated indices are summed unless specified otherwise.

As for the Euler equations, the evolution equation for the fluid velocity needs to be supplemented by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \tag{1.51}$$

Previously, we needed also to prescribe the equation of state – the relation between the pressure and the density. Now, we need to postulate, or derive from physical considerations, an expression for the stress tensor. We will decompose it as

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}. \tag{1.52}$$

The first term comes from the pressure – it leads to a force acting on a surface element in the direction normal to the surface element. The second term comes from the shear stress, and comes from the friction inside the fluid. It is natural to assume that it depends locally on ∇u – if the flow is uniform there is no shearing force. In order to understand this dependence, recall that, given a flow

$$\frac{dX}{dt} = u(t, X(t)), \quad X(0) = \alpha, \tag{1.53}$$

the deformation tensor $H_{ij} = \partial X_i / \partial \alpha_j$ obeys

$$\frac{dH_{ij}}{dt} = \frac{\partial u_i}{\partial x_m} H_{mj}, \quad H_{ij}(0) = \delta_{ij}. \tag{1.54}$$

Therefore, the skew-symmetric part of the matrix ∇u (locally in time and space) leads to a rigid-body rotation and does not contribute to the shearing force. Hence, it is also natural to assume that the shear stress σ_{ij} depends only on the symmetric part of ∇u:

$$D_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{1.55}$$

In a Newtonian fluid, the shear stress depends linearly on the deformation tensor D_{ij}:

$$\sigma = L(D),$$

for some linear map L between symmetric matrices. The map L should not depend on the point x and it should be isotropic: for each rotation matrix Q we should have

$$L(QDQ^*) = QL(D)Q^*. \tag{1.56}$$

Exercise. Show that the above conditions imply that the map L has to have the form

$$[L(D)]_{ij} = 2\mu D_{ij} + \lambda \delta_{ij} \mathrm{Tr}(D), \tag{1.57}$$

with some constants λ and μ. These constants are called the Lamé parameters in the context of the elasticity theory.

For an incompressible fluid, we have

$$\mathrm{Tr} D = \nabla \cdot u = 0, \tag{1.58}$$

hence the stress tensor has a simpler form

$$\sigma_{ij} = 2\mu D_{ij}. \tag{1.59}$$

We will make an additional assumption that μ and λ are constants that do not depend on other physical parameters such as temperature, density or pressure. Then the force term in (1.50) can be written as

$$\begin{aligned}[\nabla\cdot\tau]_k &= \frac{\partial\tau_{jk}}{\partial x_j} = \frac{\partial}{\partial x_j}\Big[-p\delta_{jk} + \mu(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}) + \lambda(\nabla\cdot u)\delta_{jk}\Big] \\ &= -\frac{\partial p}{\partial x_k} + \mu\Delta u_k + (\mu+\lambda)\frac{\partial}{\partial x_k}(\nabla\cdot u).\end{aligned} \tag{1.60}$$

This leads to the Navier-Stokes equations of compressible fluid dynamics

$$\frac{\partial u}{\partial t} + u\cdot\nabla u + \frac{1}{\rho}\nabla p = \frac{\mu}{\rho}\Delta u + \frac{(\mu+\lambda)}{\rho}\nabla(\nabla\cdot u) \tag{1.61}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) = 0, \tag{1.62}$$

$$p = F(\rho). \tag{1.63}$$

As with the Euler equations, the equation of state may also involve the temperature, and then the evolution equation for the temperature should also be prescribed.

The incompressibility constraint $\nabla\cdot u = 0$, or, equivalently, the constant density approximation $\rho = \rho_0$, simplifies the system (1.61)-(1.63) to the incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} + u\cdot\nabla u + \frac{1}{\rho_0}\nabla p = \frac{\mu}{\rho_0}\Delta u \tag{1.64}$$

$$\nabla\cdot u = 0. \tag{1.65}$$

Note that Euler's equations are formally recovered from the Navier-Stokes equations by setting the viscosity $\mu = 0$, or, equivalently, assuming that the shear stress vanishes.

From now on, unless specified otherwise, we will consider only the incompressible Euler and Navier-Stokes equations.

1.3.1 Two-dimensional flows

We will sometimes consider the two-dimensional version of the Navier-Stokes equations, which has exactly the same form as the three-dimensional equations (1.64)-(1.65) but with the fluid velocity that has only two components: $u = (u_1, u_2)$, and, in addition, the problem is posed for $x \in \mathbb{R}^2$. These can be interpreted as the solutions of the three-dimensional Navier-Stokes system of a special form $u = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$ with the pressure $p = p(x_1, x_2)$ – that is, they are independent of x_3 and the third component of the fluid velocity vanishes. It is straightforward to check that, indeed, they satisfy (1.64)-(1.65) provided that $\tilde{u} = (u_1, u_2)$ satisfies

$$\frac{\partial\tilde{u}}{\partial t} + \tilde{u}\cdot\nabla\tilde{u} + \frac{1}{\rho_0}\nabla p = \frac{\mu}{\rho_0}\Delta\tilde{u} \tag{1.66}$$

$$\nabla\cdot\tilde{u} = 0, \tag{1.67}$$

posed in $\mathbb{R}^2$ and not in $\mathbb{R}^3$.

2 The vorticity evolution

An important role in the theory of fluids is played by the fluid vorticity. It is defined in terms of the fluid velocity $u(t,x)$ as a vector

$$\omega = \mathrm{curl} u = \nabla \times u, \quad \omega_i = \varepsilon_{ijk}\partial_j u_k, \text{ in } \mathbb{R}^3, \tag{2.1}$$

in three dimensions, and as a scalar

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \text{ in } \mathbb{R}^2, \tag{2.2}$$

in two dimensions. Here, as before, ε_{ijk} is the totally anti-symmetric tensor and we use the summation convention for repeated indices. The two-dimensional vorticity can be understood as the x_3-component of the three-dimensional vorticity of the flow $(u_1(x_1,x_2), u_2(x_1,x_2), 0)$ – the other two components of the vorticity vanish for such flows. It is sometimes convenient to write also in two dimensions the vorticity as

$$\omega = \varepsilon_{ij}\partial_i u_j, \tag{2.3}$$

with the antisymmetric tensor ε_{ij} defined by $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = 1$, $\varepsilon_{21} = -1$.

Note that the vorticity vector field in three dimensions is always divergence free:

$$\nabla \cdot \omega = \varepsilon_{ijk}\partial_i\partial_j u_k = 0, \quad \text{in } \mathbb{R}^3. \tag{2.4}$$

2.1 Vorticity in two dimensions

2.1.1 Vorticity conservation in two dimensions

Let us now compute the evolution equation for the vorticity in two and three dimensions. In the two-dimensional case, we start with the Navier-Stokes equations (we will set the density $\rho_0 = 1$ for simplicity from now on, unless specified otherwise)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u, \tag{2.5}$$

and compute using (2.3):

$$\frac{\partial \omega}{\partial t} = \varepsilon_{ij}\partial_i\Big(\nu\Delta u_j - \frac{\partial p}{\partial x_j} - u_k\frac{\partial u_j}{\partial x_k}\Big) = \nu\Delta(\varepsilon_{ij}u_j) - \varepsilon_{ij}\partial_i\partial_j p - \varepsilon_{ij}\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} - \varepsilon_{ij}u_k\frac{\partial^2 u_j}{\partial x_i \partial x_k}. \tag{2.6}$$

Now, we note that

$$\varepsilon_{ij}\partial_i\partial_j p = 0, \tag{2.7}$$

because the tensor ε_{ij} is anti-symmetric, and also that

$$\begin{aligned} -\varepsilon_{ij}\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} &= -\frac{\partial u_k}{\partial x_1}\frac{\partial u_2}{\partial x_k} + \frac{\partial u_k}{\partial x_2}\frac{\partial u_2}{\partial x_k} = -\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} \\ &= \frac{\partial u_1}{\partial x_1}\Big(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\Big) + \frac{\partial u_2}{\partial x_2}\Big(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\Big) \ = -\omega\nabla \cdot u = 0. \end{aligned} \tag{2.8}$$

Using these two identities in (2.6) gives

$$\frac{\partial \omega}{\partial t} = \nu \Delta \omega - u \cdot \nabla \omega. \tag{2.9}$$

The "miracle" is that in two dimensions the term we have calculated in (2.8), and which in three dimensions will contribute to the vorticity growth, cancels out completely because of the incompressibility condition. Thus, in two dimensions, the vorticity satisfies an advection-diffusion equation

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \Delta \omega. \tag{2.10}$$

This is very remarkable, as (2.10) obeys the maximum principle: with appropriate decay conditions at infinity if (2.10) is posed in the whole space $\mathbb{R}^2$, or in the periodic case, we can immediately conclude that

$$\|\omega(t, \cdot)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}, \tag{2.11}$$

where $\omega_0(x) = \omega(0, x)$ is the initial condition for the vorticity, as long as $u(t, x)$ satisfies some very basic regularity assumptions.

Furthermore, in an inviscid fluid, when $\nu = 0$ the vorticity is simply advected along the flow lines; solution of

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0 \tag{2.12}$$

is simply

$$\omega(t, x) = \omega_0(t, A(t, x)), \tag{2.13}$$

where $A(t, x)$ is the "back-to-labels" map for (1.1). This will help us later to prove the regularity of the solutions of the Euler and Navier-Stokes equations in two dimensions, though it will not imply the regularity immediately.

2.1.2 The Biot-Savart law in two dimensions

Note also that the pressure term is nowhere to be seen in the vorticity equation (2.10). Thus, in order to close the problem, we only need to supplement the evolution equation (2.10) for vorticity by an expression for the fluid velocity $u(t, x)$ in terms of the vorticity $\omega(t, x)$. To this end, observe, that, as $u(t, x)$ is divergence free, and the problem is posed in all of $\mathbb{R}^2$, there exists a function $\psi(t, x)$, called the stream function, so that $u(t, x)$ has the form

$$u(t, x) = \nabla^\perp \psi(t, x) = (-\psi_{x_2}(t, x), \psi_{x_1}(t, x)). \tag{2.14}$$

To see this, note that, because of the divergence-free condition for $u(t, x)$, the flow

$$v(t, x) = (u_2, -u_1), \tag{2.15}$$

satisfies

$$\frac{\partial v_1}{\partial x_2} = \frac{\partial v_2}{\partial x_1}, \tag{2.16}$$

hence there exists a function $\psi(t, x)$ so that $v(t, x) = \nabla \psi(t, x)$, which is equivalent to (2.14).

The vorticity can be expressed in terms of the stream function as

$$\Delta\psi = \omega, \tag{2.17}$$

or, more explicitly,

$$\psi(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|)\omega(t,y)dy. \tag{2.18}$$

Differentiating (2.18) formally, we obtain an expression for the fluid velocity in terms of its vorticity

$$u(t,x) = \int_{\mathbb{R}^2} K_2(x-y)\omega(t,y)dy, \tag{2.19}$$

with the vector-valued integral kernel

$$K_2(x) = \frac{1}{2\pi}\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big). \tag{2.20}$$

Thus, the Navier-Stokes equations in two dimensions can be formulated purely in terms of vorticity as the advection-diffusion equation for the scalar vorticity

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \nu\Delta\omega, \tag{2.21}$$

with the velocity $u(t,x)$ given in terms of $\omega(t,x)$ by (2.19).

A potential danger is that the function $K_2(x)$ is singular, homogeneous of degree (-1) in x. Thus, it is not obvious that (2.20) gives a sufficiently regular velocity field $u(t,x)$ for the coupled problem to have a smooth solution even if the initial conditin $\omega_0(x) = \omega(0,x)$ is smooth and rapidly decaying at infinity. However, the "$1/x$" singularity in two dimensions is sufficiently mild: writing (2.19) in the polar coordinates gives (with $x^\perp = (-x_2, x_1)$)

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}(-\sin\phi,\cos\phi)\omega(x_1-r\cos\phi, x_2-r\sin\phi)d\phi dr,$$

There is no longer a singularity in (2.22), and the expression for the velocity "makes sense".

The system (2.19), (2.20), (2.21) is an example of an active scalar – the vorticity $\omega(t,x)$ is a solution of an advection-diffusion equation with the velocity coupled to the advected scalar itself.

2.2 Vorticity evolution in three dimensions

2.2.1 Vorticity equation in three dimensions

The situation in three dimensions is very different. In order to compute the evolution equation for the vorticity vector, first, note that the advection term in the Navier-Stokes equations can be written as

$$(u\cdot\nabla u)_i = u_j\frac{\partial u_i}{\partial x_j} = u_j\Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big) + u_j\frac{\partial u_j}{\partial x_i}, \tag{2.22}$$

and that

$$
\begin{aligned}
(\omega\times u)_i &= \varepsilon_{ijk}\omega_j u_k=\varepsilon_{ijk}\varepsilon_{jmn}(\partial_m u_n)u_k=(\delta_{in}\delta_{km}-\delta_{im}\delta_{kn})(\partial_m u_n)u_k\\
&=(\partial_k u_i)u_k-(\partial_i u_k)u_k.
\end{aligned}
\tag{2.23}
$$

We used above the identity

$$
\varepsilon_{jik}\varepsilon_{jmn}=\delta_{im}\delta_{kn}-\delta_{in}\delta_{km}
\tag{2.24}
$$

and anti-symmetry of ε_{ijk}. We see that

$$
u\cdot\nabla u=\omega\times u+\nabla\Big(\frac{|u|^2}{2}\Big).
\tag{2.25}
$$

Therefore, the Navier-Stokes equations can be written as

$$
u_t+\omega\times u+\nabla\Big(\frac{|u|^2}{2}+p\Big)=\nu\Delta u.
\tag{2.26}
$$

The formula

$$
\mathrm{curl}(a\times b)=-a\cdot\nabla b+b\cdot\nabla a+a(\nabla\cdot b)-b(\nabla\cdot a),
\tag{2.27}
$$

together with the incompressibility condition $\nabla\cdot u=0$ and (2.4) helps us to take the curl of (2.26), leading to the vorticity equation:

$$
\omega_t+u\cdot\nabla\omega=\nu\Delta\omega+V(t,x)\omega,
\tag{2.28}
$$

with

$$
V(t,x)\omega=\omega\cdot\nabla u,\quad V_{ij}=\frac{\partial u_i}{\partial x_j}.
\tag{2.29}
$$

We can decompose the matrix V into its symmetric and anti-symmetric parts:

$$
V=D+\Omega,\quad D=\frac{1}{2}(V+V^T),\ \ \Omega=\frac{1}{2}(V-V^T),
\tag{2.30}
$$

and observe that, for any $h\in\mathbb{R}^3$

$$
\begin{aligned}
\Omega_{ij}h_j&=\frac{1}{2}[\partial_j u_i-\partial_i u_j]h_j=\frac{1}{2}\partial_m u_k[\delta_{ik}\delta_{jm}-\delta_{im}\delta_{jk}]h_j=\frac{1}{2}\varepsilon_{lij}\varepsilon_{lkm}(\partial_m u_k)h_j\\
&=-\frac{1}{2}\varepsilon_{lij}\varepsilon_{lmk}(\partial_m u_k)h_j=-\frac{1}{2}\varepsilon_{lij}\omega_l h_j=\frac{1}{2}\varepsilon_{ilj}\omega_l h_j=\frac{1}{2}[\omega\times h]_i,
\end{aligned}
\tag{2.31}
$$

that is,

$$
\Omega h=\frac{1}{2}\omega\times h.
\tag{2.32}
$$

The matrix Ω has an explicit form

$$
\Omega=\frac{1}{2}\begin{pmatrix}0&-\omega_3&\omega_2\\ \omega_3&0&-\omega_1\\ -\omega_2&\omega_1&0\end{pmatrix}.
\tag{2.33}
$$

As a consequence, we have $\Omega\omega = 0$, thus $V\omega = D\omega$, and the vorticity equation has the form

$$\omega_t + u \cdot \nabla\omega = \nu\Delta\omega + D(t,x)\omega, \tag{2.34}$$

with

$$D_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{2.35}$$

The term $D\omega$ in the vorticity equation is known as the vortex stretching term, and it is maybe the main reason why the solutions of the three- dimensional Navier-Stokes equations exhibit such rich behavior and complexity. As we have done in two dimensions, it is possible to express the velocity $u(t,x)$ in terms of the vorticity – this relation is known as the Biot-Savart law, leading to the "pure vorticity" formulation of the Navier-Stokes equations, but we will postpone this computation until slightly later.

2.2.2 An analogy to the Burgers' equation

The vorticity equation (2.34) has a quadratic term in ω in the right side. Such quadratic nonlinearities may potentially lead to a blow up. This is easily seen on the simple ODE example

$$\dot{z} = z^2, \quad z(0) = z_0. \tag{2.36}$$

Its explicit solution is

$$z(t) = \frac{z_0}{1 - z_0 t}. \tag{2.37}$$

If $z_0 > 0$, the solution becomes infinite at the time

$$t_c = \frac{1}{z_0}. \tag{2.38}$$

At a slightly more sophisticated level, we can look at the familiar Burgers' equation on the line:

$$u_t + uu_x = 0, \quad u(0,x) = u_0(x). \tag{2.39}$$

Its solutions develop a finite time singularity if the initial condition $u_0(x)$ is decreasing on some interval. Such discontinuities are known as shocks. In order to make a connection to the vorticity equation, note that the function $\omega = -u_x$ satisfies

$$\omega_t + u\omega_x = \omega^2, \quad \omega(0,x) = \omega_0(x) = -u_0'(x). \tag{2.40}$$

This equation is analogous to the vorticity equation with $\nu = 0$, except the nonlinearity has a different form: $D(\omega)\omega$ is replaced by ω^2. As in the case of the quadratic ODE (2.36), the function $\omega(t,x)$ becomes infinite in a finite time if there are points where $\omega_0(x) > 0$. One should mention that there are two regularizations of the inviscid Burgers' equation (2.39): first, adding a diffusive (dissipative) term gives the viscous Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \quad u(0,x) = u_0(x), \tag{2.41}$$

which has global in time smooth solutions if $u_0(x)$ is smooth. A natural question which we may revisit later is why is the u_{xx} term sufficiently regularizing? More precisely, one may consider equations of the form

$$u_t + uu_x = Au, \quad u(0,x) = u_0(x), \tag{2.42}$$

where A is a linear dissipative operator in the sense that

$$(Au, u) = \int_{\mathbb{R}} (Au(x))u(x)dx \le 0. \tag{2.43}$$

If A commutes with differentiation, the "vorticity" equation will have the form

$$\omega_t + u\omega_x = A\omega + \omega^2, \quad \omega(0,x) = \omega_0(x) = -u_0'(x). \tag{2.44}$$

Then, the dissipative effect of $A\omega$ will compete with the growth caused by ω^2 in the right side. The issue of when the dissipation will win is rather delicate – we will revisit it later if we have time.

There is a different approach to the blow up in the Burgers' equation that illustrates a general strategy of trying to control some integral functionals of the solution rather than solutions themselves. Let us consider, for simplicity, the solution of the Burgers' equation on the line with a periodic initial condition $u_0(x)$:

$$u_0(x + 2\pi) = u_0(x).$$

Then the solution to

$$u_t + uu_x = 0, \quad u(0,x) = u_0(x) \tag{2.45}$$

will stay periodic for all $t > 0$ (as long as it exists):

$$u(t, x + 2\pi) = u(t,x). \tag{2.46}$$

If, in addition, the initial data is odd: $u_0(-x) = -u_0(x)$, then the solution remains odd as well: we have $u(t,x) = -u(t,x)$ for all $t > 0$. This means that, as long as the solution remains smooth, the functional

$$L(t) = \int_{-\pi}^{\pi} \frac{u(t,x)}{x} dx \tag{2.47}$$

is well-defined and finite – the function $u(t,x)$ vanishes at $x = 0$. Differentiating $L(t)$ in time gives

$$\frac{dL(t)}{dt} = \int_{-\pi}^{\pi} \frac{u_t(t,x)}{x} dx = -\int_{-\pi}^{\pi} \frac{1}{x} uu_x dx = -\frac{1}{2}\int_{-\pi}^{\pi} \frac{u^2(t,x)}{x^2} dx. \tag{2.48}$$

The Cauchy-Schwartz inequality implies that

$$L^2(t) = \left(\int_{-\pi}^{\pi} \frac{u(t,x)}{x} dx\right)^2 \le 2\pi \int_{-\pi}^{\pi} \frac{u^2(t,x)}{x^2} dx. \tag{2.49}$$

Hence, the function $L(t)$ satisfies a differential inequality

$$\frac{dL}{dt} \le -\frac{1}{4\pi} L^2(t). \tag{2.50}$$

Integrating this inequality in time gives

$$\frac{1}{L_0} - \frac{1}{L(t)} \leq -\frac{t}{4\pi}. \tag{2.51}$$

Hence, we have

$$L(t) \leq \frac{4\pi L_0}{4\pi + L_0 t}. \tag{2.52}$$

We conclude that if $L_0 < 0$ then $L(t) = -\infty$ at some time $t < -4\pi/L_0$, thus solution may not remain smooth past this time. The condition that $L_0 < 0$ distinguishes between the initial data that "look like" $u_0(x) = \sin x$ and like $u_0(x) = -\sin x$. The latter is decreasing at $x = 0$, hence the shock is expected to form there, thus it is reasonable to expect that $L(t)$, which has x in the denominator in the integrand, will blow-up. On the other hand, the former is increasing at $x = 0$, thus the shock would not form there, and $L(t)$ should not capture the singularity formation. A different functional should be considered to capture the blow-up.

Another very interesting regularization of the inviscid Burgers' equation is via dispersion:

$$u_t + uu_x = \mu u_{xxx}, \quad u(0,x) = u_0(x). \tag{2.53}$$

This is the Kortweg-de Vries equation which describes a regime of the shallow water waves. Its mathematics is incredibly rich and is connected by now with nearly every area of mathematics. If we have time, we will go back to it as well. For now, we just mention that solutions of (2.53) also remain smooth for all $t > 0$ provided that $u_0(x)$ is, say, a smooth rapidly decaying function. However, the mechanism for regularity is not dissipative but rather dispersive – the high frequencies spread faster, hence an oscillation will "fly away towards infinity very fast", and there u is small, hence the nonlinearity does not play a big role there. On the other hand, the balance between dispersion and nonlinearity leads to extremely interesting effects.

2.2.3 Flows with a spatially homogenous vorticity

As an example, we consider flows that have a spatially uniform vorticity $\omega(t)$. Let us choose a symmetric matrix $D(t)$ with $\mathrm{Tr}D(t) = 0$, and a vector-valued function $\omega(t) \neq 0$ such that

$$\frac{d\omega}{dt} = D(t)\omega(t), \quad \omega(0) = \omega_0. \tag{2.54}$$

We also define the anti-symmetric matrix $\Omega(t)$ via (2.33), so that

$$\Omega(t)h = \frac{1}{2}\omega(t) \times h, \text{ for any } h \in \mathbb{R}^3, \quad \Omega_{ij} = \varepsilon_{imj}\omega_m. \tag{2.55}$$

A direct computation, using the symmetry of D, the assumption $\mathrm{Tr}D = 0$, and (2.33), gives

$$\dot{\Omega} + D\Omega + \Omega D = 0. \tag{2.56}$$

The observation is that the flow

$$u(t,x) = \frac{1}{2}\omega(t) \times x + D(t)x \tag{2.57}$$

gives an exact solution of the three-dimensional Euler and Navier-Stokes equations, with the vorticity $\mathrm{curl} u = \omega$. Indeed, first, as the trace of $D(t)$ vanishes, both components in (2.57) are divergence-free:

$$\nabla \cdot u = \partial_j(\varepsilon_{jkl}\omega_k x_l) + \partial_j(D_{jk}x_k) = \varepsilon_{jkl}\omega_k\delta_{jl} + D_{jk}\delta_{jk} = 0. \tag{2.58}$$

Moreover, the second term in (2.57) is the gradient of the function $(1/2)(D(t)x \cdot x)$, hence its vorticity vanishes, while identity (2.27) means that

$$\mathrm{curl} u = \frac{1}{2}\mathrm{curl}(\omega(t) \times x) = -\frac{1}{2}\omega \cdot \nabla x + \frac{1}{2}\omega(\nabla \cdot x) = -\frac{1}{2}\omega + \frac{3}{2}\omega = \omega. \tag{2.59}$$

Next, we compute

$$u_t = \frac{1}{2}\dot{\omega} \times x + \dot{D}x, \tag{2.60}$$

and

$$\partial_j u_k = \frac{1}{2}\partial_j(\varepsilon_{kmn}\omega_m x_n) + \partial_j(D_{km}x_m) = \frac{1}{2}\varepsilon_{kmj}\omega_m + D_{kj}, \tag{2.61}$$

so that

$$u \cdot \nabla u_k = u_j \partial_j u_k = \frac{1}{2}\varepsilon_{kmj}u_j\omega_m + u_j D_{kj} = \frac{1}{2}\omega \times u + Du. \tag{2.62}$$

Putting these equations together and using (2.55) leads to

$$\begin{aligned} &u_t + u \cdot \nabla u = \frac{1}{2}\dot{\omega} \times x + \dot{D}x + \frac{1}{2}\omega \times u + Du = \frac{1}{2}\dot{\omega} \times x + \dot{D}x \\ &+\frac{1}{2}\omega \times \Big(\frac{1}{2}\omega \times x + Dx\Big) + D\Big(\frac{1}{2}\omega \times x + Dx\Big) \\ &= (\dot{D} + \dot{\Omega} + \Omega^2 + D^2 + D\Omega + \Omega D)x = (\dot{D} + \Omega^2 + D^2)x = -\nabla p(t,x) \end{aligned} \tag{2.63}$$

We have used (2.56) in the next to last equality above. The pressure is given explicitly by

$$p(t,x) = -\frac{1}{2}\Big(\frac{\partial D}{\partial t} + D^2 + \Omega^2\Big)x \cdot x. \tag{2.64}$$

We conclude that, given any symmetric trace-less matrix $D(t)$, we may construct a solution of the Euler equations as above.

Example 1. A jet flow. As the first example of using the above construction, we may take $\omega_0 = 0$, so that $\omega(t) = 0$ and $D(t) = \mathrm{diag}(-\gamma_1, -\gamma_2, \gamma_1 + \gamma_2)$ with $\gamma_1, \gamma_2 > 0$. The flow is

$$u(t,x) = (-\gamma_1 x_1, -\gamma_2 x_2, (\gamma_1 + \gamma_2)x_3). \tag{2.65}$$

The particle trajectories are

$$X(t,\alpha) = (e^{-\gamma_1 t}\alpha_1, e^{-\gamma_2 t}\alpha_2, e^{(\gamma_1+\gamma_2)t}\alpha_3), \tag{2.66}$$

and have the form of a jet, going toward the x_3-axis, and up along this line for $x_3 > 0$, and down this direction for $x_3 < 0$.

Example 2. A strain flow. Consider $D = \mathrm{diag}(-\gamma, \gamma, 0)$ with $\gamma > 0$, and, once again, vorticity $\omega = 0$, so that

$$u(t,x) = (-\gamma x_1, \gamma x_2, 0). \tag{2.67}$$

Then the particle trajectories are

$$X(t,\alpha) = (e^{-\gamma t}\alpha_1, e^{\gamma t}\alpha_2, \alpha_3). \tag{2.68}$$

The particle trajectories stay in a fixed plane orthogonal to the x_3-axis and are stretched in this plane: nearby two particles starting near the x_1-axis with $\alpha_2 > 0$ and $\alpha_2 < 0$ will separate exponentially fast in time.

2.2.4 Shear layer solutions

Here, we will generalize the second example above: we will be looking at flows of the form generalizing (2.67):

$$u(t,x) = (-\gamma x_1, \gamma x_2, w(t,x_1)), \tag{2.69}$$

that is, the third flow component depends only on x_1 and t. Such flows satisfy the Navier-Stokes equations with the pressure $p(t,x) = \gamma(x_1^2+x_2^2)/2$, provided that the vertical component of the flow w satisfies a linear advection-diffusion equation

$$\frac{\partial w}{\partial t} - \gamma x_1 \frac{\partial w}{\partial x_1} = \nu \frac{\partial^2 w}{\partial {x_1}^2}. \tag{2.70}$$

The vorticity is given by

$$\omega(t,x) = (0, -\frac{\partial w}{\partial x_1}, 0), \tag{2.71}$$

and its second component $\tilde{\omega} = -w_{x_1}$ satisfies (after dropping the tilde)

$$\frac{\partial \omega}{\partial t} - \gamma x_1 \frac{\partial \omega}{\partial x_1} = \nu \frac{\partial^2 \omega}{\partial {x_1}^2} + \gamma\omega. \tag{2.72}$$

Here, we see clearly the three competing effects in the vorticity evolution: the diffusive (dissipative) term $\nu\omega_{x_1x_1}$, the convective term $-\gamma x_1\omega_{x_1}$ and the vorticity growth term $\gamma\omega$. It is instructive to look at the three effects in this very simple setting.

First, let us note that when $\gamma > 0$, the vorticity equation (2.72) admits steady solutions:

$$-\gamma x_1 \bar{\omega}' = \nu\bar{\omega}'' + \gamma\bar{\omega}. \tag{2.73}$$

Indeed, setting $y = \lambda x_1$ leads to

$$-\gamma y \bar{\omega}_y = \lambda^2 \nu \bar{\omega}_{yy} + \gamma\bar{\omega}, \tag{2.74}$$

thus, choosing $\lambda = \sqrt{\gamma/\nu}$, we arrive at

$$-y\bar{\omega}_y = \bar{\omega}_{yy} + \bar{\omega}. \tag{2.75}$$

This equation has an explicit steady solution

$$\bar{\omega}(y) = e^{-y^2/2}, \tag{2.76}$$

hence a steady solution of (2.73) is

$$\bar{\omega}(x_1) = e^{-\gamma x_1^2/(2\nu)}. \tag{2.77}$$

Such solutions do not exist when $\gamma = 0$ – they are sustained by the stretch, and are localized in a layer of the width $O(\sqrt{\nu/\gamma})$ around the plane $\{x_1 = 0\}$. They may also not exist at zero viscosity: if $\nu = 0$ then (2.73) has no non-trivial bounded steady solutions – thus, they are a result of a balance between the stretch and the friction.

Equation (2.72) can be solved explicitly. Fitst, writing

$$\omega(t, x) = e^{\gamma t} z(t, x_1) \tag{2.78}$$

gives

$$\frac{\partial z}{\partial t} - \gamma x_1 \frac{\partial z}{\partial x_1} = \nu \frac{\partial^2 z}{\partial {x_1}^2}. \tag{2.79}$$

Next, making a change of variables:

$$z(t, x) = \eta(\tau(t), e^{\gamma t} x_1) \tag{2.80}$$

with the function $\tau(t)$ to be determined, leads to

$$\dot{\tau} \frac{\partial \eta}{\partial \tau} + \gamma e^{\gamma t} x_1 \frac{\partial \eta}{\partial \xi} - \gamma x_1 e^{\gamma t} \frac{\partial \eta}{\partial \xi} = \nu e^{2\gamma t} \frac{\partial^2 \eta}{\partial \xi^2}. \tag{2.81}$$

Taking

$$\dot{\tau} = \nu e^{2\gamma t}, \tag{2.82}$$

or

$$\tau(t) = \frac{\nu}{2\gamma} \Big(e^{2\gamma t} - 1 \Big), \tag{2.83}$$

leads to the standard heat equation

$$\frac{\partial \eta}{\partial \tau} = \frac{\partial^2 \eta}{\partial \xi^2}, \quad \tau > 0, \ \xi \in \mathbb{R}, \tag{2.84}$$

with the initial condition $\eta(0, \xi) = \omega_0(\xi)$. Therefore, the vorticity is

$$\omega(t, x_1) = e^{\gamma t} \int G\Big(\frac{\nu}{2\gamma} (e^{2\gamma t} - 1), e^{\gamma t} x_1 - y \Big) \omega_0(y) dy, \tag{2.85}$$

where $G(t, x_1)$ is the standard heat kernel:

$$G(t, x_1) = \frac{1}{\sqrt{4\pi t}} e^{-|x_1|^2/(4t)}. \tag{2.86}$$

Let us look at the long time behavior of vorticity:

$$\begin{aligned} \omega(t, x_1) &= e^{\gamma t} \Big(\frac{4\pi\nu}{2\gamma} (e^{2\gamma t} - 1) \Big)^{-1/2} \int \exp\Big\{ - \frac{|e^{\gamma t} x_1 - y|^2}{\frac{4\nu}{2\gamma}(e^{2\gamma t} - 1)} \Big\} \omega_0(y) dy \\ &\to \bar{\omega}(x) = \Big(\frac{\gamma}{2\pi\nu} \Big)^{1/2} e^{-\gamma |x_1|^2/(2\nu)} \int \omega_0(y) dy, \end{aligned} \tag{2.87}$$

provided that the initial vorticity $\omega_0 \in L^1(\mathbb{R})$. Thus, the vorticity is localized as $t \to +\infty$ around $x_1 = 0$, in a layer of the width $O(\sqrt{\nu/\gamma})$, and its long time limit is a multiple of the steady solution (2.77).

2.2.5 The Biot-Savart law in three dimensions

We now return to the vorticity equation in three dimensions

$$\omega_t + u \cdot \nabla\omega = \nu\Delta\omega + \omega \cdot \nabla u. \tag{2.88}$$

Our goal is to derive an expression for the velocity u in terms of the vorticity ω, so as to formulate the Euler and Navier-Stokes equations purely in terms of vorticity. In two dimensions, this was done using the stream function, solution of

$$\Delta\psi = \omega, \tag{2.89}$$

with u given by

$$u = \nabla^{\perp}\psi = (-\psi_{x_2}, \psi_{x_1}), \tag{2.90}$$

or, equivalently,

$$u(t,x) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y)dy, \tag{2.91}$$

with the vector-valued integral kernel

$$K_2(x) = \frac{1}{2\pi}\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big). \tag{2.92}$$

In three dimensions, given a divergence-free vector field $\omega(x)$ we need to find a divergence-free vector field $u(t,x)$ so that

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0. \tag{2.93}$$

Attempting the same strategy as in two dimensions, we define the stream vector ψ via

$$\Delta\psi = \omega, \tag{2.94}$$

and

$$u(x) = -\nabla \times \psi(x). \tag{2.95}$$

Note that, as $\nabla \cdot \omega = 0$ by assumption, we have

$$\Delta(\nabla \cdot \psi) = 0. \tag{2.96}$$

Hence, if we assume that $\nabla \cdot \psi$ is bounded, then $\nabla \cdot \psi = 0$, and ψ is also divergence-free. The flow u defined by (2.95) is divergence-free: $\nabla \cdot u = 0$, and

$$\begin{aligned}[] [\nabla \times u]_i &= \varepsilon_{ijk}\partial_j u_k = -\varepsilon_{ijk}\partial_j\varepsilon_{kmn}\partial_m\psi_n = -\varepsilon_{kij}\varepsilon_{kmn}\partial_j\partial_m\psi_n = -(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_j\partial_m\psi_n \\ &= -\partial_i\partial_j\psi_j + \Delta\psi_i, \end{aligned} \tag{2.97}$$

that is, ω is the vorticity of u:

$$\nabla \times u = -\nabla(\nabla \cdot \psi) + \Delta\psi = \omega. \tag{2.98}$$

We have an explicit expression for the stream-vector $\psi(x)$ as the solution of the Poisson equation (2.94):

$$\psi(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\omega(y)dy. \tag{2.99}$$

The velocity is then given by

$$u_i(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \varepsilon_{ijk}\partial_j\Big(\frac{1}{|x-y|}\Big)\omega_k(y)dy = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \varepsilon_{ijk}\frac{x_j - y_j}{|x-y|^3}\omega_k(y)dy, \tag{2.100}$$

so that

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} K(x-y)\times\omega(y)dy, \tag{2.101}$$

with

$$K(x) = -\frac{1}{4\pi}\frac{x}{|x|^3}. \tag{2.102}$$

As in the two-dimensional case, the integral operator defining $u(x)$ in terms of the vorticity $\omega(x)$ is not "really singular" – the singularity of the $1/|x|^2$ type is cancelled in three dimensions by the Jacobian if we pass to the spherical coordinates. However, unlike in two dimensions, the vorticity equation in three dimensions

$$\omega_t + u\cdot\nabla\omega = \nu\Delta\omega + \omega\cdot\nabla u, \tag{2.103}$$

involves not only $u(x)$ but also the gradient ∇u. Formally differentiating (2.101) leads to (this identity is not quite correct because of the singularity of the integrals involved)

$$\nabla u(x)\text{"}=\text{"}\int_{\mathbb{R}^3} \nabla K(x-y)\times\omega(y)dy. \tag{2.104}$$

The integral kernel $\nabla K(x)$ in (2.104) has the singularity of the type $x/|x|^4$, which can not be simply cancelled by the Jacobian in three dimensions if we pass to the spherical coordinates. Integral operators with a singularity of this type are known as singular integral operators, and we will deal with them in some detail later, leaving for now the vorticity equation on a formal level.

3 The conserved quantities

We will now discuss the physical quantities conserved by the Euler and Navier-Stokes equations. They are important both from the physical and mathematical points of view – a system that possesses sufficiently regular integrals of motion will not have irregular solutions if the initial condition is smooth. As we will see, the integrals of motion for the fluid equations are often insufficient to deduce the existence and regularity of solutions.

3.1 Vortex lines

In three dimensions, we say that a smooth curve Γ is a vortex line at a time $t \geq 0$ if its tangent is everywhere parallel to the vorticity vector $\omega(t,x)$. Let us show that if

$$\Gamma_0 = \{\gamma(s), \quad 0 \leq s \leq 1\} \tag{3.1}$$

is a vortex line at the time $t = 0$ then its push-forward

$$\Gamma(t) = \{X(t,\gamma(s)), \quad 0 \leq s \leq 1\} \tag{3.2}$$

is a vortex line at the time $t > 0$. For that, we have the following.

Lemma 3.1. *Let $\omega(t,x)$ be the vorticity of a solution to the Euler equations in three dimensions. Then, we have*

$$\omega(t,X(t,\alpha)) = (\nabla_\alpha X)(t,\alpha)\omega_0(\alpha). \tag{3.3}$$

Proof. Note that (3.3) holds at $t = 0$. Recall that the matrix

$$H_{ij}(t,X(t,\alpha)) = \frac{\partial X_i(t,\alpha)}{\partial \alpha_j}, \tag{3.4}$$

satisfies (1.4)

$$\frac{dH}{dt} = (\nabla u)H, \tag{3.5}$$

so that

$$\frac{d}{dt}(\nabla_\alpha X)(t,\alpha)\omega_0(\alpha) = \nabla u H \omega_0. \tag{3.6}$$

On the other hand, the Euler equations

$$\omega_t + u \cdot \nabla\omega = \omega \cdot \nabla u \tag{3.7}$$

imply that

$$\frac{d}{dt}\omega(t,X(t,\alpha)) = (\nabla_x u(t,X(t,\alpha))\omega(t,X(t,\alpha)) \tag{3.8}$$

This finishes the proof.

3.2 Kelvin's theorem

Consider a smooth, oriented, closed curve C_0, and let $C(t)$ be its image under a flow $u(t,x)$:

$$C(t) = \{X(t,\alpha) : \ \alpha \in C_0\}, \tag{3.9}$$

with

$$\frac{dX}{dt} = u(t,X), \quad X(0,\alpha) = \alpha. \tag{3.10}$$

The circulation around $C(t)$ is

$$\Gamma_{C(t)} = \oint_{C(t)} u(t,x) \cdot d\ell, \tag{3.11}$$

where $d\ell$ is the length element along $\Gamma(t)$. Recall that, generally, if a closed curve Γ is parametrized as $\Gamma = \{\gamma(s),\ 0 \le s \le 1\}$, then the circulation of a vector $w(x)$ over Γ is

$$\oint_\Gamma w \cdot d\ell = \int_0^1 w(\gamma(s)) \cdot \gamma'(s) ds. \tag{3.12}$$

Note that the right side does not depend on the parametrization of the curve Γ.

Let us parametrize the initial and evolved curves as

$$C_0 = \{\gamma(s),\ 0 \le s \le 1\}, \quad C(t) = \{X(t, \gamma(s)),\ 0 \le s \le 1\}. \tag{3.13}$$

The length element along the evolved curve has the components (prime denotes the derivative with respect to the parametrization parameter s)

$$\frac{dX_j(t,\gamma(s))}{ds} = \frac{\partial X_j(t,\gamma(s))}{\partial \gamma_k}\gamma_k(s)' ds = H(t, X(t,\gamma(s))\gamma'(s), \tag{3.14}$$

with the matrix

$$H_{ij}(t, X(t,\alpha)) = \frac{\partial X_i(t,\alpha)}{\partial \alpha_j}, \tag{3.15}$$

which, as we recall, satisfies (1.4)

$$\frac{dH}{dt} = (\nabla u) H. \tag{3.16}$$

Now, we may compute, using the parametrization (3.13) of the curve $C(t)$:

$$\begin{aligned}
&\frac{d}{dt}\oint_{C(t)} u(t,x) \cdot d\ell = \frac{d}{dt}\int_0^1 u(t, X(t,\gamma(s)) \cdot (H\gamma') ds = \int_0^1 [(\dot{u} \cdot H\gamma') + (u \cdot \dot{H}\gamma')] ds \\
&= \int_0^1 [(u_t + u\cdot\nabla u) \cdot H\gamma') + (u \cdot (\nabla u H)\gamma')] ds \\
&= \oint_{C(t)} (u_t + u\cdot\nabla u)\cdot d\ell + \oint_{C(t)} (\nabla u)^t u \cdot d\ell.
\end{aligned} \tag{3.17}$$

If u satisfies the Euler equations, the first term in the last line above can be written in terms of the pressure as

$$\oint_{C(t)} (u_t + u\cdot\nabla u)\cdot d\ell = -\oint \nabla p \cdot d\ell = 0. \tag{3.18}$$

The second term can be written as

$$\oint_{C(t)} (\nabla u)^t u \cdot d\ell = \oint_{C(t)} \frac{\partial u_k}{\partial x_j} u_k d\ell_j = \oint_{C(t)} \nabla\Big(\frac{|u|^2}{2}\Big) \cdot d\ell = 0. \tag{3.19}$$

We see that

$$\frac{d}{dt}\oint_{C(t)} u(t,x) \cdot d\ell = 0. \tag{3.20}$$

This is Kelvin's theorem for the Euler equations: the circulation of the flow along a curve that evolves with the flow is preserved in time.

3.3 Conservation of the integrals of velocity and vorticity

If u is a divergence-free velocity field, and q is a scalar function, and both of them decay sufficiently fast at infinity, we have

$$\int_{\mathbb{R}^n}(u\cdot\nabla\phi)dx=-\int(\nabla\cdot u)\phi dx=0. \tag{3.21}$$

Therefore, integrating either the Euler or the Navier-Stokes equations with solutions that decay rapidly at infinity, we conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^n}udx=0, \tag{3.22}$$

both in two and three dimensions. The same identity implies that in two dimensions the total vorticity is preserved: integrating (2.21), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2}\omega dx=-\nu\int_{\mathbb{R}^2}\Delta\omega dx-\int_{\mathbb{R}^2}(u\cdot\nabla\omega)dx=0. \tag{3.23}$$

However, in that case we know more: any regular solution of (2.21) can be decomposed as

$$\omega=\omega^+(t,x)-\omega^-(t,x),$$

where $\omega^\pm$ are the solutions of (2.21) with the initial conditions $\omega_0^\pm(x)$, respectively. It follows that

$$\int_{\mathbb{R}^2}|\omega|dx\le\int_{\mathbb{R}^2}\omega^+(t,x)dx+\int_{\mathbb{R}^2}\omega^-(t,x)dx=\int_{\mathbb{R}^2}|\omega_0|dx, \tag{3.24}$$

that is, not only the integral of the vorticity is preserved but its L^1-norm does not grow in two dimensions.

In addition, for the solutions of the Euler equations in two dimensions, vorticity satisfies the advection equation

$$\omega_t+u\cdot\nabla\omega=0. \tag{3.25}$$

Therefore, not only the integral of the vorticity but all L^p-norms of ω are preserved, with any $1\le p\le\infty$:

$$\int_{\mathbb{R}^2}|\omega(t,x)|^pdx=\int_{\mathbb{R}^2}|\omega_0(x)|^pdx. \tag{3.26}$$

In three dimensions, the vorticity vector satisfies (2.103). Integrating this equation leads to

$$\frac{d}{dt}\int_{\mathbb{R}^3}\omega_idx=\int_{\mathbb{R}^3}(\omega\cdot\nabla u_i)dx=0, \tag{3.27}$$

since $\omega(t,x)$ is also a divergence-free field. Thus, the total integral of the vorticity is preserved also in three dimensions. However, conservation of the L^p-norms does not follow, and vorticity may grow.

3.4 Evolution of energy, dissipation and enstrophy

The kinetic energy of the fluid is

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |u(t,x)|^2 dx. \tag{3.28}$$

Differentiating in time, assuming that solutions are smooth, gives

$$\begin{aligned} \frac{dE}{dt} &= \int_{\mathbb{R}^n} (u \cdot u_t) dx = \int_{\mathbb{R}^n} (-u_j u_k \frac{\partial u_j}{\partial x_k} - u \cdot \nabla p + \nu u_j \Delta u_j) dx \\ &= -\int_{\mathbb{R}^n} (u \cdot \nabla\Big(\frac{|u|^2}{2} + p\Big) - \nu \int_{\mathbb{R}^n} |\nabla u|^2 dx = -\nu \int_{\mathbb{R}^n} |\nabla u|^2 dx. \end{aligned} \tag{3.29}$$

Therefore, the energy of the solutions of the Euler equations ($\nu = 0$) is preserved in time:

$$E(t) = E(0), \tag{3.30}$$

while the energy of the solutions of the Navier-Stokes equations is dissipating:

$$\frac{dE}{dt} = -\nu \mathcal{D}(t), \tag{3.31}$$

where $\mathcal{D}(t)$ is the enstrophy

$$\mathcal{D}(t) = \int_{\mathbb{R}^n} |\nabla u|^2 dx. \tag{3.32}$$

For incompressible flows, the enstrophy can be expressed purely in terms of vorticity using the identity

$$|\omega|^2 = \varepsilon_{ijk}\varepsilon_{imn}(\partial_j u_k)(\partial_m u_n) = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})(\partial_j u_k)(\partial_m u_n) = |\nabla u|^2 - (\partial_j u_k)(\partial_k u_j). \tag{3.33}$$

Note that

$$\int_{\mathbb{R}^n} (\partial_j u_k)(\partial_k u_j) dx = -\int_{\mathbb{R}^n} u_k (\partial_k \partial_j u_j) dx = 0. \tag{3.34}$$

We used the incompressibility condition on u in the last step. This implies that the enstrophy for a divergence-free flow is

$$\mathcal{D}(t) = \int_{\mathbb{R}^n} |\omega|^2 dx. \tag{3.35}$$

Therefore, large vorticity leads to increased energy dissipation – this, however, does not automatically lead to regularity.

An important comment is that the above computations assume that the solution $u(t,x)$ of the Navier-Stokes equations is sufficiently smooth. The possibility of energy dissipation as the solutions potentially develop a singularity is an extremely important open question.

3.5 Conservation of helicity

The helicity of a flow is

$$\mathcal{H} = \int_{\mathbb{R}^3} (u \cdot \omega) dx. \tag{3.36}$$

This definition is non-trivial only in three dimensions, as in two dimensions we have, for any incompressible flow,

$$\begin{aligned} \int_{\mathbb{R}^2} u_1 \omega dx &= \int_{\mathbb{R}^2} u_1 \Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \Big) dx = - \int_{\mathbb{R}^2} \Big(u_2 \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \frac{\partial (u_1^2)}{\partial x_2} \Big) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \frac{\partial}{\partial x_2} (u_2^2 - u_1^2) dx = 0, \end{aligned} \tag{3.37}$$

with a similar computaiton for u_2. Once again we used above incompressibility of $u(t,x)$.

In three dimensions, however, helicity is a non-trivial quantity, and, for the solutions of the Euler equations, we may compute

$$\frac{d\mathcal{H}}{dt} = \int_{\mathbb{R}^3} (u_t \cdot \omega + u \cdot \omega_t) dx. \tag{3.38}$$

We have

$$u_t \cdot \omega + (u \cdot \nabla u) \cdot \omega + \omega \cdot \nabla p = 0, \tag{3.39}$$

and

$$u \cdot \omega_t + (u \cdot \nabla \omega) \cdot u = u \cdot (\omega \cdot \nabla u). \tag{3.40}$$

The last term in (3.39) integrates to zero since $\nabla \cdot \omega = 0$:

$$\int_{\mathbb{R}^3} (\omega \cdot \nabla p) dx = 0. \tag{3.41}$$

The other terms lead to

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= - \int_{\mathbb{R}^3} (u_k (\partial_k u_j) \omega_j + u_k u_j \partial_j \omega_k - u_j \omega_k \partial_k u_j) dx = \\ &= - \int_{\mathbb{R}^3} (-u_k u_j \partial_k \omega_j + u_k u_j \partial_j \omega_k + \frac{1}{2} |u|^2 \partial_k \omega_k) dx = 0. \end{aligned} \tag{3.42}$$

Here, we have integrated by parts in the first term in the right side and used incompressibility of u to show that the first two terms in the right side cancel each other, while the last term vanishes after integration by parts because $\nabla \cdot \omega = 0$. Thus, helicity is preserved for the solutions of the Euler equations. In particular, the velocity field and the vorticity can not be "too aligned" in any growth or blow-up scenario for the Euler equations.

4 The Constantin-Lax-Majda toy model

4.1 The formulation of the model

In order to appreciate the difficulties of the problem of the regularity for the solutions of the Euler and the Navier-Stokes equations, and in particular, focus on the effect vortex stretching term, we consider here a toy model studied by Constantin, Lax and Majda in 1985. The vortex stretching term in the three-dimensional vorticity equation for the Euler equation

$$\omega_t + u \cdot \nabla\omega = \omega \cdot \nabla u, \tag{4.1}$$

has the form (2.104) – once again, it should not be taken too literally because of the singularity in the integral,

$$\nabla u(x)\text{''} = \text{''} \int_{\mathbb{R}^3} \nabla K(x-y) \times \omega(y) dy, \tag{4.2}$$

with

$$K(x) = -\frac{1}{4\pi}\frac{x}{|x|^3}. \tag{4.3}$$

The Constantin-Lax-Majda model aims to imitate three important properties of the right side in the vorticity equation (4.1): first, it is quadratic in ω, second, its integral vanishes:

$$\int_{\mathbb{R}^3} \omega \cdot \nabla u \; dx = 0. \tag{4.4}$$

The third feature is that the kernel $\nabla K(x)$ has the singularity of the type $x/|x|^4$, which is of the kind $x/|x|^{n+1}$ in n dimensions that is "barely non-integrable". Integral operators with such kernels are known as Calderon-Zygmund operators. Constantin, Lax and Majda considered a one-dimensional model, with an analogous singularity in one dimension

$$\frac{\partial\omega(t,x)}{\partial t} = H[\omega]\omega, \quad x \in \mathbb{R}, \tag{4.5}$$

with the initial condition $\omega(0,x) = \omega_0(x)$. Here, $H(\omega)$ is the Hilbert transform, a singular integral operator in one dimension:

$$H[\omega](x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{\omega(y)}{x-y} dy. \tag{4.6}$$

The principal value above is understood as

$$H[\omega](x) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{|y|>\varepsilon} \frac{\omega(x-y)}{y} dy = \frac{1}{\pi}\int_{|y|>1} \frac{\omega(x-y)}{y} dy + \frac{1}{\pi}\int_{-1}^{1} \frac{\omega(x-y)-\omega(x)}{y} dy. \tag{4.7}$$

The singularity $1/x$ in the kernel of the one-dimensional Hilbert transform is analogous to the singularity $x/|x|^4$ in three dimensions that appears in the kernel ∇K in (4.2): both are odd, and their size is $1/|x|^n$.

4.2 The toyest model of all

Before proceeding with the analysis of the Constantin-Lax-Majda model, let us pause and see what would happen if we would consider the simplest model that would preserve only the quadratic nature of the nonlinearity in the vorticity equation:

$$\frac{d\omega(t,x)}{dt} = \omega^2(t,x), \quad \omega(0,x) = \omega_0(x), \quad x \in \mathbb{R}. \tag{4.8}$$

Its explicit solution is

$$\omega(t,x) = \frac{\omega_0(x)}{1 - t\omega_0(x)}. \tag{4.9}$$

If there exist $x \in \mathbb{R}$ so that $\omega_0(x) > 0$, this solution makes sense until the denominator vanishes, that is, until the time

$$T_c = \inf\Big[\frac{1}{\omega_0(x)} :\ \omega_0(x) > 0\Big]. \tag{4.10}$$

Let us assume that the function $\omega_0(x)$ attains its maximum at $x = x_m$, so that $T_c = 1/\omega_0(x_m)$. The function $\omega(t,x)$ at the time $t = T_c$ has an asymptotic expansion near the point $x = x_m$:

$$\omega(T_c,x) = \frac{\omega_0(x)}{1 - T_c\omega_0(x)} \approx \frac{\omega_0(x_m)}{-(T_c/2)\omega_0''(x_m)(x-x_m)^2}. \tag{4.11}$$

Thus, the function $\omega(t,x)$ blows up at the point x_m and the blow-up profile is $O(x-x_m)^{-2}$. As a consequence, all L^p-norms of $\omega(t,x)$ blow up as well:

$$\int_{\mathbb{R}} |\omega(t,x)|^p dx \to +\infty \text{ as } t \uparrow T_c, \tag{4.12}$$

for all $p \geq 1$. Moreover, if we define the "velocity" as the anti-derivative of vorticity:

$$v(t,x) = \int_{-\infty}^{x} \omega(t,y)dy, \tag{4.13}$$

then $v(t,x)$ also blows-up at the time T_c and its blow-up profile is $O(x-x_m)^{-1}$. Therefore, the L^p-norms of the velocity blows up as well:

$$\int_{\mathbb{R}} |v(t,x)|^p dx \to +\infty \text{ as } t \uparrow T_c, \tag{4.14}$$

for all $p \geq 1$. In particular, the kinetic energy blows up:

$$\int_{\mathbb{R}} |v(t,x)|^2 dx \to +\infty \text{ as } t \uparrow T_c. \tag{4.15}$$

This is in contrast to the energy conservation in the true Euler equations. Thus, the toy model (4.8) can not be even "toyishly" correct. This example is intended simply to show that some models are too "toy" to be even considered!

4.3 The Hilbert transform

In order to understand the Constantin-Lax-Majda model, let us first recall some basic properties of the Hilbert transform and its alternative definition in terms of complex analysis. Given a Schwartz class function $f(x) \in \mathcal{S}(\mathbb{R})$ define a function

$$u(x,y) = \int_{\mathbb{R}} e^{-2\pi y|\xi|}\hat{f}(\xi)e^{2\pi i x\xi}d\xi, \quad y \geq 0, \quad x \in \mathbb{R}.$$

Here, the Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx, \quad f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi. \tag{4.16}$$

The function $u(x, y)$ is harmonic in the upper half plane:

$$\Delta_{x,y} u = 0 \text{ in } \mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty),$$

and satisfies the boundary condition on the line $y = 0$:

$$u(x, 0) = f(x), \quad x \in \mathbb{R}.$$

We can write $u(x, y)$ as a convolution

$$u(x, y) = P_y \star f = \int P_y(x - x') f(x') dx',$$

with

$$\hat{P}_y(\xi) = e^{-2\pi y |\xi|},$$

and

$$P_y(x) = \int_{-\infty}^{\infty} e^{-2\pi y|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{2\pi(y - ix)} + \frac{1}{2\pi(y + ix)} = \frac{y}{\pi(x^2 + y^2)}.$$

Next, set $z = x + iy$ and write

$$u(z) = \int_{\mathbb{R}} e^{-2\pi y|\xi|} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \int_0^{\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi + \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi.$$

Consider the function $v(z)$ given by

$$iv(z) = \int_0^{\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi - \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi.$$

Note that, as $f(x)$ is real-valued, we have $\overline{\hat{f}(\xi)} = \hat{f}(-\xi)$, thus $v(z)$ is real-valued:

$$\begin{aligned} i\bar{v}(z) &= -\int_0^{\infty} \overline{\hat{f}(\xi)} e^{-2\pi i \bar{z} \xi} d\xi + \int_{-\infty}^{0} \overline{\hat{f}(\xi)} e^{-2\pi i z \xi} d\xi \\ &= -\int_0^{\infty} \hat{f}(-\xi) e^{-2\pi i \bar{z} \xi} d\xi + \int_{-\infty}^{0} \hat{f}(-\xi) e^{-2\pi i z \xi} d\xi = \int_0^{\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi - \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi \\ &= iv(z). \end{aligned} \tag{4.17}$$

Moreover, as the function

$$u(z) + iv(z) = \int_0^{\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi$$

is analytic in the upper half-plane $\{\mathrm{Im} z > 0\}$, the function v is the harmonic conjugate of u. It can be written as

$$v(z) = \int_{\mathbb{R}} (-i\mathrm{sgn}(\xi)) e^{-2\pi y|\xi|} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = Q_y \star f,$$

with

$$\hat{Q}_y(\xi) = -i\mathrm{sgn}(\xi) e^{-2\pi y|\xi|}, \tag{4.18}$$

and

$$Q_y(x) = -i \int_{-\infty}^{\infty} \mathrm{sgn}(\xi) e^{-2\pi y|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi} \frac{x}{x^2 + y^2}.$$

The Poisson kernel and its conjugate are related by

$$P_y(x) + iQ_y(x) = \frac{i}{\pi(x + iy)} = \frac{1}{i\pi z},$$

which is analytic in $\{\mathrm{Im} z \geq 0\}$.

In order to consider the limit of Q_y as $y \to 0$, we relate it to the principal value of $1/x$ defined as in (4.7): it is an element of the space $\mathcal{S}'(\mathbb{R})$ of the Schwartz distributions, defined by

$$\mathrm{P.V.}\frac{1}{x}(\phi) = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\phi(x)}{x} dx = \int_{|x|<1} \frac{\phi(x) - \phi(0)}{x} dx + \int_{|x|>1} \frac{\phi(x)}{x} dx, \quad \phi \in \mathcal{S}(\mathbb{R}), \tag{4.19}$$

which is well-defined for $\phi \in \mathcal{S}(\mathbb{R})$. The conjugate Poisson kernel Q_y and the principal value of $1/x$ are related as follows.

Proposition 4.1. *Let $Q_y = \frac{1}{\pi} \frac{x}{x^2 + y^2}$, then for any function $\phi \in \mathcal{S}(\mathbb{R})$*

$$\frac{1}{\pi} P.V.\frac{1}{x}(\phi) = \lim_{y \to 0} \int_{\mathbb{R}} Q_y(x)\phi(x) dx.$$

Proof. Let

$$\psi_y(x) = \frac{1}{x} \chi_{y < |x|}(x)$$

so that

$$\mathrm{P.V.}\frac{1}{x}(\phi) = \lim_{y \to 0} \int_{\mathbb{R}} \psi_y(x)\phi(x) dx.$$

Note, however, that

$$\begin{aligned}
&\int (\pi Q_y(x) - \psi_y(x))\phi(x) dx = \int_{\mathbb{R}} \frac{x\phi(x)}{x^2 + y^2} dx - \int_{|x|>y} \frac{\phi(x)}{x} dx \\
&= \int_{|x|<y} \frac{x\phi(x)}{x^2 + y^2} dx + \int_{|x|>y} \left[\frac{x}{x^2 + y^2} - \frac{1}{x} \right] \phi(x) dx \\
&= \int_{|x|<1} \frac{x\phi(xy)}{x^2 + 1} dx - \int_{|x|>y} \frac{y^2\phi(x)}{x(x^2 + y^2)} dx = \int_{|x|<1} \frac{x\phi(xy)}{x^2 + 1} dx - \int_{|x|>1} \frac{\phi(xy)}{x(x^2 + 1)} dx.
\end{aligned} \tag{4.20}$$

The dominated convergence theorem implies that both integrals on the utmost right side above tend to zero as $y \to 0$. $\Box$

It is important to note that the computation in (4.20) worked only because the kernel $1/x$ is odd – this produces the cancellation that saves the day. This would not happen, for instance, for a kernel behaving as $1/|x|$ near $x = 0$.

Thus, the Hilbert transform defined as

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy. \tag{4.21}$$

can be also written as

$$Hf(x) = \lim_{y \to 0} Q_y \star f(x). \tag{4.22}$$

In other words, we take the function $f(x)$, extend it as a harmonic function $u(x,y)$ to the upper half-plane, and find the conjugate harmonic function $v(x,y)$. Then, $Hf(x) = v(x,0)$, the restriction of $v(x,y)$ to the real axis. It follows from (4.18) that

$$\widehat{Hf}(\xi) = \lim_{t \downarrow 0} \hat{Q}_t(\xi)\hat{f}(\xi) = -i\mathrm{sgn}(\xi)\hat{f}(\xi). \tag{4.23}$$

Therefore, the Hilbert transform may be extended to an isometry $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, with

$$\|Hf\|_{L^2} = \|f\|_{L^2}, \;\; H(Hf) = -f, \tag{4.24}$$

and

$$\int (Hf)(x)g(x)dx = -\int f(x)(Hg)(x)dx. \tag{4.25}$$

4.4 Back to the Constantin-Lax-Majda model

Let us now return to the CLM model

$$\omega_t = H[\omega]\omega, \;\; \omega(0,x) = \omega_0(x). \tag{4.26}$$

The term $H[\omega]\omega$ in the right side of (4.26) is similar to the vorticity stretching term $D\omega$ in the true three-dimensional vorticity equation in the three aspects we have discussed above, below (4.3). It is quadratic in ω, it follows from (4.25) that the operator H is skew-symmetric:

$$\int_{\mathbb{R}} H[\omega](x)\omega(x)dx = 0, \tag{4.27}$$

so the right side of (4.26) integrates to zero, as in (4.4), and the kernel $1/x$ has the correct singularity – it is odd and of the size $1/|x|^n$ (where n is the dimension). It follows from (4.27) that the integral of the solution of the toy model (4.26) is preserved:

$$\frac{d}{dt}\int_{\mathbb{R}} \omega(t,x)dx = 0. \tag{4.28}$$

Given a function ϕ, let us now use the "complex analysis" definition of $\psi = H[\phi]$, and set $u(x,y)$ and $v(x,y)$ so that the function $f = u+iv$ is analytic in $\{y > 0\}$, with the boundary values $u(x,0) = \phi(x)$, $v(x,0) = \psi(x)$. As we may write

$$-if^2 = 2uv + i(v^2 - u^2), \tag{4.29}$$

it follows that the harmonic conjugate of uv is $(v^2 - u^2)/2$. Restricting this identity to the real line gives

$$H(\phi H[\phi]) = \frac{1}{2}(H[\phi])^2 - \frac{1}{2}\phi^2. \tag{4.30}$$

Applying the Hilbert transform to the toy vorticity equation gives then

$$\frac{d}{dt}H[\omega] = \frac{1}{2}(H[\omega])^2 - \frac{\omega^2}{2}. \tag{4.31}$$

Therefore, the function

$$w(t,x) = H[\omega](t,x) + i\omega(t,x) \tag{4.32}$$

satisfies the simple quadratic ODE

$$\frac{dw}{dt} = \frac{1}{2}(H[\omega])^2 - \frac{1}{2}\omega^2 + iH[\omega]\omega = \frac{1}{2}w^2. \tag{4.33}$$

Hence, the function $w(t,x)$ is given explicitly by

$$w(t,x) = \frac{w_0(x)}{1 - \frac{1}{2}tw_0(x)}. \tag{4.34}$$

Taking the imaginary part of (4.34) gives an explicit formula for the solution of the toy vorticity equation:

$$\begin{aligned} \omega(t,x) &= \mathrm{Im}\frac{w_0(x)}{1 - \frac{1}{2}tw_0(x)} = \mathrm{Im}\frac{2(H[\omega_0](x) + i\omega_0(x))}{2 - t(H[\omega_0](x) + i\omega_0(x))} \\ &= \mathrm{Im}\frac{2(H[\omega_0](x) + i\omega_0(x))(2 - tH[\omega_0](x) + it\omega_0(x))}{(2 - tH[\omega_0](x))^2 + t^2(\omega_0(x))^2} \\ &= 2\frac{t\omega_0(x)H[\omega_0](x) + \omega_0(x)(2 - tH[\omega_0](x))}{(2 - tH[\omega_0](x))^2 + t^2(\omega_0(x))^2} = \frac{4\omega_0(x)}{(2 - tH[\omega_0](x))^2 + t^2(\omega_0(x))^2}. \end{aligned} \tag{4.35}$$

The explicit formula

$$\omega(t,x) = \frac{4\omega_0(x)}{(2 - tH[\omega_0](x))^2 + t^2(\omega_0(x))^2}. \tag{4.36}$$

gives an explicit criterion for the solution of the vorticity to exist for all times $t > 0$. Namely, the solution $\omega(t,x)$ exists and remains smooth provided that there does not exist a point $x \in \mathbb{R}$ so that both $\omega_0(x) = 0$ and $H[\omega_0](x) > 0$. The explicit breakdown time for a smooth solution is then

$$T_c = \inf\Big\{\frac{2}{H[\omega_0](x)} : \;\; \omega_0(x) = 0, \; H[\omega_0](x) > 0\Big\}. \tag{4.37}$$

As an example, consider $\omega_0(x) = \cos x$, so that $H[\omega_0](x) = \sin x$, and

$$\omega(t,x) = \frac{4\cos x}{(2 - t\sin x)^2 + t^2\cos^2 x} = \frac{4\cos x}{4 + t^2 - 4t\sin x}. \tag{4.38}$$

The breakdown time $T_c = 2$, at the point $x = \pi/2$, and the corresponding "toy velocity" is

$$v(t,x) = \int_0^x \omega(t,y)dy = \frac{1}{t}\log(1 + \frac{t^2}{4} - t\sin x). \tag{4.39}$$

Therefore,

$$\int_{-\pi}^{\pi} |\omega(t,x)|^p dx \to +\infty \tag{4.40}$$

as $t \uparrow T_c$, for any $1 \le p < \infty$. On the other hand, the L^p-norms of the velocity stay finite:

$$\int_{-\pi}^{\pi} |v(t,x)|^p dx \to M_p < +\infty, \tag{4.41}$$

for all $1 \le p < +\infty$, as $t \to\uparrow T_c$. In particular, the kinetic energy does not blow-up at the time T_c:

$$\int_{-\pi}^{\pi} |v(t,x)|^2 dx \to M_2 < +\infty, \tag{4.42}$$

This is in contrast to what happens in the "most toyest" model (4.8), where, the kinetic energy blows up at the blow-up time. Thus, while the Constantin-Lax-Majda model does not necessarily capture the physics of the Euler equations, it provides a "reasonable" one-dimensional playground.

5 The weak solutions to the Navier-Stokes equations

We will now start looking at the existence and regularity of the solutions of the Navier-Stokes equations. In order to focus on the less technical points, we will consider the periodic solutions to the Navier-Stokes equations:

$$\begin{aligned} &u_t + u\cdot\nabla u - \nu\Delta u + \nabla p = f(t,x),\\ &\nabla\cdot u = 0,\\ &u(0,x) = u_0. \end{aligned} \tag{5.1}$$

Here, f is the forcing term, and $u_0(x)$ is the initial condition. We assume both to be 1-periodic in all directions: $f(t, x + e_j) = f(t,x)$, $u_0(x + e_j) = u_0(x)$, with $j = 1,2$ in $\mathbb{R}^2$ and $j = 1,2,3$ in $\mathbb{R}^3$. We will look for periodic in x solutions to (5.1) in $\mathbb{R}^n$, $n = 2,3$.

Note first that, integrating (5.1) over $\mathbb{T}^n$ and using the incompressibility of $u(t,x)$, we deduce that the integral of u is conserved if $f = 0$:

$$\langle u\rangle(t) = \int_{\mathbb{T}^n} u(t,x)dx = 0. \tag{5.2}$$

Here, $\mathbb{T}^n = [0,1]^n$ is the unit torus. When $f \neq 0$, (5.2) holds, provided that $\langle f \rangle = 0$ for all $t \geq 0$.

Generally, we have a separate equation for $\langle u \rangle$:

$$\frac{d\langle u \rangle}{dt} = \langle f \rangle, \tag{5.3}$$

hence $\bar{u}(t) = \langle u(t,\cdot) \rangle$ is explicit:

$$\bar{u}(t) = \bar{u}(0) + \int_0^t \langle f(s,\cdot) \rangle ds.$$

Then, we can set

$$X(t) = \int_0^t \bar{u}(s)ds,$$

and observe that the man-zero flow

$$v(t,x) = u(t, x + X(t)) - \bar{u}(t),$$

satisfies the forced Navier-Stokes equations

$$\begin{aligned} &v_t + v \cdot \nabla v - \nu \Delta v + \nabla p = g(t,x), \\ &\nabla \cdot v = 0, \\ &v(0,x) = v_0, \end{aligned} \tag{5.4}$$

with the force

$$g(t,x) = f(t, x + X(t)) - \langle f(t,\cdot) \rangle.$$

With that change of variable, both the initial condition $v_0(x)$ and the force $g(t,x)$ are still 1-periodic in x, but, in addition, $\langle g(t,\cdot) \rangle = 0$ for all $t \geq 0$. Thus, we may assume without loss of generality that $\langle f \rangle = 0$, and (5.2) holds.

The two and three dimensional cases are very different. In two dimensions, we will eventually be able to show existence of regular solutions for all $t > 0$, provided that the forcing $f(t,x)$ and the initial condition $u_0(x)$ are sufficiently regular. On the other hand, in three dimensions, we will only be able to show that there exists a time $T_c > 0$ that depends on the force f and the initial condition u_0 so that the solution of the Navier-Stokes equations remains regular until the time T_c. However, if both the initial data and the forcing are sufficiently small (in a sense to be made precise later), then solutions of the Navier-Stokes equations remain regular for all times $t > 0$. This will be shown using the dominance of diffusion over the nonlinearity for small data.

5.1 The definition of the weak solutions

The distinction between two and three dimensions is less dramatic if we talk about weak solutions. As is usual in the theory of weak solutions of partial differential equations, the definition of a weak solution of the Navier-Stokes equations (5.1) comes from multiplying the equation by a smooth test function and integrating by parts. First, we note that any test vector field ψ can be decomposed as a sum of a gradient field and a divergence-free field:

$$\psi(x) = \phi(x) + \nabla\eta(x), \tag{5.5}$$

with $\nabla\cdot\phi(x) = 0$. This is known as the Hodge decomposition. In the periodic case the Hodge decomposition is quite explicit: write $\psi(x)$ in terms of the Fourier transform

$$\psi(x) = \sum_{k\in\mathbb{Z}^n} \psi_k e^{2\pi i k\cdot x}, \tag{5.6}$$

and consider the potential

$$\eta(x) = \sum_{k\in\mathbb{Z}^n, k\neq 0} \frac{(\psi_k \cdot k)}{2\pi i|k|^2} e^{2\pi i k\cdot x}. \tag{5.7}$$

Its gradient is

$$\nabla\eta(x) = \sum_{k\in\mathbb{Z}^n, k\neq 0} \frac{(\psi_k \cdot k)}{|k|^2} k e^{2\pi i k\cdot x}. \tag{5.8}$$

The Fourier coefficients of the difference

$$\phi(x) = \psi(x) - \nabla\eta(x) = \sum_{k\in\mathbb{Z}^n, k\neq 0} \Big(\psi_k - \frac{(\psi_k \cdot k)}{|k|^2} k\Big) e^{2\pi i k\cdot x} \tag{5.9}$$

are

$$\phi_k = \psi_k - \frac{(\psi_k \cdot k)}{|k|^2} k. \tag{5.10}$$

They satisfy

$$\phi_k \cdot k = 0, \tag{5.11}$$

which implies that the vector field $\phi(x)$ is divergence-free:

$$\nabla\cdot\phi(x) = 0. \tag{5.12}$$

Let now $u(t,x)$ be a smooth solution of the Navier-Stokes equations

$$u_t + u\cdot\nabla u + \nabla p = \nu\Delta u + g, \tag{5.13}$$
$$\nabla\cdot u = 0. \tag{5.14}$$

We will also use the Hodge decomposition of the forcing term

$$g = f + \nabla\zeta \text{ with } \nabla\cdot f = 0. \tag{5.15}$$

The first observation is that if we multiply (5.13) by $\nabla\eta(x)$ and integrate, then we simply get the Poisson equation for the pressure. Indeed, if w is a smooth periodic vector field, and $\nabla\cdot w = 0$, then

$$\int_{\mathbb{T}^n} w(x)\cdot\nabla\eta(x)dx = -\int_{\mathbb{T}^n} \eta(x)(\nabla\cdot w)(x)dx = 0. \tag{5.16}$$

It follows that
$$\int_{\mathbb{T}^n}(u_t\cdot\nabla\eta)dx=\int_{\mathbb{T}^n}(\Delta u\cdot\nabla\eta)dx=0. \tag{5.17}$$
For the pressure we have:
$$\int_{\mathbb{T}^n}(\nabla p\cdot\nabla\eta)dx=-\int_{\mathbb{T}^n}p\Delta\eta dx, \tag{5.18}$$
while for the nonlinear term we get, after an integration by parts, using the divergence-free condition on u:
$$\int_{\mathbb{T}^n}((u\cdot\nabla u)\cdot\nabla\eta)dx=\int_{\mathbb{T}^n}u_j(\partial_j u_k)\partial_k\eta dx=-\int_{\mathbb{T}^n}u_ju_k(\partial_j\partial_k\eta)dx. \tag{5.19}$$
We deduce that, for any test function $\eta(x)$, we have
$$\int_{\mathbb{T}^n}(p\Delta\eta+u_ju_k(\partial_j\partial_k\eta)dx=\int_{\mathbb{T}^n}g\cdot\nabla\eta=\int_{\mathbb{T}^n}\nabla\zeta\cdot\nabla\eta. \tag{5.20}$$
This is the weak form of the Poisson equation
$$-\Delta p=(\partial_j u_k)(\partial_k u_j)-\Delta\zeta. \tag{5.21}$$
On the other hand, when we multiply (5.13) by a divergence-free smooth vector field $w(x)$, the pressure term disappears:
$$\int_{\mathbb{T}^n}(w\cdot\nabla p)dx=0, \tag{5.22}$$
and the nonlinear term may be written as
$$\int_{\mathbb{T}^n}((u\cdot\nabla u)\cdot w)dx=\int_{\mathbb{T}^n}u_j(\partial_j u_k)w_k dx=-\int_{\mathbb{T}^n}u_ju_k\partial_j w_k dx. \tag{5.23}$$
Thus, if w is a $C^\infty(\mathbb{T}^n)$ periodic divergence-free field, integration by parts gives
$$\int_{\mathbb{T}^n}[u_t\cdot w-u_ju_k\partial_j w_k]dx=\nu\int_{\mathbb{T}^n}(u\cdot\Delta w)dx+\int_{\mathbb{T}^n}(f\cdot w)dx. \tag{5.24}$$
For now, we say that $u(t,x)$ is a weak solution of the Navier-Stokes equations if (5.24) holds for all periodic smooth divergence-free vector fields $w(x)$. A little later, we will make this notion more precise, setting up the proper spaces in which the weak solutions live, and relaxing the C^∞ assumption on the test function. Note that this definition completes sidesteps the issue of the pressure field.

5.2 The Galerkin approximation

In order to construct the weak solutions, we will consider the Galerkin approximation of the Navier-Stokes equations. In the periodic case, this is equivalent to the projection of the equations on the divergence-free Fourier modes with $|k|\le m$, where $m>0$ is fixed. That is, given a vector-field
$$\psi(x)=\sum_{k\in\mathbb{Z}^n}a_ke^{2\pi ik\cdot x}, \tag{5.25}$$

we set

$$\psi^{(m)}(x) = P_m\psi(x) = \sum_{|k|\le m} \Big(a_k - \frac{(a_k\cdot k)}{|k|^2}k\Big)e^{2\pi i k\cdot x}, \tag{5.26}$$

so that, in particular,

$$\nabla\cdot\psi^{(m)} = 0. \tag{5.27}$$

Note that if ψ is a divergence-free vector field then $\psi^{(m)}$ is simply the projection on the Fourier modes with $|k|\le m$.

The Galerkin approximation of the Navier-Stokes equations

$$u_t + u\cdot\nabla u + \nabla p = \nu\Delta u + f, \tag{5.28}$$

with $u(0,x) = u_0(x)$, and a divergence-free force f: $\nabla\cdot f = 0$, is the system

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)}\cdot\nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)},\ \ u^{(m)}(0) = u_0^{(m)}. \tag{5.29}$$

This is a finite-dimensional constant coefficients system of quadratic ODE's for the Fourier coefficients u_m of the function $u(x)$ with $|k|\le m$. If the function f is time-independent, this system is autonomous. The goal is obtain bounds on the solution $u^{(m)}$ of the Galerkin system that would allow us to pass to the limit $m\to+\infty$, leading to a weak solution of the Navier-Stokes equations.

5.2.1 A bound on the energy and enstrophy for the Galerkin solutions

We fix an arbitrary time $T>0$ throughout the analysis of the Galerkin system. As (5.29) is a system of constant coefficient non-linear ODEs for the coefficients u_k, $|k|\le m$, it has a solution for a sufficiently small time $t>0$ (which a priori may depend on the initial data $u_0^{(m)}$, as well as on m). Unlike partial differential equations, such ODEs may lose solutions only via the blow-up of the energy

$$\|u^{(m)}\|_2^2 = \sum_{|k|\le m} |u_k|^2, \tag{5.30}$$

and that, as we will now show, can not happen in a finite time for any finite m. Indeed, we have

$$\int_{\mathbb{T}^n} (P_m(u^{(m)}\cdot\nabla u^{(m)})\cdot u^{(m)})dx = \int_{\mathbb{T}^n} ((u^{(m)}\cdot\nabla u^{(m)})\cdot u^{(m)})dx = 0. \tag{5.31}$$

We used the definition of the projection P_m in the first identity, and the incompressibility of $u^{(m)}$ in the second. Therefore, multiplying (5.29) by $u^{(m)}$ and integrating, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n} |u^{(m)}|^2dx = -\nu\int_{\mathbb{T}^n} |\nabla u^{(m)}|^2dx + \int_{\mathbb{T}^n} (f^{(m)}\cdot u^{(m)})dx. \tag{5.32}$$

We will now use the Poincaré inequality

$$4\pi^2\int_{\mathbb{T}^n} |\phi|^2dx \le \int_{\mathbb{T}^n} |\nabla\phi|^2, \tag{5.33}$$

that holds for all mean-zero periodic functions ϕ. With its help, identity (5.32) implies that $E(t) = \|u^{(m)}\|_2^2$ satisfies

$$\frac{1}{2}\frac{dE}{dt} \le -4\pi^2\nu E(t) + \|f^{(m)}\|_2\sqrt{E(t)} \le -4\pi^2\nu E(t) + 2\pi^2\nu E(t) + \frac{1}{8\pi^2\nu}\|f\|_2^2$$
$$\le -2\pi^2\nu E(t) + \frac{1}{8\pi^2\nu}\|f\|_2^2. \tag{5.34}$$

Therefore, we have the inequality

$$\frac{d}{dt}\Big(E(t)e^{4\pi^2\nu t}\Big) \le \frac{1}{4\pi^2\nu}\|f\|_2^2 e^{4\pi^2\nu t}. \tag{5.35}$$

Integrating in time leads to an estimate

$$E(t) \le E(0)e^{-4\pi^2\nu t} + \frac{1}{4\pi^2\nu}\int_0^t e^{-4\pi^2\nu(t-s)}\|f(s)\|_2^2 ds. \tag{5.36}$$

The estimate (5.36) relies only on the finiteness of the L^2-norm of the forcing f. Another way to estimate the right side in (5.32), relying only on the finiteness of a weaker norm of f, is to use the inequality

$$\Big|\int_{\mathbb{T}^n}(f\cdot g)dx\Big| = \Big|\sum_{k\in\mathbb{Z}^n} f_k g_k\Big| \le \Big(\sum_{k\in\mathbb{Z}^n} 4\pi^2k^2|g_k|^2\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}^n}\frac{|f_k|^2}{4\pi^2k^2}\Big)^{1/2} = \|\nabla g\|_2\|f\|_{H^{-1}}, \tag{5.37}$$

with the H^{-1}-norm defined as in the above inequality. Using this inequality in (5.32) gives

$$\frac{1}{2}\frac{dE}{dt} \le -\nu\|\nabla u^{(m)}\|_2^2 + \|\nabla u^{(m)}\|_2\|f\|_{H^{-1}} \le -\nu\|\nabla u^{(m)}\|_2^2 + \frac{\nu}{2}\|\nabla u^{(m)}\|_2^2 + \frac{1}{2\nu}\|f\|_{H^{-1}}^2$$
$$= -\frac{\nu}{2}\|\nabla u^{(m)}\|_2^2 + \frac{1}{2\nu}\|f\|_{H^{-1}}^2. \tag{5.38}$$

Now, we use the Poincaré inequality to obtain:

$$\frac{dE}{dt} \le -C_1\nu E + \frac{C_2}{\nu}\|f\|_{H^{-1}}^2, \tag{5.39}$$

with universal constants C_1 and C_2. Integrating this differential inequality in time leads to another estimate for $E(t)$, which involves only $\|f\|_{H^{-1}}$ and not $\|f\|_2$:

$$E(t) \le E(0)e^{-C_1\nu t} + \frac{C_2'}{\nu}\int_0^t e^{-C_1\nu(t-s)}\|f(s)\|_{H^{-1}}^2 ds. \tag{5.40}$$

The same argument provides a time-averaged bound on the enstrophy $D(t) = \|\nabla u(t)\|_2^2$. Indeed, integrating inequality (5.38) in time leads to

$$\frac{1}{2}\|u^{(m)}(T)\|_2^2 + \frac{\nu}{2}\int_0^T\int_{\mathbb{T}^n}|\nabla u^{(m)}(s,x)|^2 dxds \le \frac{1}{2}\|u_0^{(m)}\|_2^2 + \frac{1}{2\nu}\int_0^T \|f^{(m)}(s)\|_{H^{-1}}^2 ds. \tag{5.41}$$

5.2.2 The function spaces and an intermediate summary

Now, we need to introduce certain spaces. We denote by H the space of all mean-zero vector-valued functions u in the space $[L^2(\mathbb{T}^n)]^n$, with zero divergence (in the sense of distributions):

$$H = \{u \in L^2(\mathbb{T}^n) : \ \nabla \cdot u = 0, \ \langle u \rangle = 0\}, \tag{5.42}$$

with the inner product

$$(f, g) = \int_{\mathbb{T}^n} (f \cdot g) dx. \tag{5.43}$$

In other words, a vector field $u \in H$ if its Fourier coefficients in the expansion

$$u(x) = \sum_{k \in \mathbb{Z}^n} u_k e^{2\pi i k \cdot x} \tag{5.44}$$

satisfy $u_0 = 0$, $k \cdot u_k = 0$ for all $k \in \mathbb{Z}^n_* = \mathbb{Z}^n \setminus \{0\}$, and

$$\|u\|_H^2 = \sum_{k \in \mathbb{Z}^n_*} |u_k|^2 < +\infty. \tag{5.45}$$

We also denote by V the space of divergence-free functions in the Sobolev space $H^1(\mathbb{T}^n)$:

$$V = \{u \in H^1(\mathbb{T}^n) : \ \nabla \cdot u = 0, \ \langle u \rangle = 0\}, \tag{5.46}$$

with the inner product

$$\langle f, g \rangle = \int_\Omega (\frac{\partial u}{\partial x_i} \cdot \frac{\partial g}{\partial x_i}) dx, \tag{5.47}$$

for two vector-valued functions f and g. That is, $u \in V$ if its Fourier coefficients satisfy $u_0 = 0$, as well as $k \cdot u_k = 0$ for all k, and

$$\|u\|_V^2 = \sum_{k \in \mathbb{Z}^n_*} |k|^2 |u_k|^2 < +\infty. \tag{5.48}$$

The dual space to V consists of all distributions with the Fourier coefficients that satisfy

$$\|u\|_{V'}^2 = \sum_{k \in \mathbb{Z}^n_*} \frac{|u_k|^2}{|k|^2} < +\infty, \ u_0 = 0 \text{ and } k \cdot u_k = 0. \tag{5.49}$$

We will occasionally use the Sobolev spaces H^s, $s \in \mathbb{R}$, of divergence-free functions: we say that $u \in H^s(\mathbb{T}^n)$ if its Fourier coefficients u_k satisfy

$$u_0 = 0, \ k \cdot u_k = 0 \text{ and } \|u\|_{H^s} = \left(\sum_{k \in \mathbb{Z}^n_*} |k|^{2s} |u_k|^2 \right)^{1/2} < +\infty. \tag{5.50}$$

We have, with this notation $V = H^1$ and $V' = H^{-1}$. The spaces $L^2(0, T; H)$ and $L^2(0, T; V)$ have the respective norms

$$\|u\|_{L^2(0,T;H)}^2 = \int_0^T \|u(t)\|_H^2 dt, \quad \|u\|_{L^2(0,T;V)}^2 = \int_0^T \|u(t)\|_V^2 dt. \tag{5.51}$$

Summarizing our analysis of the Galerkin system so far, and rephrasing the results in terms of the spaces H, V and V', we have proved the following.

Proposition 5.1. *Assume that* $f \in L^\infty(0,T;H)$. *Then, the Galerkin system (5.29) has a unique solution* $u^{(m)} \in L^2(0,T;V) \cap L^\infty(0,T;H)$. *More precisely, there exist two universal constants* $C_1 > 0$ *and* $C_2 > 0$ *so that*

$$\|u^{(m)}(t)\|_H^2 \leq \|u_0\|_H^2 e^{-4\pi^2\nu t} + \frac{1}{4\pi^2\nu}\int_0^t e^{-4\pi^2\nu(t-s)}\|f(s)\|_H^2 ds, \tag{5.52}$$

$$\|u^{(m)}(t)\|_H^2 \leq \|u_0\|_H^2 e^{-C_1\nu t} + \frac{C_2}{\nu}\int_0^t e^{-C_1\nu(t-s)}\|f(s)\|_{V'}^2 ds \tag{5.53}$$

$$\nu\int_0^T \|u^{(m)}(s)\|_V^2 ds \leq \|u_0\|_2^2 + \frac{1}{2\nu}\int_0^T \|f(s)\|_{V'}^2 ds. \tag{5.54}$$

5.3 The Galerkin approximation: bounds on the time derivative

5.3.1 The time derivative estimate

The next step is obtain bounds on the time derivative of $u^{(m)}$. They will be needed in the passage to the limit $m \to +\infty$, to ensure that the limit is weakly continuous in time. Let us write the Galerkin approximation of the Navier-Stokes equaitons as

$$\frac{\partial u^{(m)}}{\partial t} = \nu\Delta u^{(m)} - P_m(u^{(m)}\cdot\nabla u^{(m)}) + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}. \tag{5.55}$$

We will aim to obtain the following bounds on $u_t^{(m)}$. The estimates are slightly different in two and three dimensions.

Proposition 5.2. *Assume that* $f \in L^2(0,T;V')$. *There exists a constant* C *which depends on the norm* $\|u_0\|_H$ *of the initial conditionu*$_0$, *the* $L^2(0,T;V')$*-norm of the forcing* f, *and the viscosity* ν *but not on* m, *so that the solution to the Galerkin system (5.29) in dimension* $n = 3$ *satisfies the estimate*

$$\int_0^T \Big\|\frac{\partial u^{(m)}}{\partial t}(t)\Big\|_{V'}^{4/3} \leq C. \tag{5.56}$$

and in dimension $n = 2$ *it satisfies*

$$\int_0^T \Big\|\frac{\partial u^{(m)}}{\partial t}(t)\Big\|_{V'}^{2} \leq C. \tag{5.57}$$

For the proof, we will estimate individually each of the terms in the right side of (5.55). As we assume that $f \in L^2(0,T;V')$, the forcing term in is not a problem either in dimension two or three. The Laplacian term in (5.55) is also bounded in $L^2(0,T;V')$, as follows from (5.54): the Fourier coefficients of Δu are $|k|^2 u_k$, hence

$$\|\Delta u\|_{V'}^2 = \sum_{k\in\mathbb{Z}^n}\frac{|k|^4}{|k|^2}|u_k|^2 = \|u\|_V^2, \tag{5.58}$$

thus

$$\int_0^T \|\Delta u^{(m)}(s)\|_{V'}^2 ds = \int_0^T \|u^{(m)}(s)\|_V^2 ds \leq \frac{1}{\nu}\|u_0^{(m)}\|_2^2 + \frac{1}{2\nu^2}\int_0^T \|f^{(m)}(s)\|_{V'}^2 ds. \tag{5.59}$$

The nonlinear term will require the most effort. We will establish the following bounds.

Lemma 5.3. *There exists a constant C that so that in two dimensions we have, for any function $u \in V$:*

$$\|(u \cdot \nabla u)\|_{V'} \le C\|u\|_H \|u\|_V, \quad n = 2, \tag{5.60}$$

and in three dimensions we have

$$\|(u \cdot \nabla u)\|_{V'} \le C\|u\|_H^{1/2} \|u\|_V^{3/2}, \quad n = 3. \tag{5.61}$$

Together with the uniform energy bound (5.53) and the enstrophy bound (5.54), this implies the conclusion of Proposition 5.2. Indeed, in dimension $n = 2$, (5.60) gives

$$\int_0^T \|P_m(u \cdot \nabla u)(s)\|_{V'}^2 ds \le \int_0^T \|(u \cdot \nabla u)(s)\|_{V'}^2 ds \le (\sup_{0 \le t \le T} \|u(t)\|_H^2) \int_0^T \|u(s)\|_V^2 ds \le C,$$

and in dimension $n = 3$, (5.61) leads to

$$\int_0^T \|P_m(u \cdot \nabla u)(s)\|_{V'}^{4/3} ds \le \int_0^T \|(u \cdot \nabla u)(s)\|_{V'}^{4/3} ds \le (\sup_{0 \le t \le T} \|u(t)\|_H^{2/3}) \int_0^T \|u(s)\|_V^2 ds \le C.$$

Thus the proof of Proposition 5.2 is reduced to proving Lemma 5.3.

5.3.2 The proof of Lemma 5.3: bounds on the nonlinear term

Note that

$$\|(u \cdot \nabla u)\|_{V'} = \|(-\Delta)^{-1/2}(u \cdot \nabla u)\|_H. \tag{5.62}$$

The operator $(-\Delta)^{-1/2}$ is defined via its action on the Fourier coefficients of a mean-zero function $u(x)$:

$$(-\Delta)^{-1/2} u(x) = \sum_{k \in \mathbb{Z}^n} \frac{u_k}{|k|} e^{2\pi i k \cdot x}. \tag{5.63}$$

This operator commutes with the projection P_m, as, in particular, it preserves the incompressibility of u. Hence, Lemma 5.3 can be restated as follows.

Lemma 5.4. *Let $u \in V$, then in three dimensions we have the estimate*

$$\|(-\Delta)^{-1/2}(u \cdot \nabla u)\|_H \le C\|u\|_H^{1/2} \|u\|_V^{3/2}, \tag{5.64}$$

while in two dimensions we have

$$\|(-\Delta)^{-1/2}(u \cdot \nabla u)\|_H \le C\|u\|_H \|u\|_V, \tag{5.65}$$

Proof. In this proof, we will use interchangeably the notation $\|u\|_{H^1}$ and $\|u\|_V$, since the divergence-free property plays almost no role in the proof. Take an arbitrary $u \in H$ and $w \in H$ and write, for the inner product in H:

$$((-\Delta)^{-1/2}(u \cdot \nabla u), w) = ((u \cdot \nabla u), (-\Delta)^{-1/2} w). \tag{5.66}$$

In three dimensions, we will show

Lemma 5.5. *In dimension $n = 3$, for any $u, v, w \in V$ we have*

$$|((u \cdot \nabla v), w)| \le C\|u\|_{H^{1/2}}\|v\|_{H^1}\|w\|_{H^1}. \tag{5.67}$$

Applying this estimate in (5.66) gives

$$|((-\Delta)^{-1/2}(u \cdot \nabla u), w)| = |((u \cdot \nabla u), (-\Delta)^{-1/2}w)| \le C\|u\|_{H^{1/2}}\|u\|_{H^1}\|(-\Delta)^{-1/2}w)\|_{H^1}. \tag{5.68}$$

As

$$\|(-\Delta)^{-1/2}w)\|_{H^1} = \|w\|_H, \tag{5.69}$$

and

$$\|u\|^2_{H^{1/2}} = \sum_{k\in\mathbb{Z}^n} |k||u_k|^2 \le \left(\sum_{k\in\mathbb{Z}^n} |k|^2|u_k|^2\right)^{1/2} \left(\sum_{k\in\mathbb{Z}^n} |u_k|^2\right)^{1/2} = \|u\|_H\|u\|_V, \tag{5.70}$$

we deduce from (5.66) that in three dimensions we have

$$|((-\Delta)^{-1/2}(u \cdot \nabla u), w)| \le C\|u\|_H^{1/2}\|u\|_V^{3/2}\|w\|_H. \tag{5.71}$$

As this estimate holds for all $w \in H$, (5.64) follows.

In two dimensions, we will show

Lemma 5.6. *In dimension $n = 2$, we have*

$$|((u \cdot \nabla v), u)| \le C\|u\|_H\|u\|_{H^1}\|v\|_{H^1}. \tag{5.72}$$

To see that this implies (5.65), we write, using incompressibility of u:

$$((-\Delta)^{-1/2}(u \cdot \nabla u), w) = ((u \cdot \nabla u), (-\Delta)^{-1/2}w) = -((u \cdot \nabla(-\Delta)^{-1/2}w), u). \tag{5.73}$$

Applying estimate (5.72) in (5.73) gives

$$\begin{aligned} |((-\Delta)^{-1/2}(u \cdot \nabla u), w)| &= |((u \cdot \nabla(-\Delta)^{-1/2}w), u)| \\ &\le C\|u\|_H\|(-\Delta)^{-1/2}w\|_{H^1}\|u\|_{H^1} = C\|u\|_H\|u\|_{H^1}\|w\|_H. \end{aligned} \tag{5.74}$$

As this holds for any $w \in H$, we conclude that (5.65) holds in two dimensions.

Thus, we only need to verify (5.67) in three dimensions and (5.72) in two dimensions to finish the proof of Lemma 5.4.

Proof of Lemma 5.5. In three dimensions, we use Hölder's inequality to get

$$\begin{aligned} |((u \cdot \nabla v), w)| &\le \int_{\mathbb{T}^3} |u_j(\partial_j v_k)w_k|dx \le \|u\|_{L^3(\mathbb{T}^3)}\|\nabla v\|_{L^2(\mathbb{T}^3)}\|w\|_{L^6(\mathbb{T}^3)} \\ &= \|u\|_{L^3(\mathbb{T}^3)}\|v\|_{H^1(\mathbb{T}^3)}\|w\|_{L^6(\mathbb{T}^3)}. \end{aligned} \tag{5.75}$$

The Sobolev inequality says that, for $m < n/2$,

$$\|f\|_{L^q(\mathbb{T}^n)} \le C\|f\|_{H^m(\mathbb{T}^n)}, \tag{5.76}$$

as long as

$$\frac{1}{q} \geq \frac{1}{2} - \frac{m}{n}. \tag{5.77}$$

Therefore, in dimension $n = 3$, taking $q = 3$ and $m = 1/2$ we have

$$\|u\|_{L^3(\mathbb{T}^3)} \leq C\|u\|_{H^{1/2}}, \tag{5.78}$$

while taking $q = 6$ and $m = 1$, we obtain

$$\|w\|_{L^6(\mathbb{T}^3)} \leq C\|w\|_{H^1(\mathbb{T}^3)}. \tag{5.79}$$

It follows then from (5.75) that

$$|((u \cdot \nabla v), w)| \leq \|u\|_{L^3(\mathbb{T}^3)}\|v\|_{H^1(\mathbb{T}^3)}\|w\|_{L^6(\mathbb{T}^3)} \leq C\|u\|_{H^{1/2}(\mathbb{T}^3)}\|v\|_{H^1(\mathbb{T}^3)}\|w\|_{H^1(\mathbb{T}^3)}, \tag{5.80}$$

which is (5.67).

Proof of Lemma 5.6. In two dimensions, we proceed similarly: Hölder's inequality implies

$$|((u \cdot \nabla v), w)| \leq \|u\|_{L^4(\mathbb{T}^2)}\|w\|_{L^4(\mathbb{T}^2)}\|v\|_{H^1(\mathbb{T}^2)}. \tag{5.81}$$

The Sobolev inequality (5.76) in two dimensions, with $q = 4$ and $m = 1/2$ implies that

$$\|f\|_{L^4(\mathbb{T}^2)} \leq C\|f\|_{H^{1/2}(\mathbb{T}^2)}. \tag{5.82}$$

Using this in (5.81) leads to

$$|((u \cdot \nabla v), w)| \leq \|u\|_{L^4(\mathbb{T}^2)}\|w\|_{L^4(\mathbb{T}^2)}\|v\|_{H^1(\mathbb{T}^2)} \leq C\|u\|_{H^{1/2}(\mathbb{T}^2)}\|w\|_{H^{1/2}(\mathbb{T}^2)}\|v\|_{H^1(\mathbb{T}^2)}. \tag{5.83}$$

As

$$\|u\|^2_{H^{1/2}} \leq \|u\|_H\|u\|_{H^1}, \tag{5.84}$$

we obtain

$$|((u \cdot \nabla v), w)| \leq C(\|u\|_H\|u\|_{H^1}\|w\|_H\|w\|_{H^1})^{1/2}\|v\|_{H^1(\mathbb{T}^2)}, \tag{5.85}$$

hence

$$|((u \cdot \nabla v), u)| \leq C\|u\|_H\|u\|_{H^1}\|v\|_{H^1(\mathbb{T}^2)}, \tag{5.86}$$

which is (5.72). This finishes the proof of Lemma 5.4. □

5.4 A compactness theorem

We have deduced above uniform in m a priori bounds on the solution $u^{(m)}$ of the Galerkin system

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)} \cdot \nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}. \tag{5.87}$$

The next step is to use these uniform bounds to show that the sequence $u^{(m)}$ has a (strongly) convergent subsequence in $L^2(0, T; H)$. As we will see, the limit of this subsequence will be a weak solution of the Navier-Stokes equations. We will use the following compactness result.

Proposition 5.7. *Let u_m be a sequence of functions satisfying*

$$\|u_m(t)\|_H \leq C, \tag{5.88}$$

for all $0 \leq t \leq T$,

$$\int_0^T \|u_m(s)\|_V^2 ds \leq C, \text{ for all } m = 1, 2, \dots \tag{5.89}$$

and

$$\int_0^T \left\| \frac{\partial u^{(m)}}{\partial t}(t) \right\|_{V'}^p \leq C, \text{ for all } m = 1, 2, \dots, \tag{5.90}$$

with some $C > 0$ and $p > 1$. Then there exists a subsequence u_{m_j} of u_m which converges strongly in $L^2(0,T;H)$ to a function $u \in L^2(0,T;V)$.

Proof. The uniform bound (5.89) implies that there exists a subsequence u_{m_j} which converges weakly in $L^2(0,T;V)$ to a function $u \in L^2(0,T;V)$, which also obeys the bound (5.89). In addition, using the diagonal argument, we may ensure that the sequence of time derivatives $u_t^{(m)}$ converges weakly to the derivative u_t in $L^p(0,T;V')$. Thus, the estimate (5.90) also holds for the function u. The difference

$$w_j = u_{m_j} - u$$

converges weakly to zero in $L^2(0,T;V)$, and the bounds (5.88)-(5.90) hold for w_j as well. Our goal is to prove that the convergence of w_j to zero is strong in $L^2(0,T;H)$.

Note that for any $f \in V$

$$\|f\|_H \leq (\|f\|_V \|f\|_{V'})^{1/2}, \tag{5.91}$$

hence, for any $\delta > 0$ we have

$$\|f\|_H^2 \leq \delta \|f\|_V^2 + \frac{1}{\delta} \|f\|_{V'}^2. \tag{5.92}$$

The uniform bound (5.89) for the functions w_j and (5.92) imply

$$\int_0^T \|w_j\|_H^2 dt \leq C\delta + \frac{1}{\delta} \int_0^T \|w_j\|_{V'}^2 dt. \tag{5.93}$$

Our goal is to estimate the second term in (5.93), and show that it goes to zero as $j \to +\infty$, with $\delta > 0$ fixed. Note that

$$\|w_j(t)\|_{V'} \leq \|w_j(t)\|_H \leq C. \tag{5.94}$$

Thus, the Lebesgue dominated convergence theorem shows that it suffices to show that

$$\|w_j\|_{V'} \to 0 \text{ pointwise in } t \in [0,T]. \tag{5.95}$$

To this end, given a time $\varepsilon > 0$ and $\varepsilon \leq t \leq T$, let us write

$$w_j(t,x) = w_j(s,x) + \int_s^t \frac{\partial w_j(\tau,x)}{\partial \tau} d\tau, \tag{5.96}$$

and average this identity over $s \in [t-\varepsilon, t]$:

$$w_j(t,x) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} w_j(s,x)ds + \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} ds \int_{s}^{t} \frac{\partial w_j(\tau,x)}{\partial \tau} d\tau$$
$$= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} w_j(s,x)ds + \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} (\tau - t + \varepsilon)\frac{\partial w_j(\tau,x)}{\partial \tau} d\tau. \tag{5.97}$$

In order to bound the first term, note that for any $0 \le a \le b \le T$ the integral

$$I_j(x) = \int_a^b w_j(t,x)dt \tag{5.98}$$

converges weakly to zero in V. Indeed, for any $v \in V'$, the function $\chi_{[a,b]}(t)v(x)$ is an element of $L^2(0,T;V')$, and $w_j \to 0$ weakly in $L^2(0,T;V)$, thus we have

$$\int_{\mathbb{T}^n} I_j(x)v(x)dx = \int_0^T \int_{\mathbb{T}^n} w_j(t,x)\chi_{[a,b]}(t)v(x)dxdt \to 0 \text{ as } j \to \infty. \tag{5.99}$$

As V is compactly embedded into H, weak convergence in V implies strong convergence in H: the sequence I_j converges strongly to zero in H. Thus, it also converges strongly to zero in V'. In particular, given any $\varepsilon > 0$ and $\delta > 0$, for all j sufficiently large we have

$$\frac{1}{\varepsilon}\left\|\int_{t-\varepsilon}^{t} w_j(s,x)ds\right\|_{V'} < \delta \text{ for } j \ge J(\varepsilon,\delta,t), \tag{5.100}$$

giving a pointwise in time estimate for the first term in (5.97). For the second term in (5.97), we may use the Minkowski inequality, followed by Hölder's inequality, with $1/q + 1/p = 1$:

$$\frac{1}{\varepsilon}\left\|\int_{t-\varepsilon}^{t} (\tau - t + \varepsilon)\frac{\partial w_j(\tau,x)}{\partial \tau} d\tau\right\|_{V'} \le \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} (\tau - t + \varepsilon)\left\|\frac{\partial w_j(\tau,x)}{\partial \tau}\right\|_{V'} d\tau \tag{5.101}$$
$$\le \frac{1}{\varepsilon}\left(\int_{t-\varepsilon}^{t} (\tau - t + \varepsilon)^q d\tau\right)^{1/q}\left(\int_{t-\varepsilon}^{t} \left\|\frac{\partial w_j(\tau,x)}{\partial \tau}\right\|_{V'}^p d\tau\right)^{1/p}$$
$$\le C\varepsilon^{1/q}\left(\int_0^T \left\|\frac{\partial w_j(\tau,x)}{\partial \tau}\right\|_{V'}^p d\tau\right)^{1/p} \le C\varepsilon^{1/q},$$

for all $j \ge 1$. It is here that the assumption $p > 1$ is used, so that $q < +\infty$.. It follows from the above analysis that, given any $\varepsilon > 0$ and $\delta > 0$, we may find $J(\varepsilon,\delta,t)$ so that

$$\|w_j(t)\|_{V'} \le \delta + C\varepsilon^{1/q}, \text{ for all } j \ge J(\varepsilon,\delta,t). \tag{5.102}$$

In other words, we have shown that

$$\|w_j(t)\|_{V'} \to 0 \text{ as } j \to \infty, \text{ pointwise in } t \in [0,T]. \tag{5.103}$$

As we have explained above, we may use the Lebesgue dominated convergence theorem to conclude from (5.93) that the sequence w_j converges strongly to zero in $L^2(0,T;H)$. This finishes the proof of Proposition 5.7. □

5.5 The weak solutions as limits of the Galerkin solutions

We will now construct the weak solutions of the Navier-Stokes equations as a limit of the solutions $u^{(m)}$ of the Galerkin system as $m \to \infty$. In particular, the definition of the weak solution we will adopt is motivated by the estimates on $u^{(m)}$ we have obtained above. We say that $u \in C_w(0,T;H)$ if the function $\psi(t) = (u(t), h)$ is continuous for all $h \in H$.

Definition 5.8. *A function u is a weak solution to the (periodic) Navier-Stokes equations*

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f(t,x), \quad t > 0, \ x \in \mathbb{T}^n, \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{5.104}$$

if

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap C_w(0,T;H) \text{ and } \frac{\partial u}{\partial t} \in L^1_{loc}(0,T;V'), \tag{5.105}$$

and, for any $v \in V$, we have

$$\begin{aligned} &\int_{\mathbb{T}^n} u(t,x) \cdot v(x) dx + \nu \int_0^t \int_{\mathbb{T}^n} \nabla u \cdot \nabla v dx ds + \int_0^t \int_{\mathbb{T}^n} ((u \cdot \nabla u) \cdot v) dx ds \\ &\quad = \int_{\mathbb{T}^n} u_0(x) \cdot v(x) dx + \int_0^t \int f \cdot v dx ds, \text{ for all } v \in V \text{ and } 0 \le t \le T. \end{aligned} \tag{5.106}$$

Let us check that each term in (5.106) makes sense if u satisfies (5.105), and $v \in V$. The first term is finite since $u \in L^\infty(0,T;H)$. The second is finite since $u \in L^2(0,T;V)$. The last term in the left side is finite in three dimensions because of the estimate (5.67):

$$|((u \cdot \nabla u), v)| \le C \|u\|_{H^{1/2}} \|u\|_{H^1} \|v\|_{H^1} \le C \|u\|_H^{1/2} \|u\|_V^{3/2} \|v\|_V, \tag{5.107}$$

as $\|u\|_H$ is uniformly bounded in t, and $u \in L^2(0,T;V)$. In two dimensions, this term is bounded because of the estimate (5.72):

$$|((u \cdot \nabla u), v)| = |((u \cdot \nabla v), u)| \le C \|u\|_H \|u\|_V \|v\|_V, \tag{5.108}$$

again, because $\|u\|_H$ is uniformly bounded in t, and $u \in L^2(0,T;V)$.

Finally, the right side in (5.106) is finite provided that $f \in L^2(0,T;V')$ and $u_0 \in H$. The following theorem, due to Leray, is one of the most classical results in the mathematical theory of the Navier-Stokes equations (we state here its simpler version for the periodic case).

Theorem 5.9. *Given $u_0 \in H$ and $f \in L^2(0,T;V')$, there exists a weak solution of the Navier-Stokes equations*

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \quad t > 0, \ x \in \mathbb{T}^n, \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0(x). \end{aligned} \tag{5.109}$$

In addition, this weak solution satisfies the energy inequality

$$\frac{1}{2}\int_{\mathbb{T}^n} |u(t,x)|^2 dx + \nu \int_0^t \int_{\mathbb{T}^n} |\nabla u(s,x)|^2 dxds \leq \frac{1}{2}\int_{\mathbb{T}^n} |u_0(x)|^2 dx + \int_0^t \int_{\mathbb{T}^n} f(s,x)\cdot u(s,x) dxds. \tag{5.110}$$

Moreover, we have

$$\frac{\partial u}{\partial t} \in L^{4/3}(0,T;V') \text{ in dimension } n=3, \tag{5.111}$$

and

$$\frac{\partial u}{\partial t} \in L^{2}(0,T;V') \text{ in dimension } n=2. \tag{5.112}$$

Proof. Let $u^{(m)}$ be the solutions of the Galerkin system (5.29):

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)}\cdot\nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}. \tag{5.113}$$

The estimates we have obtained in the previous section imply that, after extracting a subsequence, $u^{(m)}$ converge strongly in $L^2(0,T;H)$ and weakly in $L^2(0,T;V)$ to some u. Moreover, the functions $u^{(m)}$ satisfy a uniform continuity in time bound in V':

$$u^{(m)}(t) - u^{(m)}(s) = \int_s^t \frac{\partial u^{(m)}}{\partial \tau} d\tau, \tag{5.114}$$

thus

$$\begin{aligned}\|u^{(m)}(t) - u^{(m)}(s)\|_{V'} &\leq \int_s^t \left\|\frac{\partial u^{(m)}}{\partial \tau}\right\|_{V'} d\tau \leq (t-s)^{1/q}\left(\int_s^t \left\|\frac{\partial u^{(m)}}{\partial \tau}\right\|_{V'}^p d\tau\right)^{1/p} \\ &\leq (t-s)^{1/q}\left(\int_0^T \left\|\frac{\partial u^{(m)}}{\partial \tau}\right\|_{V'}^p d\tau\right)^{1/p} \leq C(t-s)^{1/q},\end{aligned} \tag{5.115}$$

with $p=q=2$ in dimension $n=2$, and $p=4/3$, $q=4$ in dimension $n=3$. Thus, u obeys the same uniform continuity estimate estimate, and $u\in C(0,T;V')$. We also know that

$$\frac{\partial u^{(m)}}{\partial t} \to \frac{\partial u}{\partial t},$$

weakly in $L^{4/3}(0,T;V')$ in three dimensions, and weakly in $L^2(0,T;V')$ in two dimensions.

Given any $v\in V$ we multiply the Galerkin system (5.113) by v and integrate:

$$\begin{aligned}&\int_{\mathbb{T}^n} u^{(m)}(t,x)v(x)dx + \int_0^t\int_{\mathbb{T}^n}(u^{(m)}\cdot\nabla u^{(m)})\cdot(P_m v)dxds \\ &= -\nu\int_0^t\int_{\mathbb{T}^n}\nabla u^{(m)}\cdot\nabla v dxds + \int_{\mathbb{T}^n} u_0^{(m)}(x)v(x)dx + \int_0^t\int_{\mathbb{T}^n} fv dxds.\end{aligned} \tag{5.116}$$

We pass now to the limit in this identity, looking at each term individually. The first term in the right side is easy:

$$\int_0^t\int_{\mathbb{T}^n}\nabla u^{(m)}\cdot\nabla v dxds \to \int_0^t\int_{\mathbb{T}^n}\nabla u\cdot\nabla v dxds, \tag{5.117}$$

because $u^{(m)}$ converges weakly to u in $L^2(0,T;V)$.

Next, we look at the nonlinear term: set

$$A_m = \int_0^t \int_{\mathbb{T}^n} (u^{(m)} \cdot \nabla u^{(m)}) \cdot (P_m v) dx ds - \int_0^t \int_{\mathbb{T}^n} (u \cdot \nabla u) \cdot v dx ds. \tag{5.118}$$

Let us recall (5.67):

$$|((u \cdot \nabla v), w)| \le C \|u\|_{H^{1/2}} \|v\|_{H^1} \|w\|_{H^1}. \tag{5.119}$$

This inequality holds both in two and three dimensions and implies that

$$\left| \int_0^t \int_{\mathbb{T}^n} (u \cdot \nabla u) \cdot (P_m v - v) dx ds \right| \le \left(\int_0^t \|u(s)\|_V^2 ds \right) \|P_m v - v\|_V \le C \|P_m v - v\|_V \to 0, \tag{5.120}$$

as $m \to \infty$. Hence, A_m has the same limit as $m \to \infty$ as

$$A'_m = \int_{t_0}^t \int_{\mathbb{T}^n} (u^{(m)} \cdot \nabla u^{(m)} - u \cdot \nabla u) \cdot (P_m v) dx ds = B_1 + B_2, \tag{5.121}$$

where $B_{1,2}$ correspond to the decomposition

$$\begin{aligned} u^{(m)} \cdot \nabla u^{(m)} - u \cdot \nabla u &= u^{(m)} \cdot \nabla u^{(m)} - u^{(m)} \cdot \nabla u + u^{(m)} \cdot \nabla u - u \cdot \nabla u \\ &= u^{(m)} \cdot (\nabla u^{(m)} - \nabla u) + (u^{(m)} - u) \cdot \nabla u. \end{aligned} \tag{5.122}$$

To estimate B_1, we write

$$B_1 = \int_0^t \int_{\mathbb{T}^n} (u^{(m)} \cdot (\nabla u^{(m)} - \nabla u)) \cdot (P_m v) dx ds = - \int_0^t \int_{\mathbb{T}^n} (u^{(m)} \cdot \nabla P_m v) \cdot (u^{(m)} - u) dx ds. \tag{5.123}$$

The same proof as for (5.67) shows that

$$|(u \cdot \nabla v), w)| \le \|u\|_V \|v\|_V \|w\|_{H^{1/2}}. \tag{5.124}$$

Using this in (5.123) gives

$$\begin{aligned} |B_1| &\le \int_{t_0}^t \|u^{(m)}(s)\|_V \|v\|_V \|u^{(m)}(s) - u(s)\|_{H^{1/2}} ds \\ &\le \|v\|_V \left(\int_0^t \|u^{(m)}(s)\|_V^2 ds \right)^{1/2} \left(\int_0^t \|u^{(m)}(s) - u(s)\|_{H^{1/2}}^2 ds \right)^{1/2} \\ &\le C \|v\|_V \left(\int_0^t \|u^{(m)}(s)\|_V^2 ds \right)^{1/2} \left(\int_0^t \|u^{(m)}(s) - u(s)\|_V^2 ds \right)^{1/2} \\ &\times \left(\int_0^t \|u^{(m)}(s) - u(s)\|_H^2 ds \right)^{1/4} \le C \|u^{(m)} - u\|_{L^2(0,T;H)} \to 0, \quad \text{as } m \to \infty, \end{aligned} \tag{5.125}$$

as $u^{(m)}$ converges to u strongly in $L^2(0,T;H)$. As for B_2, we write

$$\begin{aligned} |B_2| &= \left| \int_0^t \int_{\mathbb{T}^n} ((u^{(m)} - u) \cdot \nabla u) \cdot (P_m v) dx ds \right| \le \int_0^t \|u^{(m)}(s) - u(s)\|_{H^{1/2}} \|u(s)\|_V \|v\|_V ds \\ &\le \|v\|_V \|u\|_{L^2(0,T;V)} \|u^{(m)}(s) - u(s)\|_{L^2(0,T;H^{1/2})} \to 0, \end{aligned} \tag{5.126}$$

for the same reason as in (5.125).

In order to pass to the limit in the two terms in (5.116) that do not involve the time integration, we first note that $u_0^{(m)}$ converges strongly in H to u_0. Furthermore, as $u^{(m)}$ converges weakly to u in $L^2(0,T;V)$, we may extract a subsequence so that $u^{(m)}(t)$ converges weakly in V to $u(t)$ (pointwise in t), except for $t\in E$, where E is an exceptional set of times in $[0,T]$ of measure zero. Weak convergence in V implies that $u^{(m)}(t)$ converges strongly to $u(t)$ in H for $t\notin E$. Hence, taking $t\notin E$ and passing to the limit $m\to\infty$ in (5.116) we arrive at

$$\begin{aligned}\int_{\mathbb{T}^n} u(t,x)v(x)dx = \int_{\mathbb{T}^n} u_0(x)v(x)dx - \int_0^t\int_{\mathbb{T}^n}(u\cdot\nabla u)\cdot v dxds\\ -\nu\int_0^t\int_{\mathbb{T}^n}\nabla u\cdot\nabla v dxds + \int_0^t\int_{\mathbb{T}^n} f v dxds.\end{aligned} \tag{5.127}$$

Given the a priori bounds on u, the right side of (5.127) is a continuous function of t, defined for all $t\in[0,T]$, not just $t\in E$. In addition, we know that $\langle u(t),v\rangle$ is also continuous because $u\in C_w(0,T;V')$, and coincides with the aforementioned right side of (5.127) for $t\notin E$. This continuity implies that $\langle u(t),v\rangle$ coincides with the right side of (5.127) for all $0\le t\le T$, which means that it satisfies (5.127) for all $t\in[0,T]$, giving us a weak solution of the Navier-Stokes equations.

The fact that $u\in C_w(0,T;H)$, and not just $u\in C(0,T;V')$ follows from (5.127), the density of V in H and the uniform in t bound on $\|u(t)\|_H$.

To obtain the energy inequality, we start with the identity

$$\frac{1}{2}\|u^{(m)}(t)\|_H^2 + \nu\int_0^t \|u^{(m)}(s)\|_V^2 ds = \frac{1}{2}\|u_0^{(m)}\|_H^2 + \int_0^t\int_{\mathbb{T}^n} f\cdot u^{(m)} dxds. \tag{5.128}$$

The right side converges, as $m\to\infty$, to

$$\frac{1}{2}\|u_0\|_H^2 + \nu\int_{t_0}^t\int_{\mathbb{T}^n} f\cdot u dxds. \tag{5.129}$$

In the left side, we may use the Fatou lemma to conclude that, as $u^{(m)}(t)$ converges weakly in H to $u(t)$ for all $t\in[0,T]$, we have

$$\frac{1}{2}\|u(t)\|_H^2 + \nu\int_0^t \|u(s)\|_V^2 ds \le \frac{1}{2}\|u_0\|_H^2 + \int_0^t\int_{\mathbb{T}^n} f\cdot u dxds. \tag{5.130}$$

This completes the proof. □

5.6 Uniqueness of the weak solutions in two dimensions

One of the main issues with weak solutions in general in nonlinear partial differential equations is their uniqueness – it is often much easier to show that they exist than to prove their uniqueness. Uniqueness of a weak solution hints that it is a "correct" solution, while non-uniqueness means that an extra condition is needed to pick the physically meaningful solution. This happens, for instance, in the theory of conservation laws where the notion of an entropy solution guarantees uniqueness among all weak solutions. The problem of the uniqueness of the weak solutions for the Navier-Stokes equations in three dimensions is still open. In two dimensions, we know that the weak solutions of

$$\begin{aligned} & u_t + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad t > 0, \; x \in \mathbb{T}^2, \\ & \nabla \cdot u = 0, \\ & u(0,x) = u_0(x). \end{aligned} \tag{5.131}$$

are unique.

Theorem 5.10. *Let $f \in L^2(0,T;V')$ and $u_0 \in H$. If u_1 and u_2 are two weak solutions of (5.131) which both lie in $L^2(0,T;V) \cap L^\infty(0,T;H) \cap C_w(0,T;H)$, then $u_1 = u_2$.*

Proof. First, we recall, see Theorem 5.9, that if u is a weak solution of the Navier-Stokes equations (5.131) in $L^2(0,T;V) \cap L^\infty(0,T;H)$ in two dimensions, then $u_t \in L^2(0,T;V')$. Let us denote $w = u_1 - u_2$. This function satisfies

$$\begin{aligned} & w_t + u_1 \cdot \nabla w + w \cdot \nabla u_2 + \nabla p' = \nu \Delta w, \quad t > 0, \; x \in \mathbb{T}^2, \\ & \nabla \cdot w = 0, \\ & w(0,x) = 0, \end{aligned} \tag{5.132}$$

with $p' = p_1 - p_2$, and we know that $w_t \in L^2(0,T;V')$.

Multiplying (5.132) by w and integrating over the torus gives

$$\int_{\mathbb{T}^2} w_t \cdot w + \nu \int_{\mathbb{T}^2} |\nabla w|^2 dx + \int_{\mathbb{T}^2} w_k (\partial_j u_{2,m}) w_m dx = 0. \tag{5.133}$$

As $w_t \in V'$ for a.e. t, and $w \in V$ for a.e. $t \in [0,T]$, identity (5.133) holds for a.e. $t \in [0,T]$. Recall that in two dimensions we have

$$|(w \cdot \nabla u_2, w)| \le C \|w\|_H \|u_2\|_V \|w\|_V. \tag{5.134}$$

As $w \in L^\infty(0,T;H)$ and $u_2, w \in L^2(0,T,H)$, we conclude from (5.133) and (5.134) that

$$\int_0^T |(w_t(t), w(t))| dt < +\infty.$$

Now, (5.133) implies that

$$\frac{d}{dt} \|w\|_H^2 \le C \|w\|_H \|u_2\|_V \|w\|_V - \nu \|w\|_V^2 \le \frac{C}{\nu} \|u_2\|_V^2 \|w\|_H^2. \tag{5.135}$$

As

$$\int_0^T \|u_2\|_V^2 dt < +\infty,$$

Gronwall's inequality implies that

$$\|w(t)\|_H^2 \le \|w(0)\|_H^2 \exp\Big\{ \int_0^t \|u_2(s)\|_V^2 ds \Big\} = 0, \tag{5.136}$$

since $w(0)=0$. This finishes the proof. □

Note that this proof would fail in three dimensions. The reason is that in three dimensions the nonlinear term satisfies

$$|(w\cdot\nabla u_2,w)|\le C\|w\|_H^{1/2}\|u_2\|_V\|w\|_V^{3/2}, \tag{5.137}$$

rather than

$$|(w\cdot\nabla u_2),w|\le C\|w\|_H\|w\|_V\|u_2\|_V, \tag{5.138}$$

which holds in two dimensions. Thus, instead of (5.135), we would get, using Young's inequality

$$\frac{d}{dt}\|w\|_H^2\le C\|w\|_H^{1/2}\|u_2\|_V\|w\|_V^{3/2}-\nu\|w\|_V^2\le\frac{C}{\nu^3}\|u_2\|_V^4\|w\|_H^2. \tag{5.139}$$

As we do not have a uniform bound on

$$\int_0^T\|u(s)\|_V^4ds,$$

we would not be able to finish the proof using the Gronwall inequality. We will need extra assumptions for uniqueness, which is what we will discuss next.

6 Strong solutions in two and three dimensions

6.1 Uniqueness of strong solutions in three dimensions

We say that u is a strong solution of the Navier-Stokes equations (in either two or three dimensions) if u is a weak solution, and, in addition, $u\in C_w(0,T;V)$, and the following bounds hold:

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^n}|\nabla u(t,x)|^2dx<+\infty, \tag{6.1}$$

and

$$\int_0^T\int_{\mathbb{T}^n}|\Delta u(t,x)|^2dxdt<+\infty. \tag{6.2}$$

The motivation for this definition comes from two properties that we will prove: first, unlike for the weak solutions, one can show that strong solutions are unique in three dimensions (existence of strong solutions in three dimensions is an important open problem). Second, as we will show, the conditions in the definition of the strong solutions are sufficient to show that they are actually infinitely differentiable if the initial condition u_0 and the forcing f are.

First, we prove their uniqueness in three dimensions.

Theorem 6.1. *Let $u_{1,2}$ be two solutions of the Navier-Stokes equations on $\mathbb{T}^3$ with the initial condition $u_0\in H$ and $f\in L^2(0,T;H)$. If both $u_{1,2}$ satisfy (6.1) and (6.2), and they lie in $C_w(0,T;V)$ then $u_1=u_2$.*

Proof. We argue as in the proof of uniqueness of the weak solutions in two dimensions. Let $w=u_1-u_2$, so that

$$(\frac{\partial w}{\partial t},w)+\nu\|w\|_V^2+(w\cdot\nabla u_2,w)=0, \tag{6.3}$$

as in (5.133). We now use the estimate

$$|((w \cdot \nabla u, w)| \leq C\|w\|_{L^2}\|w\|_{H^1}\|u\|_{H^1}^{1/2}\|\Delta u\|_2^{1/2}. \tag{6.4}$$

It is obtained as follows: recall that in three dimensions we have

$$\|w\|_{L^3(\mathbb{T}^3)} \leq C\|w\|_{H^{1/2}}, \tag{6.5}$$

thus

$$\begin{aligned}|((w \cdot \nabla u, w)| \leq \int_{\mathbb{T}^3} |w||\nabla u||w|dx &\leq \|w\|_{L^3}\|\nabla u\|_{L^3}\|w\|_{L^3} \leq C\|w\|_{H^{1/2}}^2\|\nabla u\|_{H^{1/2}} \\ &\leq C\|w\|_{L^2}\|w\|_{H^1}\|u\|_{H^1}^{1/2}\|\Delta u\|_{L^2}^{1/2},\end{aligned} \tag{6.6}$$

which is (6.4). Using the bound (6.4) in (6.3) leads to

$$\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2) + \nu\|w\|_{H^1}^2 \leq \frac{C}{\nu}\|w\|_{L^2}^2\|u\|_{H^1}\|\Delta u\|_2 + \nu\|w\|_{H^1}^2. \tag{6.7}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2) \leq \frac{C}{\nu}\|u\|_{H^1}\|\Delta u\|_2\|w\|_{L^2}^2. \tag{6.8}$$

Now, Grownwall's inequality implies that $w(t) = 0$ provided that $w(0) = 0$, and

$$\int_0^t \|u\|_{H^1}\|\Delta u\|_2 ds < +\infty, \tag{6.9}$$

which is a consequence of (6.1)-(6.2). □

6.2 Construction of the strong solutions in two dimensions

We now use the Galerkin system in two dimensions to show existence of global in time strong solutions of the Navier-Stokes equations in two dimensions. Once again, we restrict ourselves to the simpler case of the two-dimensional torus $\mathbb{T}^2$. As in the proof of the existence of weak solutions, we will use the Galerkin system

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)} \cdot \nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}, \tag{6.10}$$

and then pass to the limit $m \to +\infty$. However, we will be able to obtain better a priori bounds on the Galerkin system in two dimensions to conclude that in the limit we actually obtain strong solutions of the Navier-Stokes equations. Since we have already shown the uniqueness of the weak solutions in the two-dimensional case, this will also show that weak solutions are actually strong in two dimensions.

6.2.1 Galerkin solutions are often not large

The first step is to show that solutions of the Galerkin system are "often not large" – this will be made precise soon. The second step will be to show that if solutions are often not too large, then they can never be large.

Taking the inner product of (6.11) with $u^{(m)}$ we obtain the familiar identity

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|_H^2 + \nu\|\nabla u^{(m)}\|_H^2 = (f, u^{(m)}). \tag{6.11}$$

We may use the Poincaré inequality

$$\int_{\mathbb{T}^2} |u(x)|^2 dx = \sum_{k\in\mathbb{Z}^n} |u_k|^2 \le \sum_{k\in\mathbb{Z}^n} |k|^2|u_k|^2 = \frac{1}{4\pi^2}\int_{\mathbb{T}^n} |\nabla u|^2 dx, \tag{6.12}$$

to conclude from (6.11) that

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|_H^2 + \nu\|\nabla u^{(m)}\|_H^2 \le \frac{1}{2\cdot 4\pi^2\nu}\|f\|_H^2 + \frac{4\pi^2\nu}{2}\|u^{(m)})\|_H^2 \le \frac{1}{8\pi^2\nu}\|f\|_H^2 + \frac{\nu}{2}\|\nabla u^{(m)})\|_H^2. \tag{6.13}$$

We deduce the bounds we have seen before: there exist two explicit constants $C_{1,2} > 0$, so that

$$\nu\int_0^t \|\nabla u^{(m)}\|_V^2 ds \le \|u_0\|_H^2 + \frac{C_1}{\nu}\int_0^t \|f\|_H^2 ds, \tag{6.14}$$

and

$$\|u^{(m)}(t)\|_H^2 \le \|u_0\|_H^2 e^{-C_2\nu t} + \frac{C_1}{\nu}\int_0^t e^{-C_2\nu(t-s)}\|f\|_H^2 ds. \tag{6.15}$$

In particular, if $f \in L^\infty(0, T; H)$, then

$$\|u^{(m)}(t)\|_H^2 \le \|u_0\|_H^2 e^{-C_2\nu t} + \frac{C_1}{\nu^2}\|f\|_\infty^2, \tag{6.16}$$

with

$$\|f\|_\infty = \sup_{t>0}\|f(t)\|_H. \tag{6.17}$$

Our next goal is to get uniform in time bounds on $\|u^{(m)}(t)\|_V$ – this is not something we have done in the construction of the weak solutions, because such bound holds only in two dimensions, and not in three, while the weak solutions can be constructed both in two and three dimensions. The first step in that direction is to show that this norm can not be large for too long a time.

Proposition 6.2. *Let $u^{(m)}(t)$ be the solution for the Galerkin system with $f \in L^\infty(0, +\infty; H)$ and $u_0 \in H$, in either two or three dimensions. Then in every time interval of length $\tau > 0$ there exists a time t_0 so that*

$$\|u^{(m)}(t_0)\|_V^2 \le \frac{2}{\tau\nu}\Big(\|u_0\|_H^2 + \frac{C_1}{\nu}(\frac{1}{\nu} + \tau)\|f\|_\infty\Big). \tag{6.18}$$

Proof. Inequality (6.15) implies that

$$\nu \int_0^t \|\nabla u^{(m)}\|_V^2 ds \leq \|u_0\|_H^2 + \frac{C_1 t}{\nu}\|f\|_\infty^2, \tag{6.19}$$

and (6.15) that

$$\|u^{(m)}(t)\|_H^2 \leq \|u_0\|_H^2 + \frac{C_1}{\nu^2}\|f\|_\infty^2. \tag{6.20}$$

Let us also integrate (6.13) between the times t and $t+\tau$, leading to

$$\nu \int_t^{t+\tau} \|u^{(m)}(s)\|_V^2 ds \leq \|u^{(m)}(t)\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty \tau \leq \|u_0\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty(\frac{1}{\nu}+\tau). \tag{6.21}$$

The right side above does not depend on the time t. Therefore, on any time interval $[t, t+\tau]$ we may estimate the Lebesgue measure of the set of times when $\|u(s)\|_V$ is large:

$$\Big|\{s: \ s \in [t, t+\tau] \text{ s.t. } \|u^{(m)}(s)\|_V \geq \rho\}\Big| \leq \frac{1}{\nu\rho^2}\Big(\|u_0\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty(\frac{1}{\nu}+\tau)\Big). \tag{6.22}$$

In particular, taking

$$\rho_0 = \left[\frac{2}{\tau\nu}\left(\|u_0\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty(\frac{1}{\nu}+\tau)\right)\right]^{1/2},$$

we arrive at the conclusion of Proposition 6.2. □

6.2.2 Galerkin solutions are never large

Next, we will get rid of the "sometimes not large" restriction in Proposition 6.2, showing that in two dimensions Galerkin solutions are never large in V. We will prove the following estimate for the solutions of the Galerkin system

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)} \cdot \nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)}, \ \ u^{(m)}(0) = u_0^{(m)}. \tag{6.23}$$

Proposition 6.3. *Let $u^{(m)}$ be the solution of the Galerkin system (6.23) with the initial condition $u_0 \in H$ and $f \in L^\infty(0,T;H)$. There exists a constant α that depends on ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m so that $u^{(m)}$ satisfies the bounds*

$$\|u^{(m)}(t)\|_V \leq \alpha \text{ for all } t \geq 1, \tag{6.24}$$

and

$$\|u^{(m)}(t)\|_V \leq \frac{\alpha}{t} \text{ for all } 0 < t < 1. \tag{6.25}$$

In addition, if $u_0 \in V$ then there exists a constant α_1 which depends on ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m so that

$$\|u^{(m)}(t)\|_V \leq \alpha_1 \text{ for all } 0 < t < 1. \tag{6.26}$$

Proof. The idea is to use Proposition 6.2 – we know that for any time $t > 1$ there is a time $t_0 \in [t-1, t]$ so that the norm $\|u^{(m)}(t_0)\|_V \le \alpha$, with the constant α which depends only on ν, $\|u_0\|_H$ and $\|f\|_\infty$. The additional ingredient in this proof will be a control of the growth of $\|u^{(m)}\|_V$ on the time intervals of length 1.

We multiply (6.23) by Δu and integrate. The first term gives

$$\int_{\mathbb{T}^2} u_t^{(m)} \cdot \Delta u^{(m)} dx = -\int_{\mathbb{T}^2} \nabla u_t^{(m)} \cdot \nabla u^{(m)} dx = -\frac{1}{2}\frac{d}{dt}\|\nabla u^{(m)}(t)\|_H^2, \tag{6.27}$$

so that the overall balance is

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^{(m)}(t)\|_H^2 + \nu\|\Delta u^{(m)}\|_H^2 - ((u^{(m)} \cdot \nabla u^{(m)}), \Delta u^{(m)}) = -(f, \Delta u^{(m)}). \tag{6.28}$$

For the nonlinear term, we will use the inequality

$$|((u \cdot \nabla u), \Delta u)| \le \|u\|_H^{1/2}\|u\|_V\|\Delta u\|_H^{3/2}, \tag{6.29}$$

which holds in two dimensions. The proof is similar to that of (5.72): we write

$$|((u \cdot \nabla v), w)| \le \int_{\mathbb{T}^n} |(u_j \partial_j v_k) w_k| dx \le \|u \cdot \nabla v\|_{L^2}\|w\|_{L^2} \le \|u\|_{L^4}\|\nabla v\|_{L^4}\|w\|_{L^2}. \tag{6.30}$$

The Sobolev inequality

$$\|f\|_{L^q(\mathbb{T}^n)} \le C\|f\|_{H^m(\mathbb{T}^n)}, \qquad \frac{1}{q} \ge \frac{1}{2} - \frac{m}{n} \tag{6.31}$$

implies that in two dimensions we have

$$\|f\|_{L^4(\mathbb{T}^2)} \le C\|f\|_{H^{1/2}(\mathbb{T}^2)}. \tag{6.32}$$

Using this in (6.30) leads to

$$\begin{aligned}|((u \cdot \nabla u), \Delta u)| &\le \|u\|_{H^{1/2}}\|u\|_{H^{3/2}}\|\Delta u\|_{L^2} \le \|u\|_H^{1/2}\|u\|_V^{1/2}\|u\|_V^{1/2}\|\Delta u\|_H^{1/2}\|\Delta u\|_H \\ &= \|u\|_H^{1/2}\|u\|_V\|\Delta u\|_H^{3/2},\end{aligned} \tag{6.33}$$

which is (6.29). It follows that the nonlinear term can be estimated, using the inequality

$$ab \le \frac{\nu}{4}a^{4/3} + \frac{C}{\nu^3}b^4$$

as

$$|((u \cdot \nabla u), \Delta u)| \le \frac{\nu}{4}\|\Delta u\|_H^2 + \frac{C}{\nu^3}\|u\|_H^2\|u\|_V^4. \tag{6.34}$$

Returning to (6.28), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u^{(m)}(t)\|_H^2 + \nu\|\Delta u^{(m)}\|_H^2 &\le |((u^{(m)} \cdot \nabla u^{(m)}), \Delta u^{(m)})| + \|f\|_\infty\|\Delta u^{(m)}\|_H \\ &\le \frac{\nu}{4}\|\Delta u^{(m)}\|_H^2 + \frac{C}{\nu^3}\|u^{(m)}\|_H^2\|u^{(m)}\|_V^4 + \frac{\nu}{4}\|\Delta u^{(m)}\|_H^2 + \frac{C}{\nu}\|f\|_\infty^2.\end{aligned} \tag{6.35}$$

We conclude that

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2+\frac{\nu}{2}\|\Delta u^{(m)}\|_H^2\le\frac{C}{\nu^3}\|u^{(m)}\|_H^2\|u^{(m)}\|_V^4+\frac{C}{\nu}\|f\|_\infty^2. \tag{6.36}$$

Let us set

$$G(t_0;t)=\frac{2C}{\nu^3}\int_{t_0}^t\|u(s)\|_H^2\|u(s)\|_V^2ds, \tag{6.37}$$

then (6.36) implies, for any $t\ge t_0$:

$$\frac{d}{dt}\Big(\|u^{(m)}\|_V^2\exp\{-G(t_0;t)\}\Big)\le\frac{C}{\nu}\|f\|_\infty^2\exp\{-G(t_0;t)\}. \tag{6.38}$$

Integrating between t_0 and t gives

$$\begin{aligned}\|u^{(m)}(t)\|_V^2&\le\|u^{(m)}(t_0)\|_V^2\exp\{G(t_0;t)\}+\frac{C}{\nu}\|f\|_\infty^2\exp\{G(t_0;t)\}\int_{t_0}^t\exp\{-G(t_0;s)\}ds\\&\le\|u^{(m)}(t_0)\|_V^2\exp\{G(t_0;t)\}+\frac{C}{\nu}\|f\|_\infty^2\int_{t_0}^t\exp\{G(s;t)\}ds\\&\le\|u^{(m)}(t_0)\|_V^2\exp\{G(t_0;t)\}+\frac{C}{\nu}\|f\|_\infty^2(t-t_0)\exp\{G(t_0;t)\}.\end{aligned} \tag{6.39}$$

Now we will use the "sometimes small" result in Proposition 6.2. Given $\tau>0$ and $t>\tau$ we may find $t_0\in[t-\tau,t]$ such that

$$\|u(t_0)\|_V\le\alpha(1+\frac{1}{\tau}), \tag{6.40}$$

with the constant $\alpha>0$ that only depends on ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m or $\|u_0\|_V$. We may also use (6.21) to estimate $G(t_0;t)$:

$$G(t_0;t)\le\alpha(1+\tau). \tag{6.41}$$

Using this in (6.39) shows that for all $t>\tau$ we have

$$\begin{aligned}\|u^{(m)}(t)\|_V^2&\le\|u^{(m)}(t_0)\|_V^2\exp\{G(t_0;t)\}+\frac{C}{\nu}\|f\|_\infty^2(t-t_0)\exp\{G(t_0;t)\}\\&\le\alpha(1+\frac{1}{\tau})e^{\alpha(1+\tau)}+\alpha\tau e^{\alpha(1+\tau)}.\end{aligned} \tag{6.42}$$

This bound is uniform in $t>\tau$. Hence, if we fix $\tau=1$, we get a uniform in m estimate for $\|u^{(m)}(t)\|_V$ for all $t>1$, giving the bound (6.24).

In order to deal with times $t<1$, we will use (6.42) on the time intervals $t\in[1/2^{k+1},1/2^k]$ with $\tau=1/2^{k+1}$. The point is that for such times t and τ are comparable: $\tau\le t\le 2\tau$. Therefore, for $t<1$ we have an estimate

$$t\|u^{(m)}(t)\|_V^2\le\alpha, \tag{6.43}$$

with the constant α that only depends on ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m or $\|u_0\|_V$, which is (6.25).

Finally, if we allow the dependence on the norm $\|u_0\|_V$, then for times $t<1$ we may simply use the first line in (6.42) with $t_0=0$, together with the estimate

$$G(t_0=0,t=1)\le 2\alpha, \tag{6.44}$$

which follows from (6.41). This gives (6.26) and finishes the proof of Proposition 6.3. □

6.2.3 The strong solutions in two dimensions

The above bounds on the solutions $u^{(m)}$ of the Galerkin system (6.23) allow us to pass to the limit $m \to \infty$ to construct solutions of the Navier-Stokes equations on a two-dimensional torus

$$\begin{aligned}
&u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \quad t > 0, \quad x \in \mathbb{T}^2, \\
&\nabla \cdot u = 0, \\
&u(0, x) = u_0(x).
\end{aligned} \tag{6.45}$$

Theorem 6.4. *Assume that $T > 0$, $u_0 \in H$ and $f \in L^\infty(0, T; H)$. Then there exists a constant $C > 0$ which depends only on ν, $\|u_0\|_H$ and $\|f\|_\infty$, and a solution of the Navier-Stokes equation (6.45) which satisfies the bounds*

$$\|u(t)\|_H \le C, \tag{6.46}$$

$$\|u(t)\|_V \le C \text{ for } t \ge 1, \text{ and } \|u(t)\| \le \frac{C}{t} \text{ for } 0 < t < 1, \tag{6.47}$$

$$\int_0^T \|u(t)\|_V^2 dt \le C. \tag{6.48}$$

In addition, for any $s > 0$ there exists C_s so that

$$\int_s^T \|\Delta u(t)\|_H^2 dt \le C_s T. \tag{6.49}$$

Moreover, if $u_0 \in V$ then there exists a constant $C > 0$ which depends only on ν, $\|u_0\|_V$ and $\|f\|_\infty$ so that

$$\|u(t)\|_V \le C \text{ for all } t \ge 0, \tag{6.50}$$

and

$$\int_0^T \|\Delta u(t)\|_H^2 dt \le CT. \tag{6.51}$$

These bounds are inherited from the solutions of the Galerkin system, we leave the details of this passage to the reader, as they are very close to what was done in the corresponding passage in the construction of the weak solutions. We only mention that the $L^2(0, T; H)$ estimate for Δu follows from (6.36). Note that we do not yet claim that if u_0 is an infinitely differentiable function, then the solution $u(t, x)$ is also smooth but only that u is a strong solution in the sense that the aforementioned bounds on $u(t, x)$ hold. We will improve them soon, assuming that u_0 is smooth.

6.3 Existence of strong solutions in three dimensions

6.3.1 Strong solutions in three dimensions: small data

While existence of global in time strong solutions in three dimensions is not known, strong solutions do exist if the initial condition and the forcing are small.

Theorem 6.5. *Let $u_0 \in V$ and $f \in L^2(0,T;H)$. There exists a constant $C > 0$ which depends only on ν, so that if*

$$\|u_0\|_V + \int_0^T \|f(t)\|_H^2 dt \leq C, \tag{6.52}$$

then the Navier-Stokes equations

$$\begin{aligned} &u_t + u\cdot\nabla u + \nabla p = \nu\Delta u + f, \quad t > 0, \quad x \in \mathbb{T}^3, \\ &\nabla\cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.53}$$

have a strong solution on the time interval $[0,T]$ that satisfies

$$\|u(t)\|_V^2 + \int_0^T \|\Delta u(t)\|_H^2 dt \leq \frac{1}{C}, \tag{6.54}$$

for all $0 \leq t \leq T$.

In particular, this theorem says that if $f = 0$, then there exists $C > 0$ so that a unique strong solution exists for all $t > 0$ if $\|u_0\|_H \leq C$.

The proof of Theorem 6.5, once again, relies on the estimates for the Galerkin solutions

$$u_t^{(m)} + P_m(u^{(m)}\cdot\nabla u^{(m)}) = \nu\Delta u^{(m)}, \quad u^{(m)}(0,x) = u_0^{(m)}(x), \quad t > 0, \quad x \in \mathbb{T}^3. \tag{6.55}$$

Taking the inner product with $\Delta u^{(m)}$, as we did in the two-dimensional case, we obtain, as in (6.28):

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2 + \nu\|\Delta u^{(m)}\|_H^2 - (u^{(m)}\cdot\nabla u^{(m)}, \Delta u^{(m)}) = -(f, \Delta u^{(m)}). \tag{6.56}$$

In three dimensions, we may not use the two-dimensional estimate (6.29) for the nonlinear term. Instead, we will bound it as

$$|(u\cdot\nabla u, \Delta u)| \leq C\|u\|_V^{3/2}\|\Delta u\|_H^{3/2} \leq \frac{C}{\nu^3}\|u\|_V^6 + \frac{\nu}{4}\|\Delta u\|_H^2. \tag{6.57}$$

This comes from the estimate

$$|(u\cdot\nabla u, \Delta u)| \leq C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\Delta u\|_{L^2}. \tag{6.58}$$

The Sobolev inequality implies that in three dimensions we have

$$\|u\|_{L^3} \leq C\|u\|_{H^{1/2}}, \quad \|u\|_{L^6} \leq C\|u\|_{H^1}. \tag{6.59}$$

Using this in (6.58) gives

$$|(u\cdot\nabla u, \Delta u)| \leq C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\Delta u\|_{L^2} \leq C\|u\|_{H^1}\|\nabla u\|_{H^{1/2}}\|\Delta u\|_{L^2} \leq C\|u\|_{H^1}^{3/2}\|\Delta u\|_{L^2}^{3/2}, \tag{6.60}$$

which is (6.57). We will estimate the forcing term in (6.56) as

$$|(f, \Delta u)| \leq \frac{4}{\nu}\|f\|_H^2 + \frac{\nu}{4}\|\Delta u\|_H^2. \tag{6.61}$$

Altogether, with the above estimates, (6.56) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|_V^2+\nu\|\Delta u^{(m)}\|_H^2&=(u^{(m)}\cdot\nabla u^{(m)},\Delta u^{(m)})-(f,\Delta u^{(m)})\\&\le\frac{C}{\nu^3}\|u^{(m)}\|_V^6+\frac{\nu}{4}\|\Delta u^{(m)}\|_H^2+\frac{C}{\nu}\|f\|_H^2+\frac{\nu}{4}\|\Delta u^{(m)}\|_H^2.\end{aligned}\tag{6.62}$$

This gives

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|_V^2\le\frac{C}{\nu^3}\|u^{(m)}\|_V^6-\frac{\nu}{2}\|\Delta u^{(m)}\|_H^2+\frac{C}{\nu}\|f\|_H^2\le\frac{C}{\nu^3}\|u^{(m)}\|_V^6-\frac{\nu}{2}\|u^{(m)}\|_V^2+\frac{C}{\nu}\|f\|_H^2.\tag{6.63}$$

Therefore, the function $y(t)=\|u^{(m)}(t)\|_V^2$ satisfies a differential inequality

$$\frac{dy}{dt}\le\frac{C}{\nu^3}y^3-\nu y+\frac{C}{\nu}\|f\|_H^2.\tag{6.64}$$

Hence, as long as

$$y(s)\le\frac{\nu^2}{\sqrt{C}},\ \text{ for all } 0<s<t,\tag{6.65}$$

we have

$$\frac{dy}{dt}\le\frac{C}{\nu}\|f\|_H^2,\tag{6.66}$$

and

$$y(t)\le y(0)+\frac{C}{\nu}\int_0^t\|f(s)\|_H^2ds.\tag{6.67}$$

It follows that if

$$\|u_0\|_V^2+\frac{C}{\nu}\int_0^\infty\|f(s)\|_H^2ds\le\frac{\nu^2}{\sqrt{C}},\tag{6.68}$$

with a universal constant $C>0$, then

$$\|u^{(m)}(t)\|_V^2\le\frac{\nu^2}{\sqrt{C}},\tag{6.69}$$

for all $t>0$. This is part of the bound (6.54) on $\|u^{(m)}\|_V$. In order to get the bound on $\Delta u^{(m)}$ in $L^2(0,T;H)$, we go back to (6.62):

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2+\frac{\nu}{2}\|\Delta u^{(m)}\|_H^2\le\frac{C}{\nu^3}\|u^{(m)}\|_V^6+\frac{C}{\nu}\|f\|_H^2\le C\nu\|u^{(m)}\|_V^2+\frac{C}{\nu}\|f\|_H^2,\tag{6.70}$$

leading to

$$\frac{\nu}{2}\int_0^T\|\Delta u^{(m)}(t)\|_H^2dt\le\|u_0^{(m)}\|_V^2+C\nu\int_0^T\|u^{(m)}(t)\|_V^2dt+\frac{C}{\nu}\int_0^T\|f(t)\|_H^2dt.\tag{6.71}$$

As we also have

$$\nu\int_0^T\|u^{(m)}(t)\|_V^2dt\le\|u_0\|_H^2+\frac{C}{\nu}\int_0^T\|f(t)\|_H^2dt,\tag{6.72}$$

we deduce that under the assumptions (6.52) we have

$$\int_0^T\|\Delta u^{(m)}(t)\|_H^2dt\le C.\tag{6.73}$$

Passing to the limit $m\to\infty$ we construct a strong solution $u(t,x)$ to the Navier-Stokes equations that satisfies the same estimates (6.54). Uniqueness of the strong solution finishes the proof.

6.3.2 Strong solutions in three dimensions: short times

Next, we show that strong solutions of the Navier-Stokes exist for a sufficiently short time even if the data are not small.

Theorem 6.6. *Let $u_0 \in V$ and $f \in L^2(0,T;H)$. There exists a constant $C_0 > 0$ which depends on ν and $\|u_0\|_V$, so that if*

$$T_0 + \int_0^{T_0} \|f(t)\|_H^2 dt \leq C_0, \tag{6.74}$$

then the Navier-Stokes equations

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \quad t > 0, \quad x \in \mathbb{T}^3, \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.75}$$

have a strong solution on the time interval $[0, T_0]$ that satisfies

$$\|u(t)\|_V^2 \leq C_0^{-1}, \tag{6.76}$$

for all $0 \leq t \leq T_0$.

For the proof, we recall (6.70):

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2 + \frac{\nu}{2}\|\Delta u^{(m)}\|_H^2 \leq \frac{C}{\nu^3}\|u^{(m)}\|_V^6 + \frac{C}{\nu}\|f\|_H^2, \tag{6.77}$$

which, in particular, implies that the function $y(t) = \|u^{(m)}(t)\|_V^2$ satisfies a differential inequality

$$\dot{y}(t) \leq Cy(t)^3 + C\|f\|_H^2, \tag{6.78}$$

with the constant C that depends on ν. Dividing by $(1+y)^3$ we get

$$\frac{\dot{y}}{(1+y)^3} \leq \frac{Cy^3 + C\|f\|_H^2}{(1+y)^3} \leq C + C\|f\|_H^2, \tag{6.79}$$

Integrating in time leads to

$$\frac{1}{(1+y_0)^2} - \frac{1}{(1+y(t))^2} \leq Ct + C\int_0^t \|f(s)\|_H^2 ds. \tag{6.80}$$

Therefore, as long as the time t is such that (6.80) holds, or, rather, as long as T_0 satisfies

$$CT_0 + C\int_0^{T_0} \|f(s)\|_H^2 ds \leq \frac{1}{2(1+\|u_0\|_V^2)^2} \leq \frac{1}{2(1+y_0)^2}, \tag{6.81}$$

we have, for all $0 \leq t \leq T_0$:

$$\frac{1}{(1+y(t))^2} \geq \frac{1}{2(1+y_0)^2} \geq \frac{1}{2(1+\|u_0\|_V^2)^2}. \tag{6.82}$$

Therefore, as long as the time t is sufficiently small, so that (6.80) holds, we have

$$\|u^{(m)}(t)\|_V^2 \leq 2(1+\|u_0\|_V^2). \tag{6.83}$$

As usual, this uniform bound on the Galerkin approximations $u^{(m)}(t)$ implies that, passing to the limit $m \to +\infty$, we construct a strong solution of the Navier-Stokes equations for times $0 \leq t \leq T_0$. □

6.3.3 Strong solutions are smooth if the data are smooth

We now show that if the initial condition u_0 and the forcing f are smooth, then the strong solution to the Navier-Stokes equations (if it exists) is also infinitely differentiable. We consider only the three-dimensional case but the analysis applies essentially verbatim to the two-dimensional case as well.

Theorem 6.7. *Let $u(t,x)$ be the strong solution of the Navier-Stokes equations*

$$\begin{aligned} &u_t + u\cdot\nabla u + \nabla p = \nu\Delta u + f, \quad 0<t\le T, \quad x\in\mathbb{T}^3, \\ &\nabla\cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.84}$$

in the sense that there exists $C>0$ so that

$$\sup_{0\le t\le T}\|u(t)\|_V \le C, \quad \int_0^T \|\Delta u(s)\|_H^2 ds \le C. \tag{6.85}$$

Assume that $u_0\in C^\infty(\mathbb{T}^3)$ and $f\in C^\infty(0,T;\mathbb{T}^3)$, then $u\in C^\infty(0,T;\mathbb{T}^3)$.

The strategy of the proof will be to estimate $\|\Delta^m u(t)\|_H$ for all $m\in\mathbb{N}$, and show that, as long u satisfies the assumptions of Theorem 6.7, these norms remain finite for $0\le t\le T$, and all $m\in\mathbb{N}$. As $m\in\mathbb{N}$ will be arbitrary, the Sobolev embedding theorem will imply that u is infinitely differentiable in x, while the Navier-Stokes equations themselves will imply that u is infinitely differentiable in time (using the projection on the divergence free fields, the reader should check that the pressure term is not a problem).

Multiplying (6.84) by $(-\Delta)^m u$ and integrating over $\mathbb{T}^3$ gives

$$(u_t,(-\Delta)^m u) - (u\cdot\nabla u,(-\Delta)^m u) = -\nu(-\Delta u,(-\Delta)^m u) + (f,(-\Delta)^m u). \tag{6.86}$$

Integrating by parts leads to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 - ((-\Delta)^{m/2}(u\cdot\nabla u),(-\Delta)^{m/2}u) + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 \\ \le \|(-\Delta)^{m/2}f\|_H\|(-\Delta)^{m/2}u\|_H. \end{aligned} \tag{6.87}$$

The key inequality we will need for the nonlinear term is given by the following lemma.

Lemma 6.8. *For every $m>3/2$ there exists a constant $C>0$ so that for any vector-valued functions u, v such that $u_0=v_0=0$, and $\nabla\cdot u=\nabla\cdot v=0$, and $u_k=v_k=0$ for all $k>M$, with some $M>0$, we have*

$$\|(-\Delta)^{m/2}\mathcal{P}(u\cdot\nabla v)\|_H \le C\|(-\Delta)^{m/2}u\|_H\|(-\Delta)^{(m+1)/2}v\|_H. \tag{6.88}$$

Here, $\mathcal{P}$ is the projection on divergence-free fields.

Postponing the proof of this lemma, we apply it in (6.87):

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 \le \|(-\Delta)^{m/2}f\|_H\|(-\Delta)^{m/2}u\|_H \\ + C\|(-\Delta)^{m/2}u\|_H^2\|(-\Delta)^{(m+1)/2}u\|_H. \end{aligned} \tag{6.89}$$

Next, we use Young's inequality in the right side together with the Poincare inequality in the form

$$\|(-\Delta)^{m/2}u\|_H \leq C\|(-\Delta)^{(m+1)/2}u\|_H. \tag{6.90}$$

This leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 \leq \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{\nu}{4}\|(-\Delta)^{(m+1)/2}u\|_H^2 \\
&+\frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4 + \frac{\nu}{4}\|(-\Delta)^{(m+1)/2}u\|_H^2 \\
&\leq \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4 + \frac{\nu}{2}\|(-\Delta)^{(m+1)/2}u\|_H^2.
\end{aligned} \tag{6.91}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \frac{\nu}{2}\|(-\Delta)^{(m+1)/2}u\|_H^2 \leq \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4. \tag{6.92}$$

Looking at this as the differential inequality for $y(t) = \|(-\Delta)^{m/2}u\|_H^2$, we deduce that

$$\dot{y} \leq \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^2 y(t) \leq C_f + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^2 y(t), \tag{6.93}$$

with a finite constant C_f as $f \in C^\infty(0,T;\mathbb{T}^3)$. Grownwall's inequality implies now that $y(t)$ obeys an upper bound

$$y(t) \leq y(0)\exp\left[\frac{C}{\nu}\int_0^t \|(-\Delta)^{m/2}u(s)\|_H^2 ds\right] + C_f\int_0^t \exp\left[\frac{C}{\nu}\int_s^t \|(-\Delta)^{m/2}u(\tau)\|_H^2 d\tau\right]ds. \tag{6.94}$$

In other words, if we know that

$$\int_0^T \|(-\Delta)^{m/2}u(s)\|_H^2 ds < +\infty, \tag{6.95}$$

then

$$\sup_{0\leq t\leq T} \|(-\Delta)^{m/2}u(s)\|_H^2 ds < +\infty. \tag{6.96}$$

This, in turn, implies that

$$\int_0^T \|(-\Delta)^{m/2}u(s)\|_H^4 ds < C, \tag{6.97}$$

which can be inserted into (6.92) to conclude that

$$\int_0^T \|(-\Delta)^{(m+1)/2}u(s)\|_H^2 ds < +\infty, \tag{6.98}$$

allowing us to build an induction argument and continue forever, meaning that

$$\sup_{0\leq t\leq T} \|(-\Delta)^{m/2}u(s)\|_H^2 ds < +\infty, \text{ for any } m \in \mathbb{N}. \tag{6.99}$$

This will, in turn, imply that $u \in C^\infty$ by the Sobolev embedding theorem. However, this argument uses the bound (6.88) which applies only for $m > 3/2$, and the "free" estimate for the weak solution is

$$\int_0^T \|\nabla u(s)\|_H^2 ds = \int_0^T \|(-\Delta)^{1/2} u(s)\|_H^2 ds < +\infty, \tag{6.100}$$

which corresponds to $m = 1$, and for which we may not use this argument. Hence, to start the induction we need the assumption that

$$\int_0^T \|\Delta u(s)\|_H^2 ds < +\infty, \tag{6.101}$$

which corresponds to taking $m = 2 > 3/2$, allowing us to proceed. This is the reason behind the requirement that strong solutions satisfy (6.101).

The proof of Lemma 6.8

Recall that

$$\|(-\Delta)^{m/2}\mathcal{P}(u \cdot \nabla v)\|_H = \sup_{w \in H, \|w\|_H = 1} ((-\Delta)^{m/2}(u \cdot \nabla v), w). \tag{6.102}$$

Let us write

$$u \cdot \nabla v(x) = \sum_{k \in \mathbb{Z}^3} (2\pi i)\Big(\sum_{j+l=k} (l \cdot u_j) v_l\Big) e^{2\pi i k \cdot x}, \tag{6.103}$$

so that

$$\begin{aligned} ((-\Delta)^{m/2}(u \cdot \nabla v), w) &= \sum_{k \in \mathbb{Z}^3} (2\pi i)(4\pi^2 |k|^2)^{m/2} \Big(\sum_{j+l=k} (l \cdot u_j) v_l\Big) \cdot w_{-k} \\ &= \sum_{j+l+k=0} (2\pi i)(4\pi^2|k|^2)^{m/2} (l \cdot u_j)(v_l \cdot w_k). \end{aligned} \tag{6.104}$$

Next, we will use the inequality

$$|j+l|^m \le (|j| + |l|)^m \le C_m(|j|^m + |l|^m), \tag{6.105}$$

which implies

$$\begin{aligned} |((-\Delta)^{m/2}(u \cdot \nabla v), w)| &\le C_m \sum_{j+l+k=0} |k|^m |l| |u_j| |v_l| |w_k| \\ &\le C_m \sum_{j+l+k=0} (|j|^m + |l|^m) |l| |u_j| |v_l| |w_k| \\ &\le C_m \sum_{j+l+k=0} |l|^{m+1} |u_j| |v_l| |w_k| + C_m \sum_{j+l+k=0} |j|^m |l| |u_j| |v_l| |w_k| = A + B. \end{aligned} \tag{6.106}$$

For the first term, we may estimate

$$\begin{aligned} A &= C_m \sum_{j+l+k=0} |l|^{m+1} |u_j| |v_l| |w_k| = \sum_{j \in \mathbb{Z}^3} |u_j| \sum_{l \in \mathbb{Z}^3} |l|^{m+1} |v_l| |w_{-j-l}| \\ &\le \sum_{j \in \mathbb{Z}^3} |u_j| \Big(\sum_{l \in \mathbb{Z}^3} |l|^{2m+2} |v_l|^2\Big)^{1/2} \Big(\sum_{l \in \mathbb{Z}^3} |w_l|^2\Big)^{1/2} = \|(-\Delta)^{(m+1)/2} v\|_H \|w\|_H \sum_{j \in \mathbb{Z}^3} |u_j|. \end{aligned} \tag{6.107}$$

For the last sum above we may use the estimate

$$\sum_{j\in\mathbb{Z}^3}|u_j|\le\Big(\sum_{j\in\mathbb{Z}^3}|j|^{2m}|u_j|^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}^3}\frac{1}{|j|^{2m}}\Big)^{1/2}\le C\Big(\sum_{j\in\mathbb{Z}^3}|j|^{2m}|u_j|^2\Big)^{1/2}=C\|(-\Delta)^{m/2}u\|_H. \tag{6.108}$$

We used in the last step the assumption that $m>3/2$ (in a dimension n we would have needed to assume that $m>n/2$).

For the second term in (6.106) we write

$$\begin{aligned}B&=C\sum_{j+l+k=0}|j|^m|l||u_j||v_l||w_k|=\sum_{l\in\mathbb{Z}^3}|l||v_l|\sum_{j\in\mathbb{Z}^3}|j|^m|u_j||w_{-l-j}|\\&\le C\|(-\Delta)^{m/2}u\|_H\|w\|_H\sum_{l\in\mathbb{Z}^3}|l||v_l|,\end{aligned} \tag{6.109}$$

and

$$\sum_{l\in\mathbb{Z}^3}|l||v_l|\le\Big(\sum_{l\in\mathbb{Z}^3}|l|^{2+2m}|v_l|^2\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}^3}\frac{1}{|l|^{2m}}\Big)^{1/2}\le C\|(-\Delta)^{(m+1)/2}v\|_H, \tag{6.110}$$

as $m>3/2$. This shows that for any $w\in H$ we have

$$|((-\Delta)^{m/2}(u\cdot\nabla u),w)|\le C\|(-\Delta)^{m/2}u\|_H\|(-\Delta)^{(m+1)/2}v\|_H\|w\|_H, \tag{6.111}$$

and thus finishes the proof of Lemma 6.8. □

6.3.4 Local in time existence in higher Sobolev spaces

The arguments of the previous section imply also that the Navier-Stokes equations are locally well-posed in the higher Sobolev spaces $H^m(\mathbb{T}^3)$. For simplicity, we state the result for the case $f=0$.

Theorem 6.9. *Let $u_0\in H^m$, with $m\ge 2$, and $f=0$. There exist a time $T_m>0$ and $C_0>0$ that depend on ν, $m\ge 1$ and $\|u_0\|_{H^m}$, so that the Navier-Stokes equations*

$$\begin{aligned}&u_t+u\cdot\nabla u+\nabla p=\nu\Delta u,\quad t>0,\quad x\in\mathbb{T}^3,\\&\nabla\cdot u=0,\\&u(0,x)=u_0(x),\end{aligned} \tag{6.112}$$

have a strong solution on the time interval $[0,T_m]$ that satisfies

$$\|u(t)\|_{H^m}^2\le C_0^{-1}, \tag{6.113}$$

for all $0\le t\le T_m$.

The proof is familiar: we start with (6.114) with $f=0$:

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2+\frac{\nu}{2}\|(-\Delta)^{(m+1)/2}u\|_H^2\le\frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4. \tag{6.114}$$

Looking at this as the differential inequality for $y(t)=\|(-\Delta)^{m/2}u\|_H^2$, we deduce that

$$\dot y\le\frac{C}{\nu}y^2(t). \tag{6.115}$$

As a consequence, $y(t)$ remains finite for a time that depends only on $y(0)$. □

6.4 Infinite time blow-up implies a finite time blow-up

The problem of blow-up of solutions of a nonlinear partial differential equation usually consists in two separate problems: (1) can solutions blow-up in a finite time, and (2) can they blow-up in an infinite time, in the sense that the norm of the solutions tends to infinity as $t \to +\infty$? The second notion is usually much weaker. For example, solutions to the heat equation with a linear growth term

$$u_t = \Delta u + u, \quad t > 0, x \in \mathbb{R}^n, \tag{6.116}$$

have the long time behavior

$$u(t,x) \sim \frac{e^t \|u_0\|_{L^1}}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}, \tag{6.117}$$

and thus "blow-up in an infinite time" – all its L^p-norms, $p \geq 1$ tend to infinity as $t \to +\infty$. However, one does not normally think of these solutions as really "blowing-up" – they just grow in time.

The situation is different for the Navier-Stokes equations: an infinite time blow-up implies a finite-time blow-up. More precisely, let us assume that there exists a strong solution $u(t,x)$ of the Navier-Stokes equations

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad 0 < t \leq T, \quad x \in \mathbb{T}^3, \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.118}$$

such that $u_0 \in H$, and

$$\lim_{t \to +\infty} \|u(t)\|_V = +\infty. \tag{6.119}$$

Assuming that such u exists, and given any $T > 0$, we will now construct an initial condition $v_0 \in V$ so that the solution to (6.118) with $v(0,x) = v_0(x)$, blows up before the time $T > 0$. That is, there will be a time $T_1 \in (0,T]$ such that

$$\lim_{t \to T_1^-} \|v(t)\|_V = +\infty. \tag{6.120}$$

The idea is to combine the blow-up assumption that there exists a sequence of times $t_j \to +\infty$ such that

$$\|u(t_j)\|_V \geq 2^j, \tag{6.121}$$

with the main result of Proposition 6.2: solutions to the Navier-Stokes equations are often not large. Given an initial condition $u_0 \in H$ and a sequence t_j as in (6.121), we may use the aforementioned Proposition to find a time $s_j \in [t_j - T, t_j]$ so that

$$\|u(s_j)\|_V \leq C\Big(1 + \frac{1}{T}\Big) = C'. \tag{6.122}$$

The constant C depends only on $\|u_0\|_H$, and $\nu > 0$. Thus, if we take $u(s_j)$ as the initial condition for the Navier-Stokes equations at the time $t = 0$, then the corresponding solution to the Cauchy problem will have reached the V-norm that is larger than 2^j by the time T. As $\|u(s_j)\|_V$ is uniformly bounded in j, we may choose a subsequence $j_k \to +\infty$ so that $v_k^0(x) = u(s_{j_k}, x)$ converges weakly in V and strongly in H to a function $v_0 \in V$. Consider now the Cauchy problem with the initial condition v_0:

$$
\begin{aligned}
&v_t + v \cdot \nabla v + \nabla p = \nu \Delta v, \quad 0 < t \le T, \quad x \in \mathbb{T}^3, \\
&\nabla \cdot v = 0, \\
&v(0, x) = v_0(x).
\end{aligned} \tag{6.123}
$$

This problem has a strong solution on some time interval $[0, T_0]$, which depends only on $\|v_0\|_V$ and ν.

We will now show that (6.123) may not have a strong solution on the time interval $[0, T]$. To this end, assume that such solution exists on $[0, T]$, denote

$$
r = \sup_{0 \le t \le T} \|v(t)\|_V, \tag{6.124}
$$

and consider the functions $v_k(t) = u(t + s_{j_k})$, which are the solutions to

$$
\begin{aligned}
&\frac{\partial v_k}{\partial t} + v_k \cdot \nabla v_k + \nabla p_k = \nu \Delta v_k, \quad 0 < t \le T, \quad x \in \mathbb{T}^3, \\
&\nabla \cdot v_k = 0, \\
&v_k(0, x) = v_k^0(x) = u(s_j, x).
\end{aligned} \tag{6.125}
$$

Writing $w_j = v_j - v$, and expanding

$$
v_j \cdot \nabla v_j - v \cdot \nabla v = (v + w_j) \cdot \nabla (v + w_j) - v \cdot \nabla v = w_j \cdot \nabla v + v \cdot \nabla w_j + w_j \cdot \nabla w_j, \tag{6.126}
$$

we see that w_j satisfies (as in the proof of the uniqueness of the solutions of the Navier-Stokes equations):

$$
\begin{aligned}
&\frac{\partial w_j}{\partial t} + w_j \cdot \nabla v + v \cdot \nabla w_j + w_j \cdot \nabla w_j + \nabla p' = \nu \Delta w_j, \quad 0 < t \le T, \quad x \in \mathbb{T}^3, \quad (6.127)\\
&\nabla \cdot w_j = 0, \\
&w_j(0, x) = v_j^0(x) - v_0(x),
\end{aligned}
$$

with $p' = p_j - p$. Multiplying by w_j and integrating leads to

$$
\frac{1}{2} \frac{d}{dt} \|w_j\|_H^2 + \nu \|w_j\|_V^2 = -(w_j \cdot \nabla v, w_j). \tag{6.128}
$$

In three dimensions, we can estimate the right side as

$$
\begin{aligned}
|(w_j \cdot \nabla v, w_j)| &\le \|w_j\|_{L^3} \|\nabla v\|_{L^2} \|w_j\|_{L^6} \le C \|w_j\|_{H^{1/2}} \|v\|_V \|w_j\|_{H^1} \\
&\le C \|w_j\|_H^{1/2} \|w_j\|_V^{1/2} \|v\|_V \|w_j\|_V = C \|v\|_V \|w_j\|_H^{1/2} \|w_j\|_V^{3/2} \\
&\le \frac{\nu}{2} \|w_j\|_V^2 + \frac{C}{\nu^3} \|v\|_V^4 \|w_j\|_H^2.
\end{aligned} \tag{6.129}
$$

We used Young's inequality in the last step, with $p = 4/3$, $q = 4$. Using this in (6.128) gives

$$
\frac{1}{2} \frac{d}{dt} \|w_j\|_H^2 + \frac{\nu}{2} \|w_j\|_V^2 \le \frac{C}{\nu^3} \|v\|_V^4 \|w_j\|_H^2. \tag{6.130}
$$

As v is a strong solution to (6.123) and $\|v\|_V$ is uniformly bounded by r, by the assumption (6.124), it follows from (6.130) that there exists $C > 0$, which depends on ν and r that appears in (6.124), so that

$$\|w_j(t)\|_H \leq \|w_j(0)\|_H e^{Ct}. \tag{6.131}$$

As $w_j(0) \to 0$ strongly in H, we conclude that $w_j(t) \to 0$ strongly in H, for all $0 \leq t \leq T$. Another consequence of (6.130) and the uniform bound (6.124) is that

$$\frac{\nu}{2}\int_0^T \|w_j(t)\|_V^2 dt \leq \|w_j(0)\|_H^2 + C\int_0^T \|w_j(t)\|_H^2 dt, \tag{6.132}$$

and since $\|w_j(t)\|_H \to 0$, pointwise in t, while $\|w_j(t)\|_H \leq C$, we conclude that

$$\int_0^T \|w_j(t)\|_V^2 dt \to 0 \text{ as } j \to \infty. \tag{6.133}$$

In particular, possibly after extracting another subsequence, we know that

$$\|w_j(t)\|_V \to 0 \text{ for a.e. } t \in [0,T]. \tag{6.134}$$

Thus, given any $\delta > 0$ we can choose a sequence of times $\tau_k \in [0,T]$ such that $0 < \tau_{k+1} - \tau_k < \delta$, and $\|w_j(\tau_k)\|_V \leq 1$. Next, note that if $\|w_j(t)\|_V \leq 1$, then

$$\|v_j(t)\|_V \leq \|w_j(t)\|_V + \|v(t)\|_V \leq 1 + r, \tag{6.135}$$

with $r > 0$ as in (6.124). The local in time existence theorem implies that there exists a time T_1, which depends only on ν and r, so that if $\|v_j(t)\|_V \leq 1 + r$, then

$$\|v_j(s)\|_V \leq 10(1 + r), \tag{6.136}$$

for all $s \in [t, t + T_1]$. Taking $\delta = T_1$, we deduce that (6.136) holds for all $0 \leq t \leq T$. This, however, contradicts the assumption that $0 \leq t_{j_k} - s_{j_k} \leq T$ and

$$\|v_k(t_{j_k} - s_{j_k})\|_V = \|u(s_{j_k})\|_V \geq 2^{j_k}.$$

Thus, $v(s,x)$ can not be a strong solution on the time interval $[0,T]$.

6.5 The Beale-Kato-Majda regularity criterion

We now describe a sufficient condition for the solution to remain smooth. This time, we will work in the whole space $\mathbb{R}^3$ but the existence and regularity results we have proved for the three-dimensional torus apply essentially verbatim to the whole space as well. As we have seen in Theorem 6.9, if the H^m-norms of a smooth solution $u(t,x)$ remain finite on a time interval $[0,T]$, for some $m > 3/2$, then the solution may be extended past the time T. In other words, a time T is the maximal time of existence of a smooth solution $u(t,x)$ if and only if

$$\lim_{t\uparrow T} \|u(t)\|_{H^m} = +\infty. \tag{6.137}$$

The Beale-Kato-Majda criterion reformulates this condition in terms of the vorticity (this also requires only one derivative of u, not 3/2 derivatives).

Theorem 6.10. *Let $u_0 \in C_c^\infty(\mathbb{R}^3)$, so that there exists a classical solution v to the Navier-Stokes equations with $f = 0$. If for any $T > 0$ we have*

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < +\infty, \tag{6.138}$$

then the smooth solution u exists globally in time. If the maximal existence time of the smooth solution is $T < +\infty$, then necessarily we have

$$\lim_{t\uparrow T} \int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \tag{6.139}$$

6.5.1 A bound in terms of $\|\nabla u\|_{L^\infty}$

The starting point in the proof of Theorem 6.10 is an estimate for the evolution of the H^m-norms, assuming that we have the control of $\|Du\|_{L^\infty}$.

We take m to be an even integer for convenience. Recall the identity (6.87) with $f = 0$:

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 = ((-\Delta)^{m/2}(u\cdot\nabla u), (-\Delta)^{m/2}u). \tag{6.140}$$

Our goal is to show the following inequality:

$$|((-\Delta)^{m/2}(u\cdot\nabla u), (-\Delta)^{m/2}u)| \leq C_m\|D^m u\|_{L^2}\|Du\|_{L^\infty}, \tag{6.141}$$

with the notation $D = (-\Delta)^{1/2}$. An important preliminary point is that the term in the inner product that has the highest order derivative of u, of the order $(m+1)$, vanishes

$$((u\cdot\nabla(-\Delta)^{m/2}u), (-\Delta)^{m/2}u) = 0,$$

because $\nabla\cdot u = 0$. Hence, the left side in (6.141) can be estimated by

$$|((-\Delta)^{m/2}(u\cdot\nabla u), (-\Delta)^{m/2}u)| \leq C_m\sum_{k=1}^m \|D^k u\|_{L^{p_k}}\|D^{(m+1-k)}u\|_{L^{q_k}}, \tag{6.142}$$

with $1 \leq p_k, q_k \leq \infty$ such that

$$\frac{1}{p_k} + \frac{1}{q_k} = \frac{1}{2}. \tag{6.143}$$

We recall a Gagliardo-Nirenberg type inequality for $\mathbb{R}^d$: for any with $0 \leq j < m$, there exists $C > 0$ so that we have

$$\|D^j f\|_{L^p} \leq C\|D^m f\|_2^a\|f\|_{L^\infty}^{1-a}, \tag{6.144}$$

with

$$\frac{1}{p} = \frac{j}{d} + \frac{j}{m}\Big(\frac{1}{2} - \frac{m}{d}\Big),$$

and $a = j/m$. We will use it for $f = Du$ and $j = k-1 < m$. This gives

$$\|D^k u\|_{L^{p_k}} = \|D^{k-1}Du\|_{L^{p_k}} \leq C_k\|D^{m-1}Du\|_{L^2}^{a_k}\|Du\|_\infty^{1-a_k} = C_k\|D^m u\|_{L^2}^{a_k}\|Du\|_\infty^{1-a_k}, \ \ 1 \leq k \leq m, \tag{6.145}$$

with

$$a_k = \frac{k-1}{m-1},$$

and

$$\frac{1}{p_k} = \frac{k-1}{d} + \frac{k-1}{m-1}\left(\frac{1}{2} - \frac{m-1}{d}\right) = \frac{k-1}{2(m-1)} = \frac{a_k}{2}.$$

This gives the estimate

$$\|D^k u\|_{L^{p_k}} \le C_k \|D^m u\|_{L^2}^{a_k} \|Du\|_{\infty}^{1-a_k}, \quad 1 \le k \le m. \tag{6.146}$$

The paired term $\|D^{m+1-k}u\|_{q_k}$ that appears in (6.142) can be estimated similarly:

$$\|D^{m+1-k}u\|_{L^{q_k}} = \|D^{m-k}Du\|_{L^{q_k}} \le C_k \|D^{m-1}Du\|_{L^2}^{b_k} \|Du\|_{\infty}^{1-b_k} = C_k \|D^m u\|_{L^2}^{b_k} \|Du\|_{\infty}^{1-b_k}, \tag{6.147}$$

with

$$b_k = \frac{m-k}{m-1},$$

and

$$\frac{1}{q_k} = \frac{m-k}{d} + \frac{m-k}{m-1}\left(\frac{1}{2} - \frac{m-1}{d}\right) = \frac{m-k}{2(m-1)} = \frac{b_k}{2}.$$

Luckily, we both have

$$a_k + b_k = \frac{k-1}{m-1} + \frac{m-k}{m-1} = 1, \tag{6.148}$$

and (6.143) holds with the above choice of p_k and q_k:

$$\frac{1}{p_k} + \frac{1}{q_k} = \frac{a_k + b_k}{2} = \frac{1}{2},$$

so that these p_k and q_k can be taken in (6.142). It follows from (6.146), (6.147) and (6.148) that

$$\|D^k u\|_{L^{p_k}} \|D^{(m+1-k)}u\|_{L^{q_k}} \le C_k \|D^m u\|_{L^2} \|Du\|_{L^\infty}.$$

When $k = m$ or $k = 1$, we simply use $p = 1/2$ and $q = \infty$, getting the estimate

$$\|D^m u\|_{L^2} \|Du\|_{L^\infty}$$

for those terms. Inserting this into (6.142) gives (6.141).

With (6.141) in hand, going back to (6.140), we conclude that

$$\frac{1}{2}\frac{d}{dt}\|D^m u\|_H^2 \le C_m \|D^m u\|_H^2 \|\nabla u\|_{L^\infty}. \tag{6.149}$$

Summing over m, we conclude that for any $s \in \mathbb{N}$ we have

$$\frac{d}{dt}\|u\|_{H^s} \le C_s \|\nabla u\|_{L^\infty} \|u\|_{H^s}. \tag{6.150}$$

Therefore, if $u_0 \in C_c^\infty(\mathbb{R}^3)$, then for any of the H^s-norms to become infinite by a time T it is necessary that

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt = +\infty, \tag{6.151}$$

and, in general, we have

$$\|u\|_{H^s} \le \|u_0\|_{H^s} \exp\Big\{C_s \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\Big\}. \tag{6.152}$$

In a similar vein, multiplying the vorticity equation

$$\omega_t + u\cdot\nabla\omega = \nu\Delta\omega + \omega\cdot\nabla u \tag{6.153}$$

by ω and integrating, we see that

$$\frac{d}{dt}\|\omega(t)\|_{L^2} \le \|\nabla u\|_{L^\infty}\|\omega\|_{L^2}, \tag{6.154}$$

so that

$$\|\omega(t)\|_{L^2} \le \|\omega_0\|_{L^2} \exp\Big\{\int_0^t \|\nabla u(s)\|_{L^\infty} ds\Big\}. \tag{6.155}$$

6.5.2 Bounding $\|\nabla u\|_{L^\infty}$ by $\|\omega\|_{L^\infty}$

The above bounds show that the conclusion of Theorem 6.10 would follow from (6.151) and (6.152) if we would know that

$$\text{“}\|\nabla u\|_{L^\infty} \le C\|\omega\|_{L^\infty}\text{”}. \tag{6.156}$$

One may expect this to be true based on the validity of a similar identity for the L^2-norms: recall (3.35)

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx = \int_{\mathbb{R}^3} |\omega|^2 dx, \tag{6.157}$$

because

$$|\omega|^2 = \varepsilon_{ijk}\varepsilon_{imn}(\partial_j u_k)(\partial_m u_n) = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})(\partial_j u_k)(\partial_m u_n) = |\nabla u|^2 - (\partial_j u_k)(\partial_k u_j), \tag{6.158}$$

and

$$\int_{\mathbb{R}^n} (\partial_j u_k)(\partial_k u_j) dx = -\int_{\mathbb{R}^n} u_k(\partial_k\partial_j u_j) dx = 0. \tag{6.159}$$

Identity (6.156), however, is not quite true for the L^∞-norms – the relation between the gradient of the velocity and the vorticity is in terms of a singular integral operator which maps every $L^p \to L^p$ for $1 < p < +\infty$ but does not map L^∞ to L^∞. However, it is “almost true” as shown by the following lemma.

Lemma 6.11. *Let $u(x)$ be a smooth divergence free velocity field in $L^2\cap L^\infty$, and let $\omega = \nabla\times u$. There exists a constant $C > 0$ so that*

$$\|\nabla u\|_{L^\infty} \le C(1 + \log^+\|u\|_{H^3} + \log^+\|\omega\|_{L^2})(1 + \|\omega\|_{L^\infty}). \tag{6.160}$$

Here, for $z > 0$, we set $\log^+ z = \log z$ if $\log z > 0$, and $\log^+ z = 0$ otherwise. The L^2-norm of $\omega(t)$ that appears in (6.160) can be estimated from (6.155) as

$$\log^+ \|\omega(t)\|_{L^2} \leq \log^+ \|\omega_0\|_{L^2} + \int_0^t \|\nabla u(s)\|_{L^\infty} ds. \tag{6.161}$$

Similarly, the H^3-norm of $u(t)$ can be bounded as in (6.152):

$$\log^+ \|u(t)\|_{H^3} \leq \log^+ \|u_0\|_{H^3} + C \int_0^t \|\nabla u(s)\|_{L^\infty} ds. \tag{6.162}$$

Assuming the result of Lemma 6.11, we deduce that $\|\nabla u\|_\infty$ satisfies the inequality

$$\|\nabla u(t)\|_{L^\infty} \leq C_0\Big(1 + \int_0^t \|\nabla u(s)\|_{L^\infty} ds\Big)(1 + \|\omega(t)\|_{L^\infty}, \tag{6.163}$$

with a constant C_0 that depends on the initial data u_0. Setting

$$G(t) = \int_0^t \|\nabla u(s)\|_{L^\infty} ds, \quad \beta(t) = 1 + \|\omega(t)\|_{L^\infty},$$

we have from (6.163):

$$\frac{dG}{dt} \leq C_0(1 + G(t))\beta(t),$$

so that

$$\frac{d}{dt}\Big(G(t) \exp\Big\{ - C_0 \int_0^t \beta(s) ds)\Big\}\Big) \leq C_0 \beta(t) \exp\Big\{ - C_0 \int_0^t \beta(s) ds\Big\}.$$

Integrating in time gives

$$G(t) \exp\Big\{ - C_0 \int_0^t \beta(s) ds)\Big\} \leq 1 - \exp\Big\{ - C_0 \int_0^t \beta(s) ds\Big\}, \tag{6.164}$$

so that

$$G(t) \leq \exp\Big\{ C_0 \int_0^t \beta(s) ds)\Big\}.$$

In other words, we have

$$\int_0^t \|\nabla u(s)\|_{L^\infty} ds \leq \exp\Big\{ C_0 t + C_0 \int_0^t \|\omega(s)\|_{L^\infty} ds)\Big\}. \tag{6.165}$$

As a consequence, as long as

$$\int_0^t \|\omega(s)\|_{L^\infty} ds < +\infty, \tag{6.166}$$

all H^m-norms of the velocity remain finite, hence $u(t) \in C^\infty(\mathbb{R}^3)$. Therefore, the proof of Theorem 6.10 boils down to Lemma 6.11.

6.5.3 The proof of the estimate on $\|\nabla u\|_{L^\infty}$

We now prove Lemma 6.11 using the ideas from the theory of singular integral operators. The velocity field is related to vorticity by the Biot-Savart law:

$$u(x) = -\int_{\mathbb{R}^3} K(x-y)\omega(y)dy = \int_{\mathbb{R}^3} K(y)\omega(x+y)dy, \tag{6.167}$$

with

$$K(x)h = \frac{1}{4\pi|x|^3} x \times h, \tag{6.168}$$

for any $h \in \mathbb{R}^3$. As the singularity in $\nabla K(x)$ is of the order $1/|x|^3$ which is not integrable in three dimensions, we have to be careful about computing the gradient of u. Let us write

$$u(x+z) - u(x) = \int_{\mathbb{R}^3} K(y)[\omega(x+z+y) - \omega(x+y)]dy. \tag{6.169}$$

As $K \in L^1_{loc}(\mathbb{R}^3)$, if, say, $\omega \in C_0^\infty(\mathbb{R}^3)$, then, passing to the limit $z \to 0$, we get

$$\frac{\partial u_k(x)}{\partial x_j} = \int_{\mathbb{R}^3} K_{km}(y)\partial_j \omega_m(x+y)dy. \tag{6.170}$$

Because of the singularity in K we can not immediately integrate by parts. Let us write this integral as

$$\begin{aligned}
\frac{\partial u_k(x)}{\partial x_j} &= \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} K_{km}(y)\partial_j \omega_m(x+y)dy = \\
&= -\lim_{\varepsilon \to 0} \int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy - \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} [\partial_j K_{km}(y)]\omega_m(x+y)dy \\
&= A_{kj} + B_{kj}.
\end{aligned} \tag{6.171}$$

The first integral can be re-written as

$$\begin{aligned}
A_{kj} &= -\lim_{\varepsilon \to 0} \int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy = -\lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{|y|=\varepsilon} \frac{1}{|y|^3}[y \times \omega(x+y)]_k \frac{y_j}{|y|}dy \\
&= -\lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{|z|=1} \frac{1}{\varepsilon^3|z|^3}[\varepsilon z \times \omega(x+\varepsilon z)]_k \frac{z_j}{|z|}\varepsilon^2 dz = -\frac{1}{4\pi} \int_{|z|=1} [z \times \omega(x)]_k z_j dz \\
&= -\frac{1}{4\pi}\epsilon_{kmn} \int_{|z|=1} z_m \omega_n(x) z_j dz = \frac{\epsilon_{kmn}}{3}\omega_n(x)\delta_{mj} = -\frac{1}{3}\epsilon_{kjn}\omega_n(x).
\end{aligned} \tag{6.172}$$

Thus, we have

$$|A_{kj}| \leq \frac{1}{3}\|\omega\|_{L^\infty},$$

and the main focus is on the second term. We have

$$K_{km}(y) = \frac{\epsilon_{krm}}{4\pi|y|^3} y_r,$$

so that

$$\partial_j K_{km}(y) = -\frac{3\epsilon_{krm}}{4\pi|y|^5} y_j y_r + \frac{\epsilon_{kjm}}{4\pi|y|^3}.$$

We conclude that for any $h \in \mathbb{R}^3$ we have

$$\begin{aligned}(Bh)_k &= -\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[-\frac{3\epsilon_{krm}}{4\pi|y|^5}y_jy_r + \frac{\epsilon_{kjm}}{4\pi|y|^3}\Big]\omega_m(x+y)h_j dy \\ &= \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x+y)\times h]_k\Big)dy.\end{aligned} \tag{6.173}$$

We shall split B further as follows: take a smooth cut-off function $\rho(r)$ so that $\rho(r) = 0$ for $r > 2R$, and $\rho(r) = 1$ for $r < R$, with R to be chosen later, and write

$$\begin{aligned}(Bh)_k &= \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x+y)\times h]_k\Big)\rho(|y|)dy \\ &+ \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x-y)\times h]_k\Big)(1-\rho(|y|)dy \\ &= C_k + D_k.\end{aligned} \tag{6.174}$$

The Cauchy-Schwartz inequality implies that

$$|D_k| \leq C|h|\|\omega\|_{L^2}\Big(\int_R^\infty \frac{1}{r^6}r^2dr\Big)^{1/2} \leq \frac{C}{R^{3/2}}\|\omega\|_{L^2}|h|. \tag{6.175}$$

The key estimate is for C_k: we will show that for any $\delta > 0$ and any Hölder regularity exponent $\gamma \in (0,1)$ we have

$$|C_k| \leq C\Big\{\delta^\gamma\|\omega\|_{C^\gamma} + \|\omega\|_{L^\infty}\max\big(1, \log\frac{R}{\delta}\big)\Big\}|h|. \tag{6.176}$$

Here, $\|\omega\|_{C^\gamma}$ is the Hölder norm. Let us assume momentarily that (6.176) holds. The Sobolev inequality in dimension n

$$\|f\|_{C^\gamma(\mathbb{R}^n)} \leq C\|f\|_{H^{s+\gamma}(\mathbb{R}^n)}, \quad s > \frac{n}{2}$$

implies that in three dimensions we have, for all $0 < \gamma < 1/2$:

$$\|\omega\|_{C^\gamma} \leq C\|\omega\|_{H^2},$$

so that if (6.176) holds then

$$|C_k| \leq C\Big\{\delta^\gamma\|\omega\|_{H^2} + \|\omega\|_{L^\infty}\max\big(1,\log\frac{R}{\delta}\big)\Big\}|h| \leq C\Big\{\delta^\gamma\|u\|_{H^3} + \|\omega\|_{L^\infty}\max\big(1,\log\frac{R}{\delta}\big)\Big\}|h|. \tag{6.177}$$

Altogether, we have

$$\|\nabla u\|_{L^\infty} \leq C\Big(\|\omega\|_{L^\infty} + \frac{C}{R^{3/2}}\|\omega\|_{L^2} + \Big\{\delta^\gamma\|u\|_{H^3} + \|\omega\|_{L^\infty}\max\big(1,\log\frac{R}{\delta}\big)\Big\}\Big). \tag{6.178}$$

Thus, we set the cut-off R to be

$$R = \|\omega\|_{L^2}^{2/3}.$$

As far δ is concerned, if $\|u\|_{H^3} \leq 1$, we can take $\delta = 1$, while if $\|u\|_{H^3} \geq 1$, we can take

$$\delta = \|u\|_{H^3}^{-\gamma}.$$

In both cases, we have

$$\|\nabla u\|_{L^\infty} \leq C(1 + \log^+ \|u\|_{H^3} + \log^+ \|\omega\|_{L^2})(1 + \|\omega\|_{L^\infty}), \tag{6.179}$$

which is the claim of Lemma 6.11. It remains, therefore, only to prove the estimate (6.176).

6.5.4 A nearly $L^\infty \to L^\infty$ estimate for singular integral operators

We now prove estimate (6.176) for C_k, which we write as

$$\begin{aligned}
C_k &= \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x+y)\times h]_k\Big)\rho(|y|)dy \\
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(3(\hat{y}\cdot h)[\hat{y}\times\omega(x+y)]_k + [\omega(x+y)\times h]_k\Big)\rho(|y|)\frac{dy}{|y|^3} \\
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(3\hat{y}_m h_m \varepsilon_{kjr}\hat{y}_j\omega_r(x+y) + \varepsilon_{krm}\omega_r(x+y)h_m\Big)\rho(|y|)\frac{dy}{|y|^3} \\
&= \frac{h_m}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(3\hat{y}_m\varepsilon_{kjr}\hat{y}_j + \varepsilon_{krm}\Big)\omega_r(x+y)\rho(|y|)\frac{dy}{|y|^3} \\
&= \frac{h_m}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} P_{mkr}(y)\omega_r(x+y)\rho(|y|)dy.
\end{aligned} \tag{6.180}$$

We have denoted here

$$P_{mkr} = \frac{1}{|y|^3}(3\hat{y}_m\varepsilon_{kjr}\hat{y}_j + \varepsilon_{krm}). \tag{6.181}$$

The kernel $Q(y) = P_{mkr}(y)$ (we fix for the moment the indices m, k and r) is homogenous of degree (-3):

$$Q(\lambda y) = \frac{1}{\lambda^3}Q(y), \quad \text{for all } \lambda > 0 \text{ and } y \in \mathbb{R}^3,\ y \neq 0. \tag{6.182}$$

Thus, $Q(y)$ is "barely not in L^1": if it were slightly less singular it would have been in L^1. In addition, the average of $Q(y)$ over the unit sphere (and thus over any sphere centered at $y = 0$) vanishes:

$$\int_{|y|=1} Q(y)dy = \int_{|y|=1}(3\hat{y}_m\varepsilon_{kjr}\hat{y}_j + \varepsilon_{krm})dy = 4\pi[\varepsilon_{kjr}\delta_{mj} + \varepsilon_{krm}] = 4\pi[\varepsilon_{kmr} + \varepsilon_{krm}] = 0. \tag{6.183}$$

Consider now the term (again, with an index r fixed)

$$\mathcal{Q}\omega(x) = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} Q(y)\omega_r(x+y)\rho(|y|)dy. \tag{6.184}$$

We split the integration in the definition of $\mathcal{Q}\omega$ as follows:

$$\mathcal{Q}\omega(x) = \lim_{\varepsilon\to 0}\int_{\varepsilon\le|y|\le\delta} Q(y)\omega_r(x+y)\rho(|y|)dy + \int_{|y|\ge\delta} Q(y)\omega_r(x+y)\rho(|y|)dy = A+B. \tag{6.185}$$

The second term above is (recall that $\rho(|y|)=0$ for $|y|>2R$):

$$B = \int_{\delta\le|y|\le 2R} Q(y)\omega_r(x+y)\rho(|y|)dy, \tag{6.186}$$

which can be estimated as

$$|B| \le C\|\omega\|_{L^\infty}\int_\delta^{2R}\frac{r^{n-1}}{r^n}dr \le C\|\omega\|_{L^\infty}\log\frac{2R}{\delta}. \tag{6.187}$$

The first term in (6.185) is estimated using the Hölder continuity of ω: the mean-zero property (6.183) means that we can write

$$A = \lim_{\varepsilon\to 0}\int_{\varepsilon\le|y|\le\delta} Q(y)\omega_r(x+y)\rho(|y|)dy = \lim_{\varepsilon\to 0}\int_{\varepsilon\le|y|\le\delta} Q(y)[\omega_r(x+y)-\omega_r(x)]\rho(|y|)dy. \tag{6.188}$$

The Hölder continuity of ω implies that the integrand in the last expression above has an upper bound

$$|Q(y)[\omega_r(x-y)-\omega_r(x)]\rho(|y|)| \le \frac{C}{|y|^n}|y|^\gamma\|\omega\|_{C^\gamma} = \frac{C}{|y|^{n-\gamma}}\|\omega\|_{C^\gamma}, \tag{6.189}$$

which is integrable in y at $y=0$ for $\gamma>0$. Therefore, we have

$$A = \int_{0\le|y|\le\delta} Q(y)[\omega_r(x-y)-\omega(x)]\rho(|y|)dy, \tag{6.190}$$

and

$$|A| \le C\|\omega\|_{C^\gamma}\int_0^\delta\frac{r^{n-1}}{r^{n-\gamma}}dy \le C\|\omega\|_{C^\gamma}\delta^\gamma. \tag{6.191}$$

Putting the bounds for A and B together gives (6.176).

7 The Yudovich theory for two-dimensional Euler equations

In this section, we will study some of the basic questions concerning the behavior of solutions to the two-dimensional incompressible Euler equations.

$$\begin{aligned} &u_t + (u\cdot\nabla)u + \nabla p = 0,\\ &\nabla\cdot u = 0. \end{aligned} \tag{7.1}$$

The system (7.1) should be supplemented by the initial condition $u(0,x)=u_0(x)$. Moreover, if it is posed in a domain D, we also need to impose a boundary condition on the flow $u(t,x)$. If the boundary is impenetrable, then the natural boundary condition is

$$u\cdot\nu|_{\partial D} = 0. \tag{7.2}$$

Here ν is the normal at the boundary ∂D. The Euler equations are also often considered in the whole space $\mathbb{R}^d$, with the decay conditions at infinity, or on a torus – which is equivalent to taking periodic initial data in $\mathbb{R}^d$.

7.1 The vorticity formulation of the two-dimensional Euler equations

The theory of the existence, uniqueness and regularity of the solutions to the Euler equations is quite different in two and three spatial dimensions. In the two dimensional case, for smooth initial data there exists a unique global in time smooth solution, while for the three dimensional case an analogous result is only known locally in time. The question of the global existence of smooth solutions to the Euler equations in three dimensions is a major open problem. This difference can be illustrated on a basic level by rewriting the Euler equations in the vorticity form.

An important quantity in the fluid mechanics is the vorticity $\omega = \nabla \times u$, which describes the rotational motion of the fluid. In three dimensions, if we apply the curl operator to the system (7.1), we obtain the Euler equation in the vorticity form:

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \tag{7.3}$$

with the initial condition $\omega(0, x) = \omega_0(x)$.

The vector field u can be recovered from ω via the Biot-Savart law. In order to obtain this law in $\mathbb{R}^3$, consider the (vector-valued) stream function ψ defined (in terms of the vorticity) as the solution of the Poisson equation

$$-\Delta\psi = \omega, \quad \text{in } \mathbb{R}^3. \tag{7.4}$$

Then, one can show via vector algebra that u is given by

$$u = \nabla \times \psi. \tag{7.5}$$

That is, if u and ω are related via (7.4) and (7.5), and ω is incompressible (as it should be), then $\omega = \nabla \times u$. Together, (7.4) and (7.5) form the Biot-Savart law which expresses the velocity u via the vorticity ω.

On the other hand, in the two dimensional case the term in the right side of (7.3) vanishes. This term is often called "vortex stretching term" as it can amplify the size of the vorticity. To see that the vortex stretching term is absent in two dimensions, observe that the solutions of the two-dimensional Euler equations can be thought of as solutions of the three-dimensional equations of the special form $(u_1(x_1, x_2), u_2(x_1, x_2), 0)$, $P(x_1, x_2)$. In that case, the vorticity vector has only one non-zero component:

$$\omega = (0, 0, \partial_1 u_2 - \partial_2 u_1),$$

and can be regarded as a scalar. Then, the term in the right side of (7.3) is simply

$$(\omega \cdot \nabla)u = \omega_3 \partial_3 u,$$

but the two dimensional u does not depend on x_3. Thus, in two dimensions, the vorticity equation simplifies. We will use the notation

$$\omega = \partial_1 u_2 - \partial_2 u_1, \tag{7.6}$$

instead of ω_3.

Given a smooth bounded domain D, let us define the operator $(-\Delta_D)^{-1}$ as follows: given a function ω, we denote by $\psi = (-\Delta_D)^{-1}\omega$ the unique solution of the Dirichlet boundary value problem

$$\begin{aligned} -\Delta\psi &= \omega, \quad \text{in } D, \\ \psi &= 0, \qquad \text{on } \partial D. \end{aligned} \tag{7.7}$$

The vorticity formulation of the two-dimensional Euler equations is the system

$$\partial_t\omega + (u\cdot\nabla)\omega = 0, \tag{7.8}$$

$$u = \nabla^\perp(-\Delta_D)^{-1}\omega, \tag{7.9}$$

$$\omega(0,x) = \omega_0(x),$$

where $\nabla^\perp = (\partial_2, -\partial_1)$. Note that the flow u defined by (7.9) automatically satisfies the boundary condition

$$u\cdot\nu = 0 \text{ on } \partial D. \tag{7.10}$$

This is because the gradient of the stream function $\psi = (-\Delta_D)^{-1}\omega$ is normal to ∂D due to the boundary condition, and hence $u = \nabla^\perp\psi$ is tangent to it.

Exercise 7.1. Verify that if $u(t,x)$ satisfies the Euler equations in two dimensions, then the vorticity $\omega(t,x)$ given by (7.6) satisfies (7.8), and $u(t,x)$ and $\omega(t,x)$ are related via (7.9).

The vorticity formulation of the Euler equations in two dimensions leads to several important observations. As we will shortly see, any L^p norm of the vorticity is conserved for smooth solutions of (7.8). In particular, $\|\omega\|_{L^\infty}$ does not change. In contrast, in three dimensions, the amplitude of vorticity can and often does grow due to the vortex stretching term in the right side of (7.3).

The Yudovich theory addresses existence and uniqueness of the solutions to the 2D Euler equations with a bounded initial vorticity. The L^∞ class for vorticity is very natural since it is preserved by the evolution, and is likely close to being sharp. In addition, many phenomena in nature, such as hurricanes or tornados, feature vorticities with a very abrupt variation, hence the theory of solutions with rough vorticities is not a purely mathematical issue. As we will see, if the initial condition is more regular, this regularity is reflected in the additional regularity of the solution, even though the quantitative estimates can deteriorate very quickly.

It is not immediately clear how one can define the low regularity solutions (such as L^∞) of the vorticity equation (7.8) since we need to take derivatives. A "canonical" way around that is to define a weak solution of a nonlinear equation via the multiplication of the equation by a test function and integration by parts, and then to try to obtain some a priori bounds and use compactness arguments to show that such weak solution exists. Indeed, this is the original approach of Yudovich. However, there is an arguably more elegant approach for the two-dimensional Euler equations, via a reformulation of the problem that allows us to define a weak solution in a different way. Given a divergence-free flow $u(t,x)$, recall our definition of the particle trajectories $\Phi_t(x)$:

$$\frac{d\Phi_t(x)}{dt} = u(t,\Phi_t(x)), \;\; \Phi_0(x) = x. \tag{7.11}$$

As we have seen, if u is sufficiently regular and incompressible, (7.11) defines a volume preserving map $x \to \Phi_t(x)$ for each t.

A direct calculation, using the method of characteristics, shows that if $\omega(t,x)$ is a smooth solution of (7.8), then

$$\omega(t, \Phi_t(x)) = \omega_0(x), \quad \text{thus} \;\; \omega(t,x) = \omega_0(\Phi_t^{-1}(x)). \tag{7.12}$$

The inverse map is well-defined since trajectories cannot intersect if u is sufficiently regular (we will discuss it in more detail below). In addition, denote, as before, by $G_D(x,y)$ the Green's function for the Dirichlet Laplacian in a domain D, in the sense that the solution to (7.7) is given by

$$\psi(x) = \int_D G_D(x,y)\omega(y)dy, \quad x \in D, \tag{7.13}$$

and set

$$K_D(x,y) = \nabla_x^\perp G_D(x,y). \tag{7.14}$$

Then the Biot-Savart law in two dimensions can be written as

$$u(t,x) = \int_D K_D(x,y)\omega(t,y)\,dy. \tag{7.15}$$

A classical C^1 solution of the two-dimensional Euler equations (7.8) satisfies the system (7.11), (7.12) and (7.15). On the other hand, a direct computation shows that a smooth solution to (7.11), (7.12) and (7.15) gives rise to a classical solution of (7.8). Thus, for smooth solutions the two formulations are equivalent.

We will generalize the notion of the solution to the 2D Euler equations by saying that a triple $(\omega, u, \Phi_t(x))$ solves the 2D Euler equations if it satisfies (7.11), (7.12) and (7.15). The obvious next task is to make sense of the solutions of the latter system with the only requirement that $\omega_0 \in L^\infty$. A well known theorem on solutions to systems of ordinary differential equations yields uniqueness if $u(t,x)$ is Lipschitz in x. Thus, if it were true that for $\omega(t,x) \in L^\infty$, the Biot-Savart law would give a Lipschitz function $u(t,x)$, then it would be very reasonable to expect (7.11), (7.12) and (7.15) to be a well-posed system. This looks possible – (7.9) indicates that u is "one derivative better than ω", but in fact it is not quite true – the regularity for $u(t,x)$ when $\omega \in L^\infty$ is slightly lower than Lipschitz. Nevertheless, we will see that this lower regularity is sufficient to define unique trajectories of the ODE (7.11), making the system well-posed.

7.2 The regularity of the flow

In order to construct the solutions of the 2D Euler equations in the trajectory formulation (7.11)-(7.15) with the vorticity $\omega_0 \in L^\infty$, we first need to establish the regularity of the fluid velocity given by (7.15) for a vorticity in L^∞. This question is clearly related to the regularity of the kernel $K_D(x,y)$. The following proposition summarizes some well known properties of the Dirichlet Green's function.

Proposition 7.2. *If $D \subset \mathbb{R}^2$ is a compact domain with a smooth boundary, the Dirichlet Green's function $G_D(x,y)$ has the form*

$$G_D(x,y) = \frac{1}{2\pi}\log|x-y| + h(x,y).$$

Here, for each $y \in D$, $h(x,y)$ is a harmonic function solving

$$\Delta_x h = 0, \quad h|_{x\in\partial D} = -\frac{1}{2\pi}\log|x-y|. \tag{7.16}$$

We have $G_D(x,y) = G_D(y,x)$ for all $(x,y) \in D$, and $G_D(x,y) = 0$ if either x or y belongs to ∂D. In addition, we have the estimates

$$|G_D(x,y)| \le C(D)\left(|\log|x-y|| + 1\right) \tag{7.17}$$

$$|\nabla G_D(x,y)| \le C(D)|x-y|^{-1}, \tag{7.18}$$

$$|\nabla^2 G_D(x,y)| \le C(D)|x-y|^{-2}. \tag{7.19}$$

The following lemma outlines a key regularity property of the Green's function which allows to construct the unique solutions of the Euler equations for the initial vorticity in L^∞.

Lemma 7.3. *The kernel $K_D(x,y) = \nabla^\perp G_D(x,y)$ satisfies*

$$\int_D |K_D(x,y) - K_D(x',y)|\,dy \le C(D)\phi(|x-x'|), \tag{7.20}$$

where

$$\phi(r) = \begin{cases} r(1-\log r) & r < 1 \\ 1 & r \ge 1, \end{cases} \tag{7.21}$$

with a constant $C(D)$ which depends only on the domain D.

Proof. In what follows, $C(D)$ denotes constants that may depend only on the domain D, and may change from line to line. To show (7.20), we may assume that $r = |x-x'| < 1$. Indeed, otherwise (7.20) follows from the simple observation that

$$|K_D(x,y)| \le C(D)|x-y|^{-1},$$

so that

$$\int_D |K_D(x,y)|dy \le C(D),$$

which implies (7.20) for $x, x' \in D$ such that $|x-x'| \ge 1$.

Assume now that $r < 1$ and suppose first that the interval connecting the points x and x' lies entirely inside D. Let us set

$$A = \{y \in D : \ |y-x| \le 2r\}.$$

The estimate (7.18) implies

$$\begin{aligned}\int_{D\cap A} |K_D(x,y) - K_D(x',y)|\,dy &\le C(D)\int_{B_{2r}(x)}\Big(\frac{1}{|x-y|} + \frac{1}{|x'-y|}\Big)\,dy \\ &\le C(D)\int_{B_{2r}(x)} \frac{1}{|x-y|}dy + C(D)\int_{B_{5r}(x')} \frac{1}{|x'-y|}\,dy \le C(D)r.\end{aligned} \tag{7.22}$$

We used above the fact that $|x - x'| < r$ implies that $B_{2r}(x) \subset B_{5r}(x')$.

To bound the remainder of the integral, observe that for every y,

$$|K_D(x,y) - K_D(x',y)| \le r|\nabla K_D(x''(y),y)|, \tag{7.23}$$

where the point $x''(y)$ lies on the interval connecting x and x'. Note also that choice of the set A ensures that the distances $|x-y|$, $|x'-y|$ and $|x''-y|$ are all comparable if $y \in A^c$. Then, by (7.19) and the above considerations we have

$$\begin{aligned}\int_{D\cap A^c} |K_D(x,y) - K_D(x',y)|\,dy &\le C(D)r\int_{D\cap A^c}\frac{dy}{|x''(y)-y|^2} \le C(D)r\int_{D\cap A^c}\frac{dy}{|x-y|^2} \\ &\le C(D)r\int_r^{C(D)} s^{-1}\,ds \le C(D)r(1-\log r).\end{aligned} \tag{7.24}$$

The case where the interval connecting x and x' does not lie entirely in D is similar, one just needs to replace this interval by a curve connecting x and x' with the length of the order r. We briefly sketch the argument. The following lemma can be proved by standard methods using the compactness of the domain and the regularity of the boundary, so we do not present its proof.

Lemma 7.4. *Fix an arbitrary $\varepsilon > 0$ and let $D \subset \mathbb{R}^2$ be bounded domain with a smooth boundary. Then there exists $r_0 = r_0(D,\varepsilon) > 0$ such that if $x_0 \in \partial D$, and $r \le r_0$, then $B_r(x_0) \cap \partial D$ is a curve that, by a rotation and a translation of the coordinate system, can be represented as a graph $x_2 = f(x_1)$, with $x_0 = (0,0)$. The function f is C^∞, and $f'(0) = 0$. Moreover, the part of the boundary ∂D within $B_r(x_0)$ lies in the narrow angle between the the lines $x_2 = \pm\epsilon x_1$.*

With this lemma, suppose we have x and x' such that the interval connecting these points does not lie in D. It is enough to consider the case where $|x-x'| = r < r_0/2$, where r_0 is as in Lemma 7.4 corresponding to a sufficiently small ε. Indeed, the larger values of $|x-x'|$ can be handled by adjusting $C(D)$ in (7.20). Find a point $x_0 \in \partial D$ closest to x (it does not have to be unique). Note that by the assumption that the interval (x,x') crosses the boundary, we must have $|x-x_0| \le r_0/2$ and $|x'-x_0| < r_0$. Thus, both x and x' lie in the disk $B(x_0,r_0)$ where ∂D lies between the lines $x_2 = \pm\varepsilon x_1$. It is also not hard to see that x must lie on the vertical x_2-axis of a system of coordinates centered at x_0, with the horizontal x_1-axis tangent to ∂D at x_0. We also know that x' must lie in the narrow angle between the lines $x_2 = \pm\varepsilon x_1$. Otherwise, the interval (x,x') could not have crossed the boundary. Now take a curve connecting x and x' consisting of a straight vertical interval from x' to a point on one of the lines $x_2 = \pm\varepsilon x_1$ which is closest to x, and then an interval connecting this point to x. We can smooth out this curve without changing its length by much. It is easy to see that the length of this curve does not exceed $2r$ if ε is small enough. The rest of the proof goes through as before. □

Now we can state the regularity result for the fluid velocity.

Corollary 7.5. *The fluid velocity u satisfies*

$$\|u\|_{L^\infty} \le C(D)\|\omega\|_{L^\infty}, \tag{7.25}$$

and

$$|u(x) - u(x')| \leq C\|\omega\|_{L^\infty}\phi(|x - x'|), \tag{7.26}$$

with the function $\phi(r)$ defined in (7.21).

Proof. By (7.18), we have, for any $x, y \in D$,

$$|K_D(x,y)| \leq C(D)|x-y|^{-1},$$

so that

$$\left|\int_D K_D(x,y)\omega(y)\,dy\right| \leq C(D)\|\omega\|_{L^\infty}\int_D \frac{1}{|x-y|}\,dy \leq C(D)\|\omega\|_{L^\infty},$$

which is (7.25). The proof of (7.26) is immediate from Lemma 7.3, as

$$u(t,x) = \int_D K_D(x,y)\omega(t,y)dy,$$

and we are done. □

We say that u is log-Lipschitz if it satisfies (7.26): there exists $M > 0$ so that

$$|u(t,x) - u(t,x')| \leq M\phi(|x - x'|). \tag{7.27}$$

We will see that this bound is in fact sharp: there are velocities that correspond to bounded vorticities which are just log-Lipschitz and in particular fail to be Lipschitz.

7.3 Trajectories for log-Lipschitz velocities

7.3.1 Existence and uniqueness of trajectories

As the fluid velocity with an L^∞-vorticity is not necessarily Lipschitz but only log-Lipschitz, we may not use the classical results on the existence and uniqueness of the solutions of systems of ODEs with Lipschitz velocities. Nevertheless, as we show next, the log-Lipschitz regularity is sufficient to determine the fluid trajectories uniquely.

Lemma 7.6. *Let D be a bounded smooth domain in $\mathbb{R}^d$. Assume that the velocity field $b(t,x)$ satisfies, for all $t \geq 0$:*

$$b \in C([0,\infty) \times \bar{D}), \;\; |b(t,x) - b(t,y)| \leq C\phi(|x-y|), \;\; b(t,x)\cdot\nu|_{\partial D} = 0. \tag{7.28}$$

Here, the function $\phi(r)$ is given by (7.21) and ν is the unit normal to ∂D at the point x. Then the Cauchy problem in $\bar{D}$

$$\frac{dx}{dt} = b(t,x), \;\; x(0) = x_0, \tag{7.29}$$

has a unique global solution for all $x_0 \in \bar{D}$. Moreover, if $x_0 \notin \partial D$, then $x(t) \notin \partial D$ for all $t \geq 0$. If $x_0 \in \partial D$, then $x(t) \in \partial D$ for all $t \geq 0$.

Note that the log-Lipschitz regularity is border-line: the familiar example of the ODE

$$\dot{x} = x^\beta, \quad x(0) = 0,$$

with $\beta \in (0,1)$ does not have the uniqueness property: for example, $x(t) \equiv 0$, and

$$x(t) = \frac{t^p}{p^p}, \quad p = \frac{1}{1-\beta}$$

are both solutions. Thus, ODEs with Hölder (with an exponent smaller than one) velocities may have more than one solution. Existence of the solutions, on the other hand, does not really require the log-Lipschitz condition: uniform continuity of $b(t,x)$ and at most linear growth as $|x| \to +\infty$ would be sufficient.

Proof. Step 1. Existence of a local in time solution. Let us first show the existence of a local solution using a version of the standard Picard iteration: set

$$x_n(t) = x_0 + \int_0^t b(s, x_{n-1}(s))\, ds, \quad x_0(t) \equiv x_0.$$

Let us assume first that $x_0 \in D$. Then, as usual, we have, using the log-Lipschitz property of b:

$$|x_n(t) - x_{n-1}(t)| \le \int_0^t |b(s, x_{n-1}(s)) - b(s, x_{n-2}(s))|\, ds \le C \int_0^t \phi(|x_{n-1}(s) - x_{n-2}(s)|)\, ds. \tag{7.30}$$

Since the function $\phi(r)$ is concave, we have

$$\phi(r) \le \phi(\varepsilon) + \phi'(\varepsilon)(r - \varepsilon) = \varepsilon(1 + \log \varepsilon^{-1}) + (r - \varepsilon)\log \varepsilon^{-1} = \varepsilon + r \log \varepsilon^{-1},$$

for every $\varepsilon < 1$. Using this in (7.30) gives

$$|x_n(t) - x_{n-1}(t)| \le C \log(\varepsilon^{-1}) \int_0^t |x_{n-1}(s) - x_{n-2}(s)|\, ds + Ct\varepsilon.$$

Exercise 7.7. Use an induction argument to show that (7.30) implies, for any $0 \le t \le T$ and $\varepsilon \in (0,1)$

$$|x_n(t) - x_{n-1}(t)| \le CT\varepsilon \sum_{k=0}^{n-2} \frac{C^k (\log \varepsilon^{-1})^k t^k}{k!} + \frac{C^{n-1} t^{n-1} (\log \varepsilon^{-1})^{n-1}}{(n-1)!} \sup_{0 \le t \le T} |x_1(t) - x_0|. \tag{7.31}$$

As

$$|x_1(t) - x_0| \le Ct,$$

we have

$$|x_n(t) - x_{n-1}(t)| \le CT\varepsilon \exp(CT \log \varepsilon^{-1}) + \frac{C^n T^n (\log \varepsilon^{-1})^{n-1}}{(n-1)!},$$

for any $\varepsilon > 0$ and all $n \ge 2$, with a constant C that is independent of $\varepsilon > 0$ or n. We may now choose $\varepsilon = \exp(-n)$ and T sufficiently small so that $1 - CT > 1/2$. This leads to

$$|x_n(t) - x_{n-1}(t)| \le CT \exp(-n/2) + \frac{C^n T^n n^{n-1}}{(n-1)!}.$$

The Stirling formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

implies that if T is sufficiently small (independently of n), then

$$|x_n(t) - x_{n-1}(t)| \le \alpha^n,$$

with $\alpha < 1$. Thus, $x_n(t)$ converges uniformly to a limit $x(t)$. The uniformity of the convergence implies that the limit satisfies the integral equation

$$x(t) = x_0 + \int_0^t b(s, x(s))\, ds. \tag{7.32}$$

We also need to choose T so that $|x(t) - x_0| \le \mathrm{dist}(x_0, \partial D)$. Taking

$$T < \|b\|_{L^\infty}^{-1} \mathrm{dist}(x_0, \partial D), \tag{7.33}$$

would suffice. As b is continuous, we may differentiate (7.32) and obtain the desired ODE

$$\frac{dx(t)}{dt} = b(t, x(t)), \quad x(0) = x_0,$$

for all t on the time interval $0 \le t \le T$.

Step 2. Uniqueness of a local in time solution. Next, we show the uniqueness of this local solution – here, the log-Lipchitz property will play a crucial role. We will prove a little more general stability estimate than needed for the uniqueness, as we will need it later. Let $\sigma > 0$ be a small number. Suppose that $x(t)$ and $y(t)$ are two different solutions to (7.29) with the initial data satisfying $0 < |x_0 - y_0| < \sigma$ and set $z(t) = |x(t) - y(t)|$. Then, by the log-Lipschitz assumption on b in (7.28), we have

$$z(t)\dot{z}(t) = \frac{1}{2}\frac{d}{dt}|z(t)|^2 = (x(t) - y(t)) \cdot (b(x(t), t) - b(y(t,t)) \le Cz(t)\phi(z(t)), \tag{7.34}$$

as well as

$$z(t)\dot{z}(t) = \frac{1}{2}\frac{d}{dt}|z(t)|^2 = (x(t) - y(t)) \cdot (b(x(t), t) - b(y(t,t)) \ge -Cz(t)\phi(z(t)), \tag{7.35}$$

It follows that, as long as $z(t) > 0$ (which is true for $t > 0$ sufficiently small by the continuity of $x(t)$ and $y(t)$), we have

$$-C\phi(z(t)) \le \dot{z}(t) \le C\phi(z(t)), \;\; 0 < z(0) < \sigma.$$

In order to control $z(t)$, define $f_\sigma(t)$ as the solution of

$$\dot{f}_\sigma = 2C\phi(f_\sigma(t)), \;\; f_\sigma(0) = \sigma,$$

and $g_\sigma(t)$ as the solution of

$$\dot{g}_\sigma = -2C\phi(f_\sigma(t)), \;\; g_\sigma(0) = \frac{z(0)}{2}.$$

We claim that

$$g_\sigma(t) < z(t) < f_\sigma(t), \quad \text{for all } t > 0. \tag{7.36}$$

We will only show that $z(t) < f_\sigma(t)$, with the other inequality proved similarly. This is true for some initial time interval, simply because both $z(t)$ and $f_\sigma(t)$ are continuous and $f_\sigma(0) > z(0)$. Let $t_1 > 0$ be the smallest time such that $z(t_1) = f_\sigma(t_1)$. At this time, by the definition of t_1, we would have

$$\dot{z}(t_1) \geq \dot{f}_\sigma(t_1). \tag{7.37}$$

On the other hand, we would also have

$$\dot{z}(t_1) - \dot{f}_\sigma(t_1) \leq C\phi(z_1(t)) - 2C\phi(f_\sigma(t_1)) = -C\phi(z(t_1)) < 0,$$

contradicting the definition of t_1. Thus, no such t_1 exists and

$$z(t) < f_\sigma(t) \text{ for all } t \geq 0. \tag{7.38}$$

The proof of the lower bound in (7.36) is similar.

Now, we need an estimate on $f_\sigma(t)$. Let us show that for any $t > 0$ fixed we have

$$\lim_{\sigma \to 0^+} f_\sigma(t) = 0. \tag{7.39}$$

It suffices to consider the case where σ is small and times are small enough so that $f_\sigma(t) < 1$. Then we have

$$\frac{d}{dt} \log f_\sigma(t) = 2C(1 - \log f_\sigma(t)).$$

Solving this differential equation leads to

$$1 - \log f_\sigma(t) = (1 - \log \sigma)e^{-2Ct},$$

or

$$f_\sigma(t) = \sigma^{\exp(-2Ct)} \exp(1 - \exp(-2Ct)), \tag{7.40}$$

whence (7.39) follows. If the initial conditions for $x(t)$ and $y(t)$ are the same, then

$$0 \leq z(t) \leq f_\sigma(t) \text{ for every } \sigma > 0. \tag{7.41}$$

Now, (7.39) and (7.41) imply that $z(t) \equiv 0$, hence the solution $x(t)$ of (7.29) is unique.

Exercise 7.8. Identify the place in the uniqueness proof above where we have used the log-Lipschitz condition on the function $b(t, x)$; that is, where the proof would have failed, for example, for $\phi(r) = r^\beta$, with $\beta \in (0, 1)$.

Step 3. Global in time existence. We now address the question of the global existence. Having constructed a local solution until a time t, we can continue to extend our local solution from t to a time $t + \Delta t$, using the local in time existence we have just proved, since $x(t)$ is inside D. However, as (7.33) shows, the time step Δt depends on the distance from $x(t)$ to ∂D. Thus, in order to construct a global in time solution we need to control this distance. Let us set

$$d(t) = \text{dist}(x(t), \partial D),$$

with $d(0) \equiv d_0 > 0$ since $x_0 \in D$. Our goal is to get a lower bound on $d(t)$. Note first that since $b \in L^\infty$, the trajectory $x(t)$ is Lipschitz in time, and so is the function $d(t)$. Thus, by the Rademacher theorem, the derivative $\dot{d}(t)$ exists almost everywhere, and

$$d(t) = d_0 + \int_0^t \dot{d}(s)\, ds.$$

We will now estimate $\dot{d}(t)$ from below at any given time t for which the local solution is defined. Consider the set

$$S(t) = \{P \in \partial D :\ |x(t) - P| = d(t)\},$$

and, given $\kappa > 0$, define

$$S_\kappa(t) = \{Q \in \partial D :\ \exists P \in S(t),\ \ |Q - P| < \kappa\}.$$

We can think of the set $S_\kappa(t)$ as the points on ∂D that are very close to the points at which the distance between $x(t)$ and ∂D is realized. Therefore, we expect these points to be important for the estimate of how the distance changes. Fix some small $\varepsilon > 0$, and take $\kappa_\varepsilon > 0$ sufficiently small, so that if $Q \in S_{\kappa_\varepsilon}(t)$, then there exists $P \in S(t)$ such that

$$\left| \frac{Q - x(t)}{|Q - x(t)|} - \nu_P \right| < \frac{\varepsilon}{\|b\|_{L^\infty}}. \tag{7.42}$$

Here, ν_P is the outside unit normal to ∂D at the point P. Such κ_ε exists due to the smoothness of the boundary ∂D.

Exercise 7.9. Assume that the boundary ∂D can be represented around the point P as a graph $\partial D = (w, g(w))$ with $P = (0,0)$ and $\dot{g}(0) = 0$. Assume that the function $g(w)$ is bounded in C^2 and find an explicit bound for κ which ensures that (7.42) holds.

Let us now proceed to estimate $d(s)$ for times s slightly large than t. Consider first any point $Q \in \partial D \setminus S_{\kappa_\varepsilon}(t)$. The set $\partial D \setminus S_{\kappa_\varepsilon}(t)$ is compact, and $\mathrm{dist}(x(t), Q) > d(t)$ for every point $Q \in \partial D \setminus S_{\kappa_\varepsilon}(t)$. Therefore, there exists $\gamma_\varepsilon > 0$ such that

$$|x(t) - Q| > d + \gamma_\varepsilon, \quad \text{for all } Q \in \partial D \setminus S_{\kappa_\varepsilon}(t). \tag{7.43}$$

We deduce that

$$|x(s) - Q| \geq |x(t) - Q| - |x(t) - x(s)| \geq d(t) + \gamma_\varepsilon - \|b\|_{L^\infty}(s - t), \text{ for all } s > t \text{ and } Q \in \partial D \setminus S_{\kappa_\varepsilon}(t).$$

Thus, if

$$0 < s - t \leq \gamma_\varepsilon \|b\|_{L^\infty}^{-1},$$

then

$$|x(s) - Q| \geq d(t) \text{ for any } Q \in \partial D \setminus S_{\kappa_\varepsilon}(t). \tag{7.44}$$

Next, suppose that $Q \in S_{\kappa_\varepsilon}(t)$. We have

$$x(s) - Q = x(t) + \int_t^s b(r, x(r))\, dr - Q. \tag{7.45}$$

Denote

$$\bar{e} = \frac{x(t) - Q}{|x(t) - Q|},$$

and note that (7.42) says that

$$|\bar{e} - \nu_P| < \frac{\varepsilon}{\|b\|_{L^\infty}}. \tag{7.46}$$

Going back to (7.45), we obtain

$$|x(s)-Q| \geq (x(s)-Q)\cdot\bar{e} = |x(t)-Q| + \int_t^s b(x(r),r)\,dr\cdot\bar{e} \geq d(t) + \int_t^s b(x(r),r)\,dr\cdot\bar{e}. \tag{7.47}$$

We also recall that by the last assumption in (7.28) we have

$$b(P,t)\cdot\nu_P = 0. \tag{7.48}$$

Next, using (7.46) and (7.48), we get

$$\begin{aligned}
&\int_t^s b(x(r),r)\,dr\cdot\bar{e} = \int_t^s b(x(r),r)\cdot(\bar{e}-\nu_P)\,dr + \int_t^s (b(x(r),r) - b(P,r))\cdot\nu_P\,dr \\
&\geq \int_t^s (b(x(r),r) - b(P,r))\cdot\nu_P\,dr - \|b\|_{L^\infty}(s-t)\varepsilon\|b\|_{L^\infty}^{-1} \\
&\geq -C\int_t^s \phi(|x(r)-P|)\,dr - \varepsilon(s-t) \geq -C\phi(2d(t))(s-t) - \varepsilon(s-t).
\end{aligned} \tag{7.49}$$

In the last step we used that $|x(r) - P| \leq 2d(t)$ if $r - t$ is small enough, depending on $d(t)$ and $\|b\|_\infty$. Note also that

$$\phi(2d(t)) \leq 2\phi(d(t))$$

by concavity. To summarize (7.44), (7.47), and (7.49), we have

$$d(s) \geq d(t) - C\phi(d(t))(s-t) - \varepsilon(s-t).$$

for s sufficiently close to t. Therefore, since $\varepsilon > 0$ is arbitrary, we get

$$\dot{d}(t) \geq -C\phi(d(t)),$$

for every t for which the derivative exists. Solving this differential inequality, similarly to (7.40), we obtain

$$d(t) \geq d_0^{\exp(Ct)}\exp(1 - \exp(Ct)). \tag{7.50}$$

Therefore, the local solution can be continued indefinitely in time, and $x(t)$ will never arrive at ∂D if $x_0 \notin \partial D$.

Step 4. Starting point on the boundary. It remains to consider the case of $x_0 \in \partial D$. In this situation, take $x_n \in D$, $n = 1, \ldots,$ such that

$$\lim_{n\to\infty} x_n = x_0,$$

and consider the corresponding solutions $x_n(t)$. Due to the estimates (7.38) and (7.40), the sequence $x_n(t)$ is Cauchy in $C([0,T],\mathbb{R}^d)$ for any $T < \infty$. Therefore it has a limit $x(t)$ in this space, and this limit satisfies the integral form (7.32). We can then differentiate it in time, arriving at (7.29).

Finally, we claim that $x(t) \in \partial D$ for all times if $x_0 \in \partial D$. Indeed, suppose there exists t_0 such that $x(t_0) \notin \partial D$. Let us invert time and solve the characteristic backwards:

$$\frac{dy}{ds} = -b(t_0 - s, y(s)), \;\; y(0) = x(t_0). \tag{7.51}$$

Then $y(s)$ and $x(t_0 - s)$ satisfy the same differential equation with log-Lipschitz coefficient, so by our previous result on uniqueness, we know that $y(s) = x(t_0 - s)$. But this means that $y(s)$ starts at $x(t_0) \in D$ and arrives at $x_0 \in \partial D$ in a finite time. This contradicts our earlier estimates that apply in the same fashion to the backwards equation (7.51). □

7.3.2 The Hölder regularity of the flow map

We will now obtain a uniform continuity bound on the trajectories $\Phi_t(x)$, which are the solutions to

$$\frac{d}{dt}\Phi_t(x) = b(\Phi_t(x), t), \;\;\; \Phi_0(x) = x, \tag{7.52}$$

when the flow $b(x,t)$ is only log-Lipschitz in x. To contrast our set up with more regular situation, let us first recall the following result.

Exercise 7.10. Let $b(t,x)$ be a Lipschitz function in x: there exists $g(t) \in L^1_{\rm loc}(0,\infty)$ so that

$$|b(x,t) - b(y,t)| \le g(t)|x-y|, \text{ for all } t \ge 0 \text{ and } x, y \in \mathbb{R}^d. \tag{7.53}$$

Show that the solution to (7.52) satisfies a Lipschitz bound

$$|\Phi_t(x) - \Phi_t(y)| \le |x-y| \exp\Big\{ \int_0^t g(s)\, ds \Big\}. \tag{7.54}$$

In contrast to (7.54), we have the following Hölder estimate for the flow map when the velocity is only log-Lipschitz.

Lemma 7.11. *Suppose that $D \subset \mathbb{R}^d$ is a smooth bounded domain, and the map $\Phi_t(x)$ is generated by a log-Lipschitz vector field $b(t,x)$ satisfying assumptions of Lemma 7.6. Then, for every $x, y \in \bar{D}$ with $|x-y| \le 1/2$, and while $|\Phi_t(x) - \Phi_t(y)| \le 1/2$, we have*

$$|x-y|^{e^{Ct}} \le |\Phi_t(x) - \Phi_t(y)| \le |x-y|^{e^{-Ct}}. \tag{7.55}$$

The constant C in (7.55) only depends on the constant in the log-Lipschitz bound for b.

Of course, one can write the corresponding bounds for all $x, y \in D$ (recall that D is bounded, so $|x-y| \le C(D)$). We restrict to the $\le 1/2$ range to simplify the argument, as the bound looks different at large distances. Also note that the bound (7.55) similarly applies to $\Phi_t^{-1}(x)$.

This is a rather remarkable estimate: we can show that $\Phi_t(x)$ is Hölder continuous in space for any $t \ge 0$, but the Hölder exponent deteriorates in time. The loss of regularity compared to the result for the Lipschitz velocities in Exercise 7.10 is pretty dramatic: not only the solution is no longer Lipschitz, it cannot even keep a constant in time Hölder exponent. This is a reflection of the complexity of dynamics: the exponent in the upper bound in (7.55) tends to zero as $t \to +\infty$ because two trajectories that start very close at $t = 0$ may diverge very far at large times – much further than for Lipschitz velocities. On the other hand, the exponent in the lower bound in (7.55) grows as $t \to +\infty$ because even if at the time $t = 0$ the starting points x and y are relatively far apart, they can be extremely close at large times. This deterioration of the estimates is not an artefact of the proof – the particle trajectories corresponding to true solutions of the Euler equations can get extremely close at large times.

Proof. The result is of course closely related to the estimates (7.38) and (7.40). Let us fix x and y, and set $F(t) = |\Phi_t(x) - \Phi_t(y)|$. We compute

$$\left|\frac{d}{dt}F^2(t)\right| = 2\,|(\Phi_t(x) - \Phi_t(y)) \cdot (b(\Phi_t(x), t) - b(\Phi_t(y), t))| \le 2CF(t)\phi(F(t)),$$

with the constant $C > 0$ that depends on the domain D and $\|\omega_0\|_{L^\infty}$. Thus

$$|F'(t)| \le CF(t)\max(1, 1 - \log F(t)).$$

Recall that we only need to consider the case when $F(t) \le 1/2$. Then we have

$$|F'(t)| \le CF(t)\log F(t)^{-1},$$

which leads to

$$[\log F(0)]e^{Ct} \le \log F(t) \le [\log F(0)]e^{-Ct}.$$

The estimate (7.55) follows immediately from exponentiating this inequality and taking into account that $F(0) = |x - y|$. □

7.4 The approximation scheme

Let us return to our strategy of constructing a triple $(\omega, u, \Phi_t(x))$ solving (7.11), (7.12) and (7.15), with the initial vorticity $\omega_0 \in L^\infty$. We define an iterative sequence of approximations

$$\frac{d}{dt}\Phi_t^n(x) = u^n(t, \Phi_t^n(x)), \quad \Phi_0^n(x) = x, \tag{7.56}$$

$$u^n(t, x) = \int_D K_D(x, y)\omega^{n-1}(t, y)\,dy, \tag{7.57}$$

$$\omega^n(t, x) = \omega_0((\Phi_t^n)^{-1}(x)), \tag{7.58}$$

with $\omega^0(t, x) \equiv \omega_0(x) \in L^\infty$ for all $t \ge 0$. Note that since the velocities u^n defined by (7.57) satisfy the no flow boundary conditions at ∂D, and by Corollary 7.5 and Lemma 7.6, the solutions to the trajectory equation (7.56) exist and are unique.

Moreover, the trajectory maps $\Phi_t^n(x)$ are injective due to the uniqueness of the backward trajectories and surjective due to the global existence of these backward trajectories. Therefore, the inverse maps $(\Phi_t^n)^{-1}(x)$ in (7.58) are well-defined. Both the direct and the inverse trajectory maps are also continuous in x for each t on $\bar{D}$ due to the estimates (7.38) and (7.40), and map D to D and ∂D to ∂D. In fact, it follows from (7.40) that these maps also satisfy the Hölder regularity bounds, which we will spell out precisely in a moment.

Intuitively, each successive approximation involves solving a linear problem

$$\omega^n_t + (u^n \cdot \nabla)\omega^n = 0, \tag{7.59}$$

with the flow

$$u^n(t,x) = \int_D K_D(x,y)\omega^{n-1}(t,y)\,dy, \tag{7.60}$$

computed from the previous iteration. Note that each $\omega^n \in L^\infty$, with

$$\|\omega^n(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}. \tag{7.61}$$

However, one can not take (7.59) too literally, since we only know that ω_0 is in L^∞, and there is no reason to expect that the iterates ω^n are smooth, which is needed to make sense of (7.59) pointwise. Thus, we resort to the approximation scheme (7.56)-(7.58) as the weak formulation for (7.59)-(7.60).

The next step is to obtain uniform bounds on the solutions to the approximation scheme that will allow us to pass to the limit $n \to \infty$ and get a solution to (7.11)-(7.15).

7.4.1 The flow map corresponding to divergence free log-Lipschitz velocity is measure preserving

It will be useful for us to know that the trajectory maps corresponding to log-Lipschitz vector fields are measure preserving. We have discussed that if u is smooth and $\nabla \cdot u = 0$, then the associated trajectories map is measure preserving. However, this argument does not apply directly when the vector field $u(t,x)$ is just log-Lipschitz in the spatial variable. Taking the derivatives of the flow map to study the Jacobian is not straightforward. We will instead use an approximation argument to establish this property.

Lemma 7.12. *Let $D \in \mathbb{R}^d$ and $b(t,x)$ satisfy the assumptions of Lemma 7.6. Assume, in addition, that $\nabla \cdot b = 0$ in the distributional sense. Then, the trajectory map $\Phi_t(x)$ defined by the vector field $b(t,x)$ according to (7.11) is measure preserving on D.*

Proof. From the proof of Lemma 7.6 and Lemma 7.11, we already know that $\Phi_t(x)$ is a Hölder continuous bijection on D. It suffices to check the preservation of measure for an arbitrary d-dimensional interval lying in D, at a positive distance from ∂D. Fix such interval I and an arbitrary time $T > 0$. We will use a smooth incompressible flow that approximates $b(t,x)$ in a neighborhood of $\Phi_t(I)$. It is constructed as follows. According to the estimate (7.50), there exists $\kappa > 0$ such that

$$\mathrm{dist}(\Phi_t(I), \partial D) \ge \kappa \text{ for all } 0 \le t \le T.$$

Take any $\delta < \kappa/2$, and set

$$I_\delta := \{x \in D \,|\, \mathrm{dist}(x, I) < \delta\}.$$

Further decreasing δ if necessary, we may ensure that

$$\mathrm{dist}(\Phi_t(I_\delta), \partial D) \ge \kappa/2 \text{ for all } 0 \le t \le T.$$

Let $\eta(x)$ be a standard mollifier:

$$\eta \in C_0^\infty(\mathbb{R}^d),\ \eta(x) = 0 \text{ if } |x| \geq 1, \text{ and } \int_{\mathbb{R}^d} \eta(x)\,dx = 1.$$

Take any $\epsilon < \kappa/4$, and define

$$b_\epsilon = \eta_\epsilon * b,$$

with $\eta_\epsilon(x) = \eta(x/\epsilon)$. The flow $b_\epsilon(t,x)$ is defined for all x such that $\mathrm{dist}(x,\partial D) < \varepsilon$. In addition, it is smooth, and it is easy to check that $b_\varepsilon(t,x)$ is divergence free. Let us denote the trajectory map corresponding to $b_\epsilon(t,x)$ by $\Phi_t^\epsilon(x)$. We have

$$\begin{aligned}
|\Phi_t(x) - \Phi_t^\epsilon(x)| &\leq \left|\int_0^t (b(s,\Phi_s(x)) - b(s,\Phi_s^\epsilon(x)))\,ds\right| + \left|\int_0^t (b(s,\Phi_s^\epsilon(x))) - b_\epsilon(s,\Phi_s^\epsilon(x)))\,ds\right| \\
&\leq C\int_0^t \phi(|\Phi_s(x) - \Phi_s^\epsilon(x)|) + C\phi(\epsilon)t. \qquad (7.62)
\end{aligned}$$

Here we used the log-Lipschitz bound on b to estimate both terms. We have assumed above that $\Phi_t^\epsilon(x)$ does not come within distance ε to the boundary ∂D, and we now verify that this indeed does not happen if we choose ϵ to be small enough. One can see from (7.62) that

$$|\Phi_t(x) - \Phi_t^\epsilon(x)| \leq g(t),$$

where $g(t)$ satisfies

$$g'(t) = C\phi(g(t)) + C\phi(\epsilon),\ \ g(0) = 0.$$

Exercise 7.13. Let $h(t)$ be the solution of

$$h'(t) = C\phi(h(t)),\ \ h(0) = C\phi(\epsilon)T.$$

Show that $g(t) \leq h(t)$, for $0 \leq t \leq T$.

We can find $h(t)$ explicitly (at least while $h(t) \leq 1$):

$$h(t) = (C\phi(\epsilon)T)^{\exp(-Ct)}\exp(1 - \exp(-Ct)).$$

Therefore, there exists $\beta = \beta(T) > 0$ such that

$$|\Phi_t(x) - \Phi_t^\epsilon(x)| \leq C\epsilon^\beta \qquad (7.63)$$

for all $0 \leq t \leq T$. We can then choose ϵ so that, in particular, we have

$$|\Phi_t(x) - \Phi_t^\epsilon(x)| \leq \kappa/4 \text{ for all } 0 \leq t \leq T,$$

and so $\Phi_t^\epsilon(x)$ stays at least distance ε away from ∂D for all $x \in I_\delta$ during this time interval.

Next, take a cut-off function $f \in C_0^\infty(I_\delta)$ such that

$$0 \leq f(x) \leq 1,\ \|\nabla f(x)\|_{L^\infty} \leq C\delta^{-1}, \text{ and } f(x) = 1 \text{ if } x \in I,$$

Observe that

$$|\Phi_t^{-1}(I)| = \int_D \chi_I(\Phi_t(x))\,dx \le \int_D f(\Phi_t(x))\,dx \le \int_D \chi_{I_\delta}(\Phi_t(x))\,dx = |\Phi_t^{-1}(I_\delta)|, \tag{7.64}$$

and

$$|I| = |(\Phi_t^\epsilon)^{-1}(I)| \le \int_D f(\Phi_t^\epsilon(x))\,dx \le |(\Phi_t^\epsilon)^{-1}(I_\delta)| = |I_\delta|. \tag{7.65}$$

We used in (7.65) the fact that $(\Phi_t^\epsilon)^{-1}$ is measure preserving since this map is generated by a smooth incompressible velocity field. On the other hand, for $0 \le t \le T$ we have

$$\left|\int_D f(\Phi_t(x))\,dx - \int_D f(\Phi_t^\epsilon(x))\,dx\right| \le \|\nabla f\|_{L^\infty}|D| \sup_{x\in I_\delta, 0\le t\le T} |\Phi_t(x) - \Phi_t^\epsilon(x)| \le \frac{C(D)\epsilon^\beta}{\delta}. \tag{7.66}$$

We used (7.63) in the last step. Taking δ to zero, and simultaneously taking $\epsilon = \delta^{2/\beta}$ to zero (so that the right hand side of (7.66) goes to zero too), and using (7.64), (7.65) and (7.66), we conclude that

$$|\Phi_t^{-1}(I)| \le |I|,$$

for every interval $I \subset D$ at a positive distance from ∂D, and any $0 \le t \le T$. It follows that the same is true for any open set $\Omega \subset D : |\Phi_t^{-1}(\Omega)| \le |\Omega|$. An analogous argument using

$$\int_D f(\Phi_t^{-1}(x))\,dx, \text{ and } \int_D f((\Phi_t^\epsilon)^{-1}(x))\,dx,$$

leads to the inequality $|\Phi_t(\Omega)| \le |\Omega|$. Since Φ_t, Φ_t^{-1} are continuous and bijective, these two inequalities together imply that these maps are measure preserving. □

7.4.2 The time regularity of velocities

In order to be able to use Lemma 7.6 in the analysis of the approximation scheme, we need to establish the necessary bounds on the velocities u^n.

The space regularity estimates on u^n can be obtained using (7.61) and Corollary 7.5: it follows that all $u^n(t,x)$ are uniformly bounded and log-Lipschitz:

$$|u^n(t,x) - u^n(t,x')| \le C(D)\phi(|x-x'|). \tag{7.67}$$

In the direction of time continuity, we only need continuity but stronger control is not hard to get.

Lemma 7.14. *The velocities u^n are uniformly log-Lipschitz in time. Namely,*

$$|u^n(x,t_2) - u^n(x,t_1)| \le C\phi(|t_2-t_1|), \tag{7.68}$$

with a constant C independent of n, x, and $t_{1,2}$.

Proof. Let us take $t_2 > t_1 \ge 0$. Clearly we need to focus on the case of $|t_2 - t_1| \le 1$, since otherwise the estimate follows from uniform L^∞ bound on u^n. Now let us denote by $\Phi^n_{t_1,t_2}(z)$ the flow map generated by u^n from time t_1 to t_2, that is,

$$\Phi^n_{t_1,t_2}(z) = z + \int_{t_1}^{t_2} u^n(\Phi^n_{t_1,t}(z),t)\,dt,$$

so that

$$\Phi^n_{t_2}(y) = \Phi^n_{t_1,t_2}(\Phi^n_{t_1}(y)), \tag{7.69}$$

and

$$\Phi^n_{t_1,t_2}(y) = \Phi^n_{t_2}([\Phi^n_{t_1}]^{-1}(y)). \tag{7.70}$$

It follows that the map $\Phi^n_{t_1,t_2}$ is also measure-preserving.

We now write, first,

$$u^n(x,t_1) = \int_D K_D(x,y)\omega^{n-1}(t_1,y)\,dy = \int_D K_D(x,z)\omega_0((\Phi^{n-1}_{t_1})^{-1}(z))\,dz, \tag{7.71}$$

and, second, using the measure-preserving property of the map $\Phi^{n-1}_{t_1,t_2}$,

$$\begin{aligned}
u^n(x,t_2) &= \int_D K_D(x,y)\omega^{n-1}(t_2,y)\,dy = \int_D K_D(x,y)\omega_0((\Phi^{n-1}_{t_2})^{-1}(y))\,dy \\
&= \int_D K_D(x,\Phi^{n-1}_{t_1,t_2}(z))\omega_0((\Phi^{n-1}_{t_2})^{-1}\Phi^{n-1}_{t_1,t_2}(z))\,dz \\
&= \int_D K_D(x,\Phi^{n-1}_{t_1,t_2}(z))\omega_0((\Phi^{n-1}_{t_1})^{-1}(z))\,dz.
\end{aligned} \tag{7.72}$$

This gives

$$u^n(x,t_2) - u^n(x,t_1) = \int_D \big(K_D(x,\Phi^n_{t_1,t_2}(z)) - K_D(x,z)\big)\,\omega_0((\Phi^{n-1}_{t_1})^{-1}(z))\,dz. \tag{7.73}$$

Note that for all $z \in D$ we have

$$|\Phi^n_{t_1,t_2}(z) - z| \le \sup_n \|u^n\|_{L^\infty}|t_2 - t_1| \le C(D)\|\omega_0\|_{L^\infty}|t_2 - t_1|. \tag{7.74}$$

Let us set

$$r = 2C(D)\|\omega_0\|_{L^\infty}|t_2 - t_1|. \tag{7.75}$$

Using again the measure-preserving property of the map $\Phi^n_{t_1,t_2}$ the expression in the right side of (7.73) can be bounded by

$$\begin{aligned}
|u^n(x,t_2) - u^n(x,t_1)| &\le C(D)\|\omega_0\|_{L^\infty}\left(\int_{B_r(x)\cap D}\frac{dz}{|x-z|} + \int_{B_r(x)^c\cap D}\frac{|\Phi^n_{t_1,t_2}(z) - z|}{|x-z|^2}\,dz\right) \\
&\le C(D)\|\omega_0\|_{L^\infty}\left(r + \|u^n\|_{L^\infty}|t_2 - t_1|\log r^{-1}\right) \\
&\le C(D,\|\omega_0\|_{L^\infty})\phi(|t_2 - t_1|).
\end{aligned} \tag{7.76}$$

Thus, $u(t,x)$ is log-Lipschitz in time. □

Convergence of the approximation scheme

Let us now investigate the convergence of the sequence $(\omega^n, u^n, \Phi^n_t)$. We will first show existence of the Yudovich solution on a sufficiently small time interval $[-T,T]$. We can then iterate the arguments below to get the global solution, since the time step T will only depend on $\|\omega_0\|_{L^\infty}$ and D.

The first key step is to prove convergence of the flow $\Phi^n_t(x)$ in the $C([-T,T],L^1(D))$ topology.

Lemma 7.15. *There exists $T > 0$ and $\Phi_t(x) \in C([-T,T], L^1(D))$ such that*

$$\|\Phi_t^n - \Phi_t\|_{C([-T,T],L^1(D))} \to 0$$

as $n \to \infty$.

Proof. Let us focus on $t > 0$; the other alternative is handled by the same argument. Observe that

$$\|\Phi_t^n - \Phi_t^{n-1}\|_{L^1(D)} \leq \int_0^t \int_D |u^n(s, \Phi_s^n(x)) - u^n(s, \Phi_s^{n-1}(x))| \, dxds \tag{7.77}$$

$$+ \int_0^t \int_D |u^n(s, \Phi_s^{n-1}(x)) - u^{n-1}(s, \Phi_s^{n-1}(x))| \, dxds \equiv I_1^m(t) + I_2^m(t). \tag{7.78}$$

By Corollary 7.5, we have that

$$I_1^n(t) \leq C(D)\|\omega_0\|_{L^\infty} \int_0^t \int_D \phi(|\Phi_s^n(x) - \Phi_s^{n-1}(x)|) \, dx.$$

Since ϕ is concave on $\mathbb{R}^+$, we can apply Jensen's inequality to obtain

$$\frac{1}{|D|} \int_D \phi(|\Phi_s^n(x) - \Phi_s^{n-1}(x)|) \, dx \leq \phi\left(\frac{1}{|D|} \int_D |\Phi_s^n(x) - \Phi_s^{n-1}(x)| \, dx \right).$$

Let us define

$$\sigma_n(t) = \frac{1}{|D|}\|\Phi_t^n - \Phi_t^{n-1}\|_{L^1(D)}.$$

Then

$$I_1^n(t) \leq C(D, \|\omega_0\|_{L^\infty}) \int_0^t \phi(\sigma_n(s)) \, ds. \tag{7.79}$$

Since Φ_t^{n-1} is a measure preserving mapping, we have

$$I_2^n(t) = \int_0^t \int_D |u^n(s,z) - u^{n-1}(s,z)| \, dzds.$$

Now

$$u^n(s,z) = \int_D K_D(z,y)\omega^{n-1}(y,s) \, dy = \int_D K_D(z,y)\omega_0((\Phi_s^{n-1})^{-1}) \, dy = \int_D K_D(z, \Phi_s^{n-1}(y'))\omega_0(y') \, dy'.$$

Therefore,

$$\begin{aligned} \int_D |u^n(s,z) - u^{n-1}(s,z)| \, dz &\leq \|\omega_0\|_{L^\infty} \int_D \int_D |K_D(z, \Phi_s^{n-1}(y)) - K_D(z, \Phi_s^{n-2}(y))| \, dydz \\ &\leq C(D)\|\omega_0\|_{L^\infty} \int_D \phi(|\Phi_s^{n-1}(y) - \Phi_s^{n-2}(y)|) \, dz. \end{aligned}$$

Applying Jensen's inequality again, we obtain

$$I_2^n(t) \leq C(D)\|\omega_0\|_{L^\infty} \int_0^t \int_D \phi(|\Phi_s^{n-1}(y) - \Phi_s^{n-2}(y)|) \, dzds \leq C(D, \|\omega_0\|_{L^\infty}) \int_0^t \phi(\sigma_{n-1}(s)) \, ds.$$

Combining this last inequality with (7.77) and (7.79), we arrive at

$$\sigma_n(t) \leq C(D, \|\omega_0\|_{L^\infty}) \int_0^t (\phi(\sigma_n(s)) + \phi(\sigma_{n-1}(s)). \tag{7.80}$$

Define

$$\rho_N(t) = [\ \sup_{n \geq N} \sigma_n(t).$$

Then by (7.80), for all $n \geq N$,

$$\sigma_n(t) \leq C(D, \|\omega_0\|_{L^\infty}) \int_0^t (\phi(\rho_{N-1}(s))\, ds$$

and hence

$$\rho_N(t) \leq C(D, \|\omega_0\|_{L^\infty}) \int_0^t (\phi(\rho_{N-1}(s))\, ds.$$

We now arrived at an inequality similar to (10.13) and, similarly to the argument in the proof of Lemma 7.6, we can show that $\sigma_n(t) \leq \rho_n(t) \leq \alpha^N$ for some $1 > \alpha > 0$ for all $t \in [0, T]$ if $T > 0$ is sufficiently small. This shows that $\Phi_t^n(x)$ is a Cauchy sequence in $C([0, T], L^1(D))$, finishing the proof of the lemma. $\square$

We next upgrade the convergence of $\Phi_t^n(x)$ to $\Phi_t(x)$.

Lemma 7.16. *The sequence $\Phi_t^n(x)$ converges to $\Phi_t(x)$ uniformly on $C([-T, T] \times \bar{D})$ provided that $T > 0$ is chosen sufficiently small. Moreover, the limiting map $\Phi_t(x) \in C^{\alpha(T)}([-T, T] \times \bar{D})$ for some $\alpha(T) > 0$ and is measure preserving.*

Proof. As before, we focus on times $t > 0$. The value of T will be the same as in the previous lemma. Observe that the estimate (7.55) implies that for every $T > 0$, we have

$$\|\Phi_t^n(x)\|_{C^{\alpha(T)}([0,T] \times \bar{D})} \leq C(D, \|\omega_0\|_{L^\infty}), \tag{7.81}$$

for some $\alpha(T) > 0$, and with the norm bounded uniformly in n.

First, by the Arzela-Ascoli theorem, we can find a subsequence n_j such that $\Phi_t^{n_j}(x)$ convergences to $\Phi_t(x)$ uniformly on $[0, T] \times \bar{D}$. This implies that $\Phi_t(x)$ is continuous, and moreover a simple argument shows that it inherits the Hölder bound (7.81). Notice that for every smooth function f, we have

$$\int_D f(\Phi_t(y))\, dy = \lim_{j \to \infty} \int_D f(\Phi_t^{n_j}(y))\, dy = \int_D f(y)\, dy. \tag{7.82}$$

The last step follows since $\Phi_t^{n_j}(x)$ are measure preserving, while the first step is not hard to establish. Using (7.82), it is not hard to show that $|\Phi_t(I)| = |I|$ for every rectangle I lying in D, and this implies that $\Phi_t(x)$ is measure preserving.

Exercise 7.17. *Fill all the gaps in the previous paragraph.*

Now suppose, on the contrary, that the uniform convergence of $\Phi_t^n(x)$ to $\Phi_t(x)$ does not hold. Then we can find $\epsilon > 0$ and the sequences $n_k \to \infty$, $t_k \in [-T, T]$ and $x_k \in \bar{D}$ such that $|\Phi_{t_k}^{n_k}(x_k) - \Phi_{t_k}(x_k)| \geq \epsilon$. By (7.81) and the fact that $\Phi_t(x)$ satisfies the same bound, we can find $r > 0$ independent of k such that for all $|x - x_k| \leq r$, we have $|\Phi_{t_k}^{n_k}(x) - \Phi_{t_k}(x)| \geq \epsilon/2$. But this contradicts the $C([0,T], L^1(D))$ convergence proved in Lemma 7.15. $\square$

One can ask why do we need to worry about convergence of the whole sequence $\Phi_t^n(x)$ when we have convergence over a subsequence basically for free? Unfortunately, convergence over a subsequence does not work well with the oterative scheme. Even if we have convergence for $\Phi_t^{n_k}(x)$, we know nothing about convergence of $\Phi_t^{n_k-1}(x)$ but we would need exactly that to establish the convergence of velocities that we address next.

The lower bound in (7.55) which applies to Φ_t^n uniformly is inherited by $\Phi_t(x)$ and implies that $\Phi_t(x)$ is invertible. As Φ_t^{-1} satisfies the same estimate (7.55), it also belongs to $C^{\alpha(T)}([0,T] \times \bar{D})$. We may then define the corresponding vorticity

$$\omega(t,x) = \omega_0(\Phi_t^{-1}(x)),$$

and the fluid velocity

$$u(t,x) = \int_D K_D(x,y)\omega(t,y)\,dy.$$

Lemma 7.18. *We have $|u(t,x) - u^n(t,x)| \to 0$, as $n \to \infty$, uniformly in $[-T,T] \times \bar{D}$.*

Proof. Note that

$$|u(t,x) - u^n(t,x)| = \left| \int_D \left(K_D(x, \Phi_t(z)) - K_D(x, \Phi_t^n(z))\right) \omega_0(z)\,dz \right|. \tag{7.83}$$

Given $\epsilon > 0$, choose N so that $|\Phi_t(x) - \Phi_t^n(x)| < \delta$, for all $n \geq N$ and for all $x \in \bar{D}$, $t \in [0,T]$, with $\delta > 0$ to be determined later. Pulling $\|\omega_0\|_{L^\infty}$ out of the integral in (7.83) and setting $z = \Phi_t^{-1}(p)$ we have

$$|u(t,x) - u^n(t,x)| \leq \|\omega_0\|_{L^\infty} \int_D |K_D(x,p) - K_D(x, y(p))|\,dp. \tag{7.84}$$

Note that the map $y(p) = \Phi_t^n \circ \Phi_t^{-1}(p)$ is measure preserving, and

$$|y(p) - p| = |\Phi_t^n(\Phi_t^{-1}(p)) - \Phi_t(\Phi_t^{-1}(p))| < \delta,$$

for every p. As usual, we split the integral in (10.22) into two regions: in the first one we have

$$\int_{B_{3\delta}(x) \cap D} |K_D(x,p) - K_D(x,y(p))|\,dp \leq 2C(D) \int_{B_{3\delta}(x)} \frac{dp}{|x-p|} \leq 2C(D)\delta,$$

while in the second

$$\begin{aligned} \int_{B_{3\delta}(x)^c \cap D} |K_D(x,p) - K_D(x,y(p))|\,dp &\leq C(D)\delta \int_{B_{3\delta}(x)^c \cap D} |\nabla K_D(x, q(p))|\,dp \\ &\leq C(D)\delta \int_{B_\delta(x)^c} \frac{dp}{|x-p|^2} \leq C(D)\delta \log \delta^{-1}. \end{aligned} \tag{7.85}$$

Here, $q(p)$ is a point on a curve of length $\leq \delta$ that connects p and $y(p)$. If the interval connecting these points lies in $\bar{D}$ then this interval can be used as this curve. If not, one can use an argument similar to that in the proof of Lemma 7.3. Thus choosing δ sufficiently small we can make sure that the difference of the velocities does not exceed ϵ. $\square$

Exercise 7.19. Fill in all the details in the last step in the proof of the Lemma. Alternatively, you may first show that ω^n converges to ω in $C([-T,T],L^p(D))$ for all $p\in[1,\infty)$, and use this and Hölder inequality to prove Lemma 7.18.

We are now ready to show that

$$\frac{d}{dt}\Phi_t(x)=u(t,\Phi_t(x)).$$

Indeed, we have

$$\Phi_t^n(x)=x+\int_0^t u^n(s,\Phi_s^n(x))\,ds,$$

and, taking $n\to\infty$, using Lemma 7.18 and the definition of $\Phi_t(x)$, we obtain

$$\Phi_t(x)=x+\int_0^t u(s,\Phi_s(x))\,ds.$$

Thus, the limit triple $(\omega(t,x),u(t,x),\Phi_t(x))$ satisfies the Euler equations in our generalized sense, completing the proof of the existence of solutions.

7.5 Existence and uniqueness of the solutions

Let us now finally state the main result on the existence and uniqueness of solutions of the two-dimensional Euler equations with $\omega_0\in L^\infty$. The existence part of this theorem summarizes what has been proved above using the approximation scheme.

Theorem 7.20. *Fix any $\omega_0\in L^\infty(D)$. There exists a unique triple $(\omega(t,x),u(t,x),\Phi_t(x))$ such that for every $T>0$ the vorticity $\omega\in L^\infty([0,T],L^\infty(D))$ and is weak-$*$ continuous in time in L^∞, the fluid velocity $u(t,x)$ is uniformly bounded and log-Lipschitz in x and t, and $\Phi_t\in C^{\alpha(T)}([0,T]\times\bar D)$ is a measure preserving, invertible mapping of $\bar D$, satisfying*

$$\begin{aligned}&\frac{d\Phi_t(x)}{dt}=u(t,\Phi_t(x)),\ \ \Phi_0(x)=x,\\&\omega(t,x)=\omega_0(\Phi_t^{-1}(x)),\\&u(t,x)=\int_D K_D(x,y)\omega(t,y)\,dy.\end{aligned}\tag{7.86}$$

Here $\alpha(T)>0$ and only depends on $\|\omega_0\|_{L^\infty}$.

Proof of Theorem 7.20. We have already established existence and regularity estimates with an exception of weak-$*$ continuity. This property is key as it gives meaning to the initial value problem: $\omega(t,x)$ converges to $\omega_0(x)$ in L^∞ as $t\to 0$ in the weak-$*$ sense, that is for any test function $\eta\in L^1(D)$ we have

$$\int_D\omega(t,x)\eta(x)dx=\int_D\omega_0(\Phi_t^{-1}(x))\eta(x)dx=\int_D\omega_0(x)\eta(\Phi_t(x))dx\to\int_D\omega_0(x)\eta(x)dx,\tag{7.87}$$

as $t \to 0$. Indeed, as ω is uniformly bounded in $L^\infty(D)$, it suffices to check (10.17) for smooth functions η, for which we have

$$\int_D |\eta(\Phi_t(x)) - \eta(x)|dx \le \|\nabla\eta\|_{L^\infty} \int_D |\Phi_t(x) - x|dx \le C(D)\|\nabla\eta\|_{L^\infty}\|u\|_{L^\infty} t.$$

A similar argument works at any time $t > 0$. Note also that while we only proved existence of solutions on a small interval $[-T, T]$, the solution can be extended globally by iterating the construction as the time step only depends on D and $\|\omega_0\|_{L^\infty}$.

It remains only to prove the uniqueness. Suppose that there are two solution triples $(\omega^1, u^1, \Phi_t^1)$ and $(\omega^2, u^2, \Phi_t^2)$ satisfying the properties described in Theorem 7.20, and set

$$\eta(t) = \frac{1}{|D|}\int_D |\Phi_t^1(x) - \Phi_t^2(x)|\, dx.$$

Let us write

$$|\Phi_t^1(x) - \Phi_t^2(x)| \le \int_0^t |u^1(s, \Phi_s^1(x)) - u^1(s, \Phi_s^2(x))|\, ds + \int_0^t |u^1(s, \Phi_s^2(x)) - u^2(s, \Phi_s^2(x))|\, ds. \tag{7.88}$$

By Corollary 7.5, the first integral in the right side of (7.88) can be bounded by

$$C\|\omega_0\|_{L^\infty} \int_0^t \phi(|\Phi_s^1(x) - \Phi_s^2(x)|)\, ds.$$

For the second integral in (7.88), consider the difference

$$\begin{aligned} u^1(s, \Phi_s^2(x)) - u^2(s, \Phi_s^2(x)) &= \int_D K_D(\Phi_s^2(x), y)\omega^1(s, y)\, dy - \int_D K_D(\Phi_s^2(x), y)\omega^2(s, y)\, dy \\ &= \int_D \big(K_D(\Phi_s^2(x), \Phi_s^1(y)) - K_D(\Phi_s^2(x), \Phi_s^2(y))\big)\, \omega_0(y)\, dy, \end{aligned}$$

where we used the vorticity evolution formula in (7.86). Averaging (7.88) in x, we now obtain

$$\begin{aligned} \eta(t) &\le \frac{C\|\omega_0\|_{L^\infty}}{|D|} \int_0^t ds \int_D \phi(|\Phi_s^1(x) - \Phi_s^2(x)|)\, dx \\ &\quad + \frac{C}{|D|}\int_0^t ds \int_D |\omega_0(y)| \int_D |K_D(x, \Phi_s^1(y)) - K_D(x, \Phi_s^2(y))|\, dx dy \\ &\le C(D)\|\omega_0\|_{L^\infty} \int_0^t ds \int_D \phi(|\Phi_s^1(x) - \Phi_s^2(x)|)\frac{dx}{|D|}. \end{aligned} \tag{7.89}$$

We used Lemma 7.3 in the last step. As the function ϕ is concave, we may use Jensen's inequality to exchange ϕ and averaging in the last expression in (10.56):

$$\eta(t) \le C(D)\|\omega_0\|_{L^\infty} \int_0^t \phi(\eta(s))\, ds.$$

In addition, we have $\eta(0) = 0$. An argument very similar to the proof of uniqueness in Lemma 7.6 (based on the log-Lipschitz property of the function ϕ) can be now used to prove that $\eta(t) = 0$ for all $t \ge 0$.

Exercise 7.21. Work out the details of this argument.

This completes the proof of the theorem. □

Regularity of the solutions for regular initial data

So far, we have only assumed that $\omega_0 \in L^\infty$. Of course, the Yudovich construction also applies if the initial condition ω_0 possesses additional regularity. In that case, the solution $\omega(t,x)$ inherits this extra regularity. This is expressed by the following theorem.

Theorem 7.22. *Suppose that $\omega_0 \in C^k(\bar{D})$, $k \geq 1$. Then the solution described in Theorem 7.20, satisfies, in addition, the following regularity properties, for each $t \geq 0$:*

$$\omega(t) \in C^k(\bar{D}), \quad \Phi_t(x) \in C^{k,\alpha(t)}(\bar{D}), \text{ and } u \in C^{k,\beta}(\bar{D}),$$

for all $\beta < 1$. In addition, the kth order derivatives of u are log-Lipschitz.

The regularity of the flow $u(t,x)$ is similar in spirit to that in Theorem 7.20 – there, L^∞ initial data for vorticity led to log-Lipschitz $u(t,x)$. Here, C^k initial condition $\omega_0(x)$ leads to a flow $u(t,x)$ which has a log Lipschitz derivative of the order k. The first proof of a result similar to Theorem 7.22 goes back to the work of Wolibner and of Hölder in the early 1930s. We will provide a detailed argument for the case of $k=1$, larger values of k will be left as an exercise for the reader. We will need the following elliptic regularity result of the kind we have seen many times in Chapter **??**.

Theorem 7.23. *Suppose that D is a domain in $\mathbb{R}^d$ with smooth boundary, and let ψ be the solution of the Dirichlet problem*

$$\begin{aligned} -\Delta\psi &= \omega, \\ \psi|_{\partial D} &= 0. \end{aligned}$$

If $\omega \in C^\alpha(\bar{D})$, $\alpha > 0$, then $\psi \in C^{2,\alpha}(\bar{D})$, and

$$\|\partial_i \psi\|_{C^{1,\alpha}} \leq C(\alpha, D)\|\omega\|_{C^\alpha}.$$

This result was originally proved by Kellogg in 1931. Schauder later established a similar bound for more general elliptic operators. Such estimates are commonly called the Schauder estimates. We have not quite proved this particular estimate in Chapter **??** since it applies to a bounded domain. The reader should either treat it as a refresher exercise on the methods on Chapter **??**, or consult [**?**, **?**] for the proof. We will use this estimate for the stream function

$$\psi(t,x) = (-\Delta_D)^{-1}\omega, \quad u(t,x) = \nabla^\perp \psi(t,x).$$

We have already proved that if $\omega_0 \in L^\infty(\bar{D})$ then $\Phi_t^{-1}(x) \in C^{\alpha(t)}(\bar{D})$ for all $t \geq 0$, with $\alpha(t) = e^{-Ct}$. Since

$$\omega(t,x) = \omega_0(\Phi_t^{-1}(x)),$$

if in addition we know that $\omega_0 \in C^1(\bar{D})$, we then automatically have $\omega(t,x) \in C^{\alpha(t)}(\bar{D})$ so that the vorticity is Hölder continuous. By Theorem 7.23, we deduce that the flow $u(t,x)$ has a Hölder continuous derivative: $u(t,x) \in C^{1,\alpha(t)}(\bar{D})$. However, this a priori Hölder exponent $\alpha(t)$ decreases as t grows, while we are looking to prove that $u(t,x) \in C^{1,\beta}(\bar{D})$, for all $\beta \in (0,1)$, hence this a priori information is not sufficient.

A simple calculation starting with the trajectories equation leads to

$$\frac{d}{dt}|\Phi_t(x)-\Phi_t(y)|^2 \le 2\|\nabla u(t,\cdot)\|_{L^\infty}|\Phi_t(x)-\Phi_t(y)|^2, \tag{7.90}$$

where we now know that the derivatives of u are bounded for all t, even though their size may grow with time. Integrating (7.90) in time and using the initial condition

$$|\Phi_0(x)-\Phi_0(y)| = |x-y|,$$

we obtain

$$\exp\Big\{-\int_0^t \|\nabla u(s,\cdot)\|_{L^\infty}\,ds\Big\} \le \frac{|\Phi_t(x)-\Phi_t(y)|}{|x-y|} \le \exp\Big\{\int_0^t \|\nabla u(s,\cdot)\|_{L^\infty}\,ds\Big\}. \tag{7.91}$$

This inequality will be useful for us later. For now, we observe that it implies that $\Phi_t(x)$ is Lipschitz for every $t \ge 0$. We would like to show that, in fact, $\Phi_t(x) \in C^{1,\alpha(t)}(\bar{D})$ for all $t \ge 0$. For this purpose we need a couple of technical lemmas. In what follows, we adopt the summation convention: we sum over repeated indexes.

Lemma 7.24. *There exists a set $S \subseteq D$ of full measure so that for all $x \in S$ we have*

$$\partial_j \Phi_t^k(x) = \delta_{jk} + \int_0^t \partial_l u^k(s,\Phi_s(x))\partial_j \Phi_s^l(x)\,ds, \tag{7.92}$$

for all $t \ge 0$.

Proof. By the Rademacher theorem (see, e.g. [**?**]), it follows from (7.91) that $\Phi_t(x)$ is differentiable in x a.e. in $\bar{D}$, for each t fixed. Next, note that by the Fubini theorem, it follows that for a.e. x, $\Phi_t(x)$ is differentiable in x for a.e. t. We let S be the set of such x.

Let now $x \in S$, set

$$y = x + e_j \Delta x,$$

where e_j is a unit vector in jth direction, and consider the finite differences

$$\frac{\Phi_t^k(y)-\Phi_t^k(x)}{\Delta x} = \delta_{jk} + \int_0^t \frac{u^k(s,\Phi_s(y))-u^k(s,\Phi_s(x))}{\Delta x}\,ds. \tag{7.93}$$

We may write, explicitly listing the coordinates

$$\begin{aligned}\frac{u^k(s,\Phi_s(y))-u^k(s,\Phi_s(x))}{\Delta x} &= \frac{u^k(s,\Phi_s^1(y),\Phi_s^2(y))-u^k(s,\Phi_s^1(x),\Phi_s^2(y))}{\Phi_s^1(y)-\Phi_s^1(x)}\frac{\Phi_s^1(y)-\Phi_s^1(x)}{\Delta x}\\ &+\frac{u^k(s,\Phi_s^1(x),\Phi_s^2(y))-u^k(s,\Phi_s^1(x),\Phi_s^2(x))}{\Phi_s^2(y)-\Phi_s^2(x)}\frac{\Phi_s^2(y)-\Phi_s^2(x)}{\Delta x}.\end{aligned}$$

Since $u \in C^{1,\alpha}(\bar{D})$, it is not difficult to show, using the mean value theorem, that the first factors in the two products in the right side converge, as $\Delta x \to 0$, uniformly in x, to $\partial_l u^k(s,\Phi_s(x))$, $l = 1,2$ respectively. On the other hand, the ratios

$$\frac{\Phi_s^l(y)-\Phi_s^l(x)}{\Delta x}$$

are controlled in L^∞ by the Lipschitz estimate (7.91). Moreover, for $x \in S$, the ratio converges to $\partial_j \Phi_s^l(x)$ for a.e. $s \in [0,t]$. By the Lebesgue dominated convergence theorem, we have the convergence of the integral in (7.93) to the integral in (7.92). $\square$

Now, for $x, y \in S$ we find from (7.92) that

$$\partial_t \partial_j \Phi_t^k(x) = \partial_l u^k(t, \Phi_t(x)) \partial_j \Phi_t^l(x)$$

for all t, and similarly for y. Without loss of generality, we may confine our considerations to x, y such that $|x-y| \leq 1$. Consider the expression

$$\begin{aligned}\partial_t(\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)) = {} & (\partial_l u^k(t, \Phi_t(x)) - \partial_l u^k(t, \Phi_t(y)))\partial_j \Phi_t^l(x) \\ & + \partial_l u^k(t, \Phi_t(y))(\partial_j \Phi_t^l(x) - \partial_j \Phi_t^l(y)).\end{aligned}$$

It follows that

$$\partial_t |\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)| \leq \|\Phi_t\|_{Lip} \|\nabla u\|_{C^{\alpha(t)}} |\Phi_t(x) - \Phi_t(y)|^{\alpha(t)} + \|\nabla u\|_{L^\infty} |\partial_j \Phi_t^l(x) - \partial_j \Phi_t^l(y)|,$$

where we denote by $\|\Phi_t\|_{Lip}$ the Lipschitz bound we have on $\Phi_t(x)$ in x for a given t. Let us denote

$$F(t) = \sum_{k,j} |\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)|.$$

Then we get

$$\dot F(t) \leq \|\nabla u(\cdot, t)\|_{L^\infty} F(t) + |x-y|^{\alpha(t)} \|\Phi_t\|_{Lip}^2 \|\nabla u\|_{C^{\alpha(t)}}.$$

This inequality holds for every $t > 0$ with the corresponding value of $\alpha(t)$. Fix an arbitrary time interval $[0, T]$. By applying the Gronwall inequality, we conclude that for all $x, y \in S$ and all $t \in [0, T]$ we have

$$|\partial_j \Phi_t^k(x) - \partial_j \Phi_t^k(y)| \leq C(\|\omega_0\|_{C^1}, T)|x-y|^{\alpha(T)}. \tag{7.94}$$

Note that the dependence of the constant in (7.94) on T can be pretty complex – it is controlled by the size of norms that we showed to be finite for every time but never traced their growth. We will obtain a more clear cut, quantitative bound on the possible growth later.

Now we need one more elementary lemma.

Lemma 7.25. *Suppose that $f : \bar D \subset \mathbb{R}^d \mapsto \mathbb{R}$ is Lipschitz. Suppose there exists a set of full measure S such that $\nabla f(x)$ exists for $x \in S$, and moreover for every $x, y \in S$ we have*

$$|\nabla f(x) - \nabla f(y)| \leq C|x-y|^\gamma \tag{7.95}$$

for some fixed constant C and $0 < \gamma < 1$. Then $f \in C^{1,\gamma}(\bar D)$.

Proof. Since S is full measure, we can extend ∇f by continuity to a function $g = (g_1, \dots, g_d)$ defined on all $\bar D$. Namely, we set $g(x) = \nabla f(x)$ if $x \in S$. If $x \notin S$, then we take any sequence $x_n \in S \to x$, and define $g(x) = \lim_{n\to\infty} \nabla f(x_n)$. Note that the sequence $\nabla f(x_n)$ is Cauchy due to (7.95), so the limit is well-defined. It is also straightforward to check that the definition is unambiguous (different sequences in S lead to the same limit), and that the resulting function $g \in C^\gamma(\bar D)$. It remains to show that in fact f is everywhere differentiable and $\nabla f(x) \equiv g(x)$.

Without loss of generality, let us consider $\partial_1 f$. Let $x = (x_1, \tilde{x}) \in D$, where $\tilde{x} = (x_2, \dots, x_d)$; the case $x \in \partial D$ is similar. Given x_1, let us denote the set of $\tilde{x}$ such that $(x_1, \tilde{x}) \in D$ by F. Suppose first that $\tilde{x}$ is such that $\nabla f(y_1, \tilde{x})$ exists for a.e. y_1 such that $(y_1, \tilde{x}) \in D$. We know that a.e. $\tilde{x} \in F$ is like that, and we denote this set by G. We also know that if $\tilde{x} \in G$, then $\nabla f(y_1, \tilde{x}) = g(y_1, \tilde{x})$ for those a.e. y_1 where it exists. Then for every $(y_1, \tilde{x}) \in D$ and sufficiently close to $(x_1, \tilde{x})$, we have

$$f(y_1, \tilde{x}) = f(x_1, \tilde{x}) + \int_{x_1}^{y_1} \partial_1 f(z_1, \tilde{x})\, dz_1 = f(x_1, \tilde{x}) + \int_{x_1}^{y_1} g_1(z_1, \tilde{x})\, dz_1.$$

But this implies that $\partial_1 f(x_1, \tilde{x})$ exists and is equal to $g(x_1, \tilde{x})$. Assume now that $\tilde{x}$ belongs to the exceptional measure zero set $F \setminus G$ where $\nabla f(y_1, \tilde{x})$ fails to exist for a set of y_1 of positive measure. But then we can find $\tilde{x}_n \in G$ such that $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$. For each $\tilde{x}_n$, we have

$$f(y_1, \tilde{x}_n) = f(x_1, \tilde{x}_n) + \int_{x_1}^{y_1} g_1(z_1, \tilde{x}_n)\, dz_1$$

for all y_1 close enough to x_1. Passing to the limit in this equality, we find

$$f(y_1, \tilde{x}) = f(x_1, \tilde{x}) + \int_{x_1}^{y_1} g_1(z_1, \tilde{x})\, dz_1.$$

This implies that $\partial_1 f(x_1, \tilde{x})$ exists and is equal to $g_1(x_1, \tilde{x})$ in this case, too. $\square$

Exercise 7.26. Work out the details of the above argument in the case of $(x_1, \tilde{x}) \in \partial D$.

We conclude that the following lemma holds.

Lemma 7.27. *For every $t \geq 0$, the function $\partial_j \Phi_t^k(x)$ belongs to $C^{\alpha(t)}(\bar{D})$ and (7.92) holds for all x, t.*

Now, the proof of Theorem 7.22 in the case $k = 1$ is straightforward.
Proof. Indeed, since $\Phi_t(x)$ is measure preserving, we have

$$\det \nabla \Phi_t = 1,$$

and then the derivatives of the inverse map $\Phi_t^{-1}(x)$ in x satisfy the bounds analogous to those of Φ_t. Then, Lemma 7.27 implies immediately that

$$\omega(t, x) = \omega_0(\Phi_t^{-1}(x))$$

is $C^1(\bar{D})$ for all times. $\square$

Exercise 7.28. Carry out the analogous computations for $k > 1$, proving Theorem 7.22 in this case.

8 Vortex lines and geometric conditions for blow-up

The vorticity growth equation

Here, we investigate how vorticity alignment in the regions of high vorticity can prevent blow-up in the Navier-Stokes and Euler equations. First, we obtain an equation for the magnitude of vorticity $|\omega|$ that shows that it is plausible that the vorticity alignment in the regions of high vorticity may prevent the growth of vorticity. Recall that the vorticity of the solutions of the Navier-Stokes equations satisfies the evolution equation

$$\omega_t + u \cdot \nabla\omega - \nu\Delta\omega = \omega \cdot \nabla u \tag{8.1}$$

Multiplying by 2ω, we obtain

$$\partial_t(|\omega|^2) + u \cdot \nabla(|\omega|^2) - \nu\Delta|\omega|^2 + 2\nu|\nabla\omega|^2 = 2(\omega \cdot \nabla u) \cdot \omega. \tag{8.2}$$

The right side can be written as

$$2(\omega \cdot \nabla u) \cdot \omega = 2\omega_j(\partial_j u_k)\omega_k = 2(S\omega \cdot \omega) = 2\alpha(x)|\omega|^2,$$

with

$$\alpha(x) = (S(x)\xi(x) \cdot \xi(x)), \quad \xi(x) = \frac{\omega(x)}{|\omega(x)|}, \tag{8.3}$$

and

$$S(x) = \frac{1}{2}\big(\nabla u + (\nabla u)^t\big). \tag{8.4}$$

When $\nu = 0$ we get a particularly simple form of the vortex stretching balance for the Euler equations:

$$\partial_t|\omega| + u \cdot \nabla|\omega| = \alpha(t,x)|\omega|. \tag{8.5}$$

Thus, the vorticity growth may only appear from $\alpha(x)$ large. Our next task is to express $\alpha(x)$ in terms of the vorticity alignment. We start with the Biot-Savart law

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y)dy. \tag{8.6}$$

Let us recall that

$$\frac{\partial u_k(x)}{\partial x_j} = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} K_{km}(y)\partial_j\omega_m(x+y)dy \tag{8.7}$$

$$= -\lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy - \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}[\partial_j K_{km}(y)]\omega_m(x+y)dy = A_{kj} + B_{kj}.$$

The term A_{kj} can be simplified as

$$A_{kj} = -\lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy = -\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{|y|=\varepsilon}\frac{1}{|y|^3}[y\times\omega(x+y)]_k\frac{y_j}{|y|}dy$$

$$= -\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{|z|=1}\frac{1}{\varepsilon^3|z|^3}[\varepsilon z\times\omega(x+\varepsilon z)]_k\frac{z_j}{|z|}\varepsilon^2 dz = -\frac{1}{4\pi}\int_{|z|=1}[z\times\omega(x)]_k z_j dz$$

$$= -\frac{1}{4\pi}\epsilon_{kmn}\int_{|z|=1} z_m\omega_n(x)z_j dz = -\frac{\epsilon_{kmn}}{3}\omega_n(x)\delta_{mj} = -\frac{1}{3}\epsilon_{kjn}\omega_n(x), \tag{8.8}$$

and B can be written as

$$B_{kj} = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \Big[\frac{3\epsilon_{krm}}{4\pi |y|^5} y_j y_r - \frac{\epsilon_{kjm}}{4\pi |y|^3} \Big] \omega_m(x+y) dy.$$

Multiplying (8.7) by ϵ_{ijk} and summing over j, k, leads now to an integral equation for the vorticity:

$$\omega_i(x) = \epsilon_{ijk} \partial_j u_k = \epsilon_{ijk} A_{kj} + \epsilon_{ijk} B_{kj} = -\frac{1}{3} \epsilon_{ijk} \epsilon_{kjn} \omega_n \tag{8.9}$$
$$+ \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \epsilon_{ijk} \Big[\frac{3\epsilon_{krm}}{4\pi |y|^5} y_j y_r + \frac{\epsilon_{kmj}}{4\pi |y|^3} \Big] \omega_m(x+y) dy.$$

The first term above can be re-written as

$$-\epsilon_{ijk} \epsilon_{kjn} \omega_n = \epsilon_{ijk} \epsilon_{njk} \omega_n = 2\omega_i.$$

In the second term, we use the identities

$$\epsilon_{ijk} \epsilon_{krm} y_j y_r \omega_m = \epsilon_{kij} \epsilon_{krm} y_j y_r \omega_m = [\delta_{ir} \delta_{jm} - \delta_{im} \delta_{jr}] y_j y_r \omega_m = y_i (y \cdot \omega) - |y|^2 \omega_i,$$

and

$$\epsilon_{ijk} \epsilon_{kmj} \omega_m = \epsilon_{kij} \epsilon_{kmj} \omega_m = 2\omega_i$$

Using these transformations in (8.9), gives

$$\frac{1}{3} \omega_i(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \Big[\frac{3}{4\pi |y|^5} [y_i (y \cdot \omega(x+y)) - |y|^2 \omega_i(x+y)] + \frac{2\omega_i(x+y)}{4\pi |y|^3} \Big] dy.$$

so that

$$\omega(x) = \frac{3}{4\pi} \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \sigma(\hat{y}) \omega(x+y) \frac{dy}{|y|^3} \tag{8.10}$$

with the matrix $\sigma(\hat{y})$, $\hat{y} = y/|y|$, defined as

$$\sigma(\hat{y}) = 3(\hat{y} \otimes \hat{y}) - I. \tag{8.11}$$

Similarly, we may compute the symmetric part of ∇u:

$$S(x) = \frac{1}{2} \big(\nabla u + (\nabla u)^t \big).$$

We have

$$S_{kj} = \frac{1}{2} (A_{kj} + A_{jk}) + \frac{1}{2} (B_{kj} + B_{jk}).$$

It is easy to see that the matrix A_{kj} is anti-symmetric, thus

$$A_{kj} + A_{jk} = 0.$$

For the symmetric part of the matrix B we compute

$$B_{kj}+B_{jk}=\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[\frac{3\epsilon_{krm}}{4\pi|y|^5}y_jy_r+\frac{3\epsilon_{jrm}}{4\pi|y|^5}y_ky_r-\frac{\epsilon_{kjm}}{4\pi|y|^3}-\frac{\epsilon_{jkm}}{4\pi|y|^3}\Big]\omega_m(x+y)dy$$
$$=\frac{3}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[\epsilon_{krm}\hat{y}_j\hat{y}_r+\epsilon_{jrm}\hat{y}_k\hat{y}_r\Big]\omega_m(x+y)\frac{dy}{|y|^3}.$$

We conclude that

$$S(x)=\frac{3}{4\pi}\text{P.V.}\int M(\hat{y},\omega(x+y))\frac{dy}{|y|^3},\tag{8.12}$$

with the matrix-valued function

$$M(\hat{y},\omega)=\frac{1}{2}\big[(\hat{y}\times\omega)\otimes\hat{y}+\hat{y}\otimes(\hat{y}\times\omega)\big].\tag{8.13}$$

Going back to (8.3), we get the following expression for the vorticity stretching coefficient $\alpha(x)$:

$$\alpha(x)=(S(x)\xi(x)\cdot\xi(x))=\frac{3}{4\pi}\text{P.V.}\int(M(\hat{y},\omega(x+y))\xi(x)\cdot\xi(x))\frac{dy}{|y|^3}.\tag{8.14}$$

The integrand can be re-written as

$$M(\hat{y},\omega(x+y))\xi(x)\cdot\xi(x))=\frac{1}{2}\big[(\hat{y}\times\omega(x+y))\otimes\hat{y}+\hat{y}\otimes(\hat{y}\times\omega(x+y))\big]\xi(x)\cdot\xi(x)$$
$$=(\hat{y}\times\omega(x+y)\cdot\xi(x))(\hat{y}\cdot\xi(x))=D(\hat{y},\xi(x+y),\xi(x))|\omega(x+y)|,$$

thus

$$\alpha(x)=(S(x)\xi(x)\cdot\xi(x))=\frac{3}{4\pi}\text{P.V.}\int D(\hat{y},\xi(x+y),\xi(x))|\omega(x+y)|\frac{dy}{|y|^3}.\tag{8.15}$$

Here, we have defined, for three unit vectors e_1, e_2 and e_3:

$$D(e_1,e_2,e_3)=(e_1\cdot e_3)\text{Det}(e_1,e_2,e_3).$$

Geometrically, it follows that the regions where $\xi(x+y)$ is aligned with $\xi(x)$ contribute less to $\alpha(x)$. This applies also to the antiparallel vortex pairing, which is a physically observed phenomenon. That is, we expect that if the vorticity direction field is aligned or anti-aligned in the regions of high vorticity, the blow-up might be prevented by the vorticity alignment, though this requires a careful analysis which we will undertake next.

A priori bounds on the strain matrix

Let us first obtain some bounds on the strain matrix in terms of ω that we will need later. We have, from (8.12)-(8.13):

$$S_{kj}(x)=\frac{3}{8\pi}\text{P.V.}\int\Big[\epsilon_{krm}\hat{y}_j\hat{y}_r+\epsilon_{jrm}\hat{y}_k\hat{y}_r\Big]\omega_m(x+y)\frac{dy}{|y|^3}=\frac{3}{8\pi}\text{P.V.}\int R_{kjm}(y)\omega_m(x+y)dy,\tag{8.16}$$

with the kernel

$$R_{kjm}(y) = \frac{1}{|y|^3}\big[\epsilon_{krm}\hat{y}_j\hat{y}_r + \epsilon_{jrm}\hat{y}_k\hat{y}_r\big].$$

This kernel is of the singular integral type we have seen before in the Beale-Kato-Majda criterion: it is homogeneous of degree $(-n)$ (the dimension $n = 3$), in the sense that

$$R_{kjm}(\lambda y) = \lambda^{-3}R_{kmj}(y), \tag{8.17}$$

and its integral over any sphere centered at $y = 0$ vanishes:

$$\int_{|y|=1} R_{kjm}(y)dy = \frac{1}{3}[\epsilon_{krm}\delta_{jr} + \epsilon_{jrm}\delta_{kr}] = \frac{1}{3}[\epsilon_{kjm} + \epsilon_{jkm}] = 0. \tag{8.18}$$

Let us show that (8.17) and (8.18) imply that the Fourier transform $\hat{R}_{kjm}(\xi)$ is uniformly bounded:

$$|\hat{R}_{kjm}(\xi)| \le C. \tag{8.19}$$

Indeed, let us write

$$R_{kjm}(y) = \frac{1}{|y|^3}\Phi(\hat{y}), \quad \int_{|y|=1}\Phi(y)dy = 0.$$

As $R_{kjm}(y)$ is homogeneous of degree $(-n)$ (in dimension $n = 3$), its Fourier transform is homogeneous of degree zero. Then we have:

$$\begin{aligned}\hat{R}_{kjm}(\xi) &= \lim_{\varepsilon,\delta\to 0}\int_\varepsilon^{1/\delta}\int_{\mathbb{S}^2}\frac{1}{r^3}e^{2\pi i r(\xi\cdot\hat{y})}\Phi(\hat{y})r^2 dr d\hat{y} = \lim_{\varepsilon\to 0}\int_\varepsilon^1\int_{\mathbb{S}^2}\Big[\cos(2\pi r(\xi\cdot\hat{y})) - 1\Big]\Phi(\hat{y})\frac{dr d\hat{y}}{r}\\ &+ \lim_{\delta\to 0}\int_1^{1/\delta}\int_{\mathbb{S}^2}\cos(2\pi r(\xi\cdot\hat{y}))\Phi(\hat{y})\frac{dr d\hat{y}}{r} + i\lim_{\varepsilon,\delta\to 0}\int_\varepsilon^{1/\delta}\int_{\mathbb{S}^2}\sin(2\pi r(\xi\cdot\hat{y}))\Phi(\hat{y})\frac{dr d\hat{y}}{r}\\ &= A_1 + A_2 + A_3.\end{aligned} \tag{8.20}$$

We used the mean-zero property of $\Phi(\hat{y})$ in the second equality above. For A_3, we may write

$$\begin{aligned}A_3(\xi) &= i\lim_{\varepsilon,\delta\to 0}\int_{\mathbb{S}^2}\Phi(\hat{y})\int_\varepsilon^{1/\delta}\sin(2\pi r(\xi\cdot\hat{y}))\frac{dr d\hat{y}}{r}\\ &= i\lim_{\varepsilon,\delta\to 0}\int_{\mathbb{S}^2}\Phi(\hat{y})\mathrm{sgn}(\xi\cdot\hat{y})\Big(\int_{2\pi|\xi\cdot\hat{y}|\varepsilon}^{2\pi|\xi\cdot\hat{y}|/\delta}\frac{\sin r dr}{r}\Big)d\hat{y}.\end{aligned}$$

Recall that there exists a constant $C_0 > 0$ so that for any $a, b > 0$ we have

$$\Big|\int_a^b\frac{\sin r dr}{r}\Big| \le C_0,$$

hence $|A_3(\xi)| \le C$. For A_1+A_2, we have

$$\begin{aligned}
&A_1(\xi)+A_2(\xi)=\lim_{\varepsilon\to 0}\int_{\mathbb{S}^2}\Phi(\hat y)\Big[\int_\varepsilon^{1/\varepsilon}[\cos(2\pi r(\xi\cdot\hat y))-1]\frac{dr}{r}\Big]d\hat y\\
&=\lim_{\varepsilon\to 0}\int_{\mathbb{S}^2}\Phi(\hat y)\Big[\int_{2\pi|\xi\cdot\hat y|\varepsilon}^{2\pi|\xi\cdot\hat y|/\varepsilon}[\cos(r)-1]\frac{dr}{r}\Big]d\hat y=\int_{\mathbb{S}^2}\Phi(\hat y)\Big[\int_0^1(\cos r-1)\frac{dr}{r}+\int_1^\infty\frac{\cos r\,dr}{r}\Big]d\hat y\\
&-\lim_{\varepsilon\to 0}\int_{\mathbb{S}^2}\Phi(\hat y)\Big[\int_1^{2\pi|\xi\cdot\hat y|/\varepsilon}\frac{dr}{r}\Big]d\hat y=\int_{\mathbb{S}^2}\Phi(\hat y)\Big[\int_0^1(\cos r-1)\frac{dr}{r}+\int_1^\infty\frac{\cos r\,dr}{r}\Big]d\hat y\\
&-\lim_{\varepsilon\to 0}\int_{\mathbb{S}^2}\Phi(\hat y)\log(2\pi|\xi\cdot\hat y|/\varepsilon)d\hat y\\
&=\int_{\mathbb{S}^2}\Phi(\hat y)\Big[\int_0^1(\cos r-1)\frac{dr}{r}+\int_1^\infty\frac{\cos r\,dr}{r}\Big]d\hat y-\int_{\mathbb{S}^2}\Phi(\hat y)\log(|\hat\xi\cdot\hat y|)d\hat y. \qquad (8.21)
\end{aligned}$$

We used the mean-zero property of $\Phi(\hat y)$ in the last step. In particular, it allowed us to replace ξ by $\hat\xi$ under the logarithm sign. Now, the first integral in the last line in (8.21) does not depend on ξ and is, therefore, uniformly bounded. The second is also bounded, by an application of the Cauchy-Schwartz inequality on $\mathbb{S}^2$. We conclude that the uniform bound (8.19) holds. It follows immediately that the strain matrix satisfies an L^2-bound

$$\|S\|_{L^2}\le C\|\omega\|_{L^2}, \qquad (8.22)$$

a bound we have already seen before.

The regularized system

We will follow the paper by P. Constatin and C Fefferman for the analysis of the vorticity alignment for the Navier-Stokes equations. A similar issue for the Euler equations has been studied in their joint paper with A. Majda. We will start with a regularized Navier-Stokes system, obtained by smoothing the advecting velocity:

$$\begin{aligned}
&u_t+(\phi_\delta * u)\cdot\nabla u+\nabla p=\nu\Delta u,\quad t>0,\quad x\in\mathbb{R}^n \qquad (8.23)\\
&\nabla\cdot u=0,\\
&u(0,x)=u_0(x).
\end{aligned}$$

The convolution is performed in space only:

$$u_\delta(t,x)=\phi_\delta * u(t,x)=\int\phi_\delta(x-y)u(t,y)dy,$$

and the kernel ϕ_δ has the form

$$\phi_\delta(x)=\frac{1}{\delta^3}\phi\Big(\frac{x}{\delta}\Big),$$

with a smooth compactly supported function $\phi(x)\ge 0$ with $\|\phi\|_{L^1}=1$. Note that u_δ is also divergence-free: $\nabla\cdot u_\delta=0$. Let us explain why the regularized system (8.23) has a strong solution, which is smooth if $u_0\in C_c^\infty(\mathbb{R}^3)$. Of course, the easy bounds on $u(t,x)$ will blow-up as $\delta\to 0$. We argue as in the estimate for the evolution of the H^m-norms in the proof of the Beale-Kato-Majda criterion. First, multiplying (8.23) by u and integrating by parts we deduce that

$$\int_{\mathbb{R}^3} |u(t,x)|^2 dx + \nu \int_0^t \int |\nabla u(s,x)|^2 dx ds = \int_{\mathbb{R}^3} |u_0(x)|^2 dx, \tag{8.24}$$

hence

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2}. \tag{8.25}$$

It follows from the definition of u_δ that

$$\|u_\delta(t)\|_{C^k} \le C_k(\delta), \tag{8.26}$$

with the constants $C_k(\delta)$ that may blow-up as $\delta \to 0$. Next, multiplying (8.23) by $(-\Delta)^m u$ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 = ((-\Delta)^{m/2}(u_\delta \cdot \nabla u), (-\Delta)^{m/2}u). \tag{8.27}$$

As before, the leading order term in the right side vanishes:

$$((u_\delta \cdot \nabla(-\Delta)^{m/2}u), (-\Delta)^{m/2}u) = 0,$$

because $\nabla \cdot u_\delta = 0$. Hence, using (8.26), the right side in (8.27) can be estimated by

$$C_m \|D^m u\|_2 \sum_{i,j=1}^{3} \sum_{k=1}^{m} \|D^k u_{\delta,j}\|_{L^\infty} \|D^{(m+1-k)} u_i\|_{L^2} \le C(\delta)\|u\|_{H^m}^2. \tag{8.28}$$

Summing over m, we conclude that for any $s \in \mathbb{N}$ we have

$$\frac{d}{dt}\|u\|_{H^s} \le C_s(\delta)\|u\|_{H^s}. \tag{8.29}$$

Therefore, if $u_0 \in C_c^\infty(\mathbb{R}^3)$, then $u(t)$ remains in all $H^m(\mathbb{R}^3)$ for all $t > 0$. Of course, the Sobolev norms of $u(t)$ may blow-up as $\delta \to 0$.

Vorticity alignment prevents blow-up

We will now show that if the direction of the vorticity of the solutions of the regularized system (8.23) is sufficiently aligned then solutions of the Navier-Stokes system itself remain regular. Let us introduce some notation: given a vector e we denote by $P_e^\perp$ the projection orthogonal to e,

$$P_e^\perp v = v - (v \cdot e)e.$$

We will denote by $u(t,x)$ the solution of the regularized system (8.23), let $\omega(t,x) = \nabla \times u(t,x)$ be its vorticity and $\xi(t,x) = \omega(t,x)/|\omega(t,x)|$, while $v(t,x)$ will be the solution of the true Navier-Stokes equations

$$\begin{aligned} & v_t + v \cdot \nabla v + \nabla p = \nu \Delta v, \quad t > 0, \quad x \in \mathbb{R}^n \\ & \nabla \cdot v = 0, \\ & v(0,x) = u_0(x). \end{aligned} \tag{8.30}$$

Theorem 8.1. *Assume that there exists δ_0, $\Omega > 0$ and $\rho > 0$ so that for all $\delta \in (0, \delta_0)$ the solution $u(t, x)$ of the regularized system (8.23) satisfies*

$$\left| P^{\perp}_{\xi(t,x)}(\xi(t, x + y)) \right| \le \frac{|y|}{\rho}, \tag{8.31}$$

for all $x, y \in \mathbb{R}^3$ and $0 \le t \le T$, such that $|\omega(t, x)| > \Omega$ and $|\omega(t, x+y) > \Omega$. Then the Navier-Stokes equations (8.30) have a strong, and hence C^∞-solution on the time interval $0 \le t \le T$.

The strategy will be to get a priori bounds on $u(t, x)$ that do not depend on δ and then pass to the limit $\delta \to 0$. The passage ot the limit is very similar to what we have seen before, so we focus on the a priori bounds that follow from assumption (8.30).

The a priori bounds for the regularized system

We first get a priori bounds for the regularized system that require no assumptions on the direction of the vorticity and, in particular, are independent of (8.31). Let us set $\omega_0 = \nabla \times u_0$ and

$$Q = \int_{\mathbb{R}^3} |\omega_0(x)| dx + \frac{25}{\nu} \int_{\mathbb{R}^3} |u_0(x)|^2 dx.$$

We have then the following bounds, uniform in $\delta > 0$.

Lemma 8.2. *The following two bounds hold:*

$$\int_{\mathbb{R}^3} |\omega(t, x)| dx + \nu \int_0^t \int_{\{x : |\omega(s,x)| > 0\}} |\omega(s, x)| |\nabla \xi(s, x)|^2 dx ds \le Q, \tag{8.32}$$

for all $0 \le t \le T$, and for any $\Omega > 0$ we have

$$\int_0^T \int_{\{x : |\omega(s,x)| > \Omega\}} |\nabla \xi(s, x)|^2 dx ds \le \frac{Q}{\nu \Omega}. \tag{8.33}$$

Proof. Let us derive the equation for $\omega(t, x)$: this derivation follows that for the true Navier-Stokes equations but the vorticity equation in the presence of the regularization is not identical to that of the Navier-Stokes equations. The advection term in the regularized Navier-Stokes equations can be written as

$$u_\delta \cdot \nabla u = u \cdot \nabla u + (u_\delta - u) \cdot \nabla u = u \cdot \nabla u - v_\delta \cdot \nabla u, \tag{8.34}$$

with

$$v_\delta = u - u_\delta.$$

Recall that

$$\begin{aligned}(\omega \times u)_i &= \varepsilon_{ijk} \omega_j u_k = \varepsilon_{ijk} \varepsilon_{jmn} (\partial_m u_n) u_k = (\delta_{in} \delta_{km} - \delta_{im} \delta_{kn})(\partial_m u_n) u_k \\ &= (\partial_k u_i) u_k - (\partial_i u_k) u_k.\end{aligned} \tag{8.35}$$

We used above the identity

$$\varepsilon_{jik} \varepsilon_{jmn} = \delta_{im} \delta_{kn} - \delta_{in} \delta_{km} \tag{8.36}$$

and anti-symmetry of ε_{ijk}. Thus, as we have previously seen, the advection term can be written as

$$u \cdot \nabla u = \omega \times u + \nabla\Big(\frac{|u|^2}{2}\Big). \tag{8.37}$$

Recall also the formula

$$\nabla \times (a \times b) = -a \cdot \nabla b + b \cdot \nabla a + a(\nabla \cdot b) - b(\nabla \cdot a), \tag{8.38}$$

which now gives

$$\nabla \times (u \cdot \nabla u) = \nabla \times (\omega \times u) = -\omega \cdot \nabla u + u \cdot \nabla \omega. \tag{8.39}$$

We also had an observation that

$$\omega \cdot \nabla u = V(t,x)\omega, \quad V_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{8.40}$$

The matrix V can be split into its symmetric and anti-symmetric parts:

$$V = S + P, \quad S = \frac{1}{2}(V + V^T), \quad P = \frac{1}{2}(V - V^T), \tag{8.41}$$

The anti-symmetric part has the form

$$\begin{aligned} P_{ij}h_j &= \frac{1}{2}[\partial_j u_i - \partial_i u_j]h_j = \frac{1}{2}\partial_m u_k[\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}]h_j = \frac{1}{2}\varepsilon_{lij}\varepsilon_{lkm}(\partial_m u_k)h_j \\ &= -\frac{1}{2}\varepsilon_{lij}\varepsilon_{lmk}(\partial_m u_k)h_j = -\frac{1}{2}\varepsilon_{lij}\omega_l h_j = \frac{1}{2}\varepsilon_{ilj}\omega_l h_j = \frac{1}{2}[\omega \times h]_i, \end{aligned} \tag{8.42}$$

for any $h \in \mathbb{R}^3$. In other words, P satisfies

$$Ph = \frac{1}{2}\omega \times h, \tag{8.43}$$

and thus has an explicit form

$$P = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{8.44}$$

As a consequence, we have $P\omega = 0$, thus $V\omega = S\omega$, so that

$$\nabla \times (u \cdot \nabla u) = u \cdot \nabla \omega - S\omega. \tag{8.45}$$

This is, of course, identical to what we have obtained for the true Navier-Stokes equations. For the term in (8.34), which involves v_δ and comes from the regularization, we write

$$\begin{aligned} &[\nabla \times (v_\delta \cdot \nabla u)]_i = \varepsilon_{ijk}\partial_j[v_{\delta,m}\partial_m u_k] = v_{\delta,m}\partial_m[\varepsilon_{ijk}\partial_j u_k] + \varepsilon_{ijk}(\partial_j v_{\delta,m})(\partial_m u_k) \\ &= v_\delta \cdot \nabla \omega_i + \varepsilon_{ijk}(\partial_j v_{\delta,m})(\partial_m u_k) \end{aligned} \tag{8.46}$$

Thus, we have

$$\nabla \times (u_\delta \cdot \nabla u) = u \cdot \nabla \omega - S\omega - v_\delta \cdot \nabla \omega + (\nabla u) \odot (\nabla v_\delta) = u_\delta \cdot \nabla \omega - S\omega + (\nabla u) \odot (\nabla v_\delta). \tag{8.47}$$

Here, we have introduced the following notation: given two matrices a and b, the vector $a \odot b$ has the entries

$$(a \odot b)_i = \varepsilon_{ijk} a_{km} b_{mj}. \tag{8.48}$$

Thus, the vorticity satisfies the evolution equation

$$\omega_t + u_\delta \cdot \nabla \omega - \nu \Delta \omega = S\omega - (\nabla u) \odot (\nabla v_\delta). \tag{8.49}$$

Once again, we stress that the second term in the right side comes from the regularization. Note that the vector $\xi(t,x) = \omega(t,x)/|\omega(t,x)|$ satisfies $|\xi|^2 = 1$, which implies

$$\xi \cdot \xi_t = 0, \ \xi_j \partial_k \xi_j = 0, \ \text{for all } 1 \le k \le 3, \tag{8.50}$$

leading to

$$(\partial_k \xi_j)(\partial_k \xi_j) + \xi_j \Delta \xi_j = 0, \tag{8.51}$$

Multiplying (8.49) by $\xi(t,x)$, and using (8.50)-(8.51), we get in the left side

$$\begin{aligned}
&\xi \cdot (\omega_t + u_\delta \cdot \nabla \omega - \nu \Delta \omega) = \xi \cdot (|\omega| \xi_t + \xi |\omega|_t + |\omega| (u_\delta \cdot \nabla) \xi + \xi (u_\delta \cdot \nabla |\omega|)) \\
&- \nu (\xi \cdot \xi) \Delta |\omega| - \nu (\xi \cdot \Delta \xi) |\omega| - 2\nu \xi_k \partial_j \xi_k \partial_j |\omega| = |\omega|_t + u_\delta \cdot \nabla |\omega| - \nu \Delta |\omega| - \nu |\omega| (\xi \cdot \Delta \xi) \\
&= |\omega|_t + u_\delta \cdot \nabla |\omega| - \nu \Delta |\omega| + \nu |\omega| |\nabla \xi|^2.
\end{aligned}$$

We deduce an evolution equation for $|\omega(t,x)|$ in the region where $\omega(t,x) \neq 0$:

$$\frac{\partial |\omega|}{\partial t} + u_\delta \cdot \nabla |\omega| - \nu \Delta |\omega| + \nu |\omega| |\nabla \xi|^2 = \xi \cdot (S\omega - (\nabla u) \odot (\nabla v_\delta)). \tag{8.52}$$

Let now $f(z)$ be a C^2-function of a scalar variable z which vanishes in a neighborhood of $z = 0$. Multiplying (8.52) by $f'(|\omega|)$ and integrating gives

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^3} f(|\omega|) dx + \nu \int_{\mathbb{R}^3} f''(|\omega|) |\nabla |\omega||^2 dx + \nu \int_{\mathbb{R}^3} |\omega| f'(|\omega|) |\nabla \xi|^2 dx \\
&= \int_{\mathbb{R}^3} [\xi \cdot (S\omega - (\nabla u) \odot (\nabla v_\delta))] f'(|\omega|) dx.
\end{aligned} \tag{8.53}$$

Choose a function $\psi(y) \ge 0$ such that $\psi(y)$ vanishes for $|y| \le r_0$ and $y > \Omega_0$, and such that

$$\int_0^{\Omega_0} \psi(y) dy = 1, \tag{8.54}$$

and set

$$f(z) = \int_0^z (z - y) \psi(y) dy, \tag{8.55}$$

so that

$$f'(z) = \int_0^z \psi(y) dy, \quad f''(z) = \psi(z) \ge 0. \tag{8.56}$$

In particular, we have $0 \le f'(z) \le 1$, $f'(z) = 0$ in a neighborhood of $z = 0$, and

$$z f'(z) = z, \ \text{for } z > \Omega_0. \tag{8.57}$$

In other words, $f(z)$ is an approximation to z. Then, integrating (8.53) in time gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(|\omega(t,x)|)dx + \nu\int_0^t\int_{\{x:\omega(s,x)|>\Omega_0\}} |\omega(s,x)||\nabla\xi(s,x)|^2dx \le \int_{\mathbb{R}^3} f(|\omega_0(x)|)dx \\
&+\int_0^t\int_{\mathbb{R}^3}[\xi\cdot(S\omega-(\nabla u)\odot(\nabla v_\delta))]f'(|\omega|)dxds \qquad (8.58)\\
&\le \int_{\mathbb{R}^3}|\omega_0(x)|dx + \int_0^t\int_{\mathbb{R}^3}\Big(\frac{1}{2}|S(s,x)|^2+\frac{1}{2}|\omega(s,x)|^2+\frac{1}{2}|\nabla u|^2+\frac{1}{2}|\nabla v_\delta|^2\Big)dxds.
\end{aligned}$$

As $\nabla\cdot u = 0$, we have

$$\int_{\mathbb{R}^3}|\nabla u|^2dx = \int_{\mathbb{R}^3}|\omega|^2dx = 2\int_{\mathbb{R}^3}\mathrm{Tr}S^2dx.$$

The energy identity (8.24) means that

$$\int_{\mathbb{R}^3} f(|\omega(t,x)|)dx + \nu\int_0^t\int_{\{x:\omega(s,x)|>\Omega_0\}} |\omega(s,x)||\nabla\xi(s,x)|^2dx \le Q, \qquad (8.59)$$

with

$$Q = \int_{\mathbb{R}^3}|\omega_0(x)|dx + \frac{25}{\nu}\int_{\mathbb{R}^3}|u_0(x)|^2dx. \qquad (8.60)$$

In particular, for any $\Omega > 0$ we obtain

$$\int_0^t\int_{\{x:\omega(s,x)|>\Omega\}}|\nabla\xi(s,x)|^2dx \le \frac{Q}{\nu\Omega}. \qquad (8.61)$$

We may also let $\Omega_0 \to 0$ in (8.59), so that $f(z)\to z$, and obtain the estimate in Lemma 8.2

$$\int_{\mathbb{R}^3}|\omega(t,x)|dx + \nu\int_0^t\int_{\{x:\omega(s,x)|>0\}}|\omega(s,x)||\nabla\xi(s,x)|^2dx \le Q. \qquad (8.62)$$

This finishes the proof of this Lemma.

Enstrophy bounds when the vorticity direction is regular

Lemma 8.2 does not use assumption (8.31) on the vorticity direction. Now, we will use this assumption to obtain enstrophy bounds on the solution of the regularized system. We will show that the solution of the regularized system obeys the following a priori bounds. Here, we use assumption (8.31): there exists δ_0, $\Omega > 0$ and $\rho > 0$ so that for all $\delta\in(0,\delta_0)$ the solution $u(t,x)$ of the regularized system (8.23) satisfies

$$\Big|P^\perp_{\xi(t,x)}(\xi(t,x+y))\Big| \le \frac{|y|}{\rho}, \qquad (8.63)$$

for all $x,y\in\mathbb{R}^3$ and $0\le t\le T$, such that $|\omega(t,x)|>\Omega$ and $|\omega(t,x+y)>\Omega$.

Lemma 8.3. *There exists a constant C which depends on the initial data u_0, and Ω, ν, T, and the constant ρ in (8.63), so that*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^3}|\omega(t,x)|^2dx \le C, \tag{8.64}$$

and

$$\int_0^T\int_{\mathbb{R}^3}|\nabla\omega(t,x)|^2dx \le C, \tag{8.65}$$

for all $\delta\in(0,\delta_0)$.

With these a priori bounds in hand, one can find a subsequence $\delta_k\downarrow 0$, such that the solutions $u(t,x)$ of the regularized Navier-Stokes system converge to a solution $v(t,x)$ of the true Navier-Stokes equations which obeys the same bounds (8.64) and (8.65). These bounds imply that v is a strong solution and is therefore smooth if u_0 is smooth. Thus, our focus is on proving Lemma 8.3.

Multiplying the vorticity equation

$$\omega_t + u_\delta\cdot\nabla\omega - \nu\Delta\omega = S\omega - (\nabla u)\odot(\nabla v_\delta) \tag{8.66}$$

by ω and integrating gives

$$\frac{1}{2}\frac{d}{dt}\int|\omega|^2dx + \nu\int|\nabla\omega|^2dx = \int(S\omega\cdot\omega)dx - \int\omega\cdot((\nabla u)\odot(\nabla v_\delta))dx. \tag{8.67}$$

We will split the vorticity into the "small" and "large" components: take a cut-off function $\chi(z)$ such that $\chi(z)=1$ for $0\le z\le 1$, $\chi(z)=0$ for $z\ge 2$, and $0\le\chi(z)\le 1$ for all $z\ge 0$. We set

$$\omega(t,x) = \omega^{(1)}(t,x) + \omega^{(2)}(t,x), \tag{8.68}$$

with

$$\omega^{(1)}(t,x) = \chi\Big(\frac{|\omega(t,x)|}{\Omega}\Big)\omega(t,x), \qquad \omega^{(2)}(t,x) = \Big(1-\chi\Big(\frac{|\omega(t,x)|}{\Omega}\Big)\Big)\omega(t,x). \tag{8.69}$$

Recall that the strain matrix can be written in terms of the vorticity as

$$S(x) = \frac{3}{4\pi}\text{P.V.}\int M(\hat{y},\omega(x+y))\frac{dy}{|y|^3},\ \ \hat{y}=\frac{y}{|y|}, \tag{8.70}$$

with the matrix-valued function

$$M(\hat{y},\omega) = \frac{1}{2}\big[(\hat{y}\times\omega)\otimes\hat{y} + \hat{y}\otimes(\hat{y}\times\omega)\big]. \tag{8.71}$$

The decomposition (8.68) and (8.70) induce then the corresponding decomposition

$$S(t,x) = S^{(1)}(t,x) + S^{(2)}(t,x). \tag{8.72}$$

We can then write

$$(S\omega\cdot\omega) = \sum_{i,j,k=1}^{2}(S^{(i)}\omega^{(j)}\cdot\omega^{(k)}) = X+Y+Z, \tag{8.73}$$

where X comes from the triplets where at least one of ω is "small":

$$X = \sum_{i=1}^{2} \sum_{(j,k)\neq(2,2)} (S^{(i)}\omega^{(j)} \cdot \omega^{(k)}),$$

the term Y has S "small", and both ω "large":

$$Y = (S^{(1)}\omega^{(2)} \cdot \omega^{(2)}),$$

and, finally, Z has S and both ω "large":

$$Z = (S^{(2)}\omega^{(2)} \cdot \omega^{(2)}).$$

We also set

$$W = -\omega \cdot ((\nabla u) \odot (\nabla v_\delta)).$$

With this notation, (8.67) has the form

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2 dx + \nu \int |\nabla\omega|^2 dx = \int (X + Y + Z + W) dx. \tag{8.74}$$

We will estimate the size of each term in the right side of (8.74) separately.

In order to estimate X, we recall that for any incompressible flow v we have

$$\int |\nabla v|^2 dx = \int |\zeta|^2 dx, \quad \zeta = \nabla \times v.$$

As a consequence, the strain matrix

$$S_v = \frac{1}{2}(\nabla v + (\nabla v)^t)$$

satisfies

$$\|S_v\|_{L^2}^2 = \sum_{i,j=1}^{3} \int \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)^2 dx \leq 4 \sum_{i,j=1}^{3} \int \Big(\frac{\partial v_i}{\partial x_j}\Big)^2 dx = 4\int |\nabla v|^2 dx = 4\int |\zeta|^2 dx. \tag{8.75}$$

Then, the term X can be estimated as follows: either $\omega^{(j)}$ or $\omega^{(k)}$ is "small" and can be bounded pointwise by Ω. This allows us to use the Cauchy-Schwartz inequality and (8.75):

$$\Big|\int X(t,x)dx\Big| \leq C\Omega \|S\|_{L^2}\|\omega\|_{L^2} \leq C\Omega\|\omega\|_{L^2}^2. \tag{8.76}$$

We have used the bound (8.22)

$$\|S\|_{L^2} \leq C\|\omega\|_{L^2}. \tag{8.77}$$

in the second inequality above.

Next, we note that Y is bounded from above by

$$|Y(t,x)| \leq |S^{(1)}(t,x)||\omega(t.x)|^2, \tag{8.78}$$

so that

$$\int |Y(t,x)|dx \le \|S^{(1)}\|_{L^2}\Big(\int |\omega(t.x)|^4 dx\Big)^{1/2}. \tag{8.79}$$

The Gagliardo-Nirenberg inequality in $\mathbb{R}^n$:

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^2}^a \|u\|_{L^2}^{1-a}, \quad \frac{1}{p} = \frac{1}{2} - \frac{a}{n},$$

implies that in $\mathbb{R}^3$ we have

$$\Big(\int |\omega(x)|^4 dx\Big)^{1/2} \le C\Big(\int |\nabla\omega(x)|^2 dx\Big)^{3/4}\Big(\int |\omega(x)|^2 dx\Big)^{1/4}. \tag{8.80}$$

Using this in (8.78) gives

$$\begin{aligned}\int |Y(t,x)|dx &\le \|S^{(1)}\|_{L^2}\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{1/2} \le C\|\omega^{(1)}\|_{L^2}\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{1/2}\\ &\le \frac{\nu}{8}\|\nabla\omega\|_{L^2}^2 + \frac{C}{\nu^3}\|\omega^{(1)}\|_{L^2}^4\|\omega\|_{L^2}^2. \end{aligned} \tag{8.81}$$

We have used Young's inequality in the last step, as well as the bound (8.77) for $\|S^{(1)}\|_{L^2}$. The second term in the right side can be bounded with the help of the estimate (8.32) in Lemma 8.2 as

$$\|\omega^{(1)}\|_{L^2}^2 \le 2\Omega \int |\omega(t,x)|dx \le 2\Omega Q. \tag{8.82}$$

Thus, the term Y can be estimated as

$$\int |Y(t,x)|dx \le \frac{\nu}{8}\|\nabla\omega\|_{L^2}^2 + \frac{C}{\nu^3}(\Omega Q)^2\|\omega\|_{L^2}^2. \tag{8.83}$$

Before looking at Z, which is the most difficult term, we bound W:

$$W = -\omega\cdot((\nabla u)\odot(\nabla v_\delta)).$$

This term is only there because of the regularization and should disappear as $\delta \to 0$. Note that

$$\begin{aligned}\|v_\delta\|_{L^2}^2 = \|u-u_\delta\|_{L^2}^2 = \|u-\phi_\delta * u\|_{L^2}^2 = \int |1-\hat\phi_\delta(\xi)|^2|\hat u(\xi)|^2 d\xi = \int |1-\hat\phi(\delta\xi)|^2|\hat u(\xi)|^2 d\xi\\ \le C\delta^2\int |\xi|^2|\hat u(\xi)|^2 d\xi = C\delta^2\|\nabla u\|_{L^2}^2 = C\delta^2\|\omega\|_{L^2}^2. \end{aligned} \tag{8.84}$$

The integral of W is

$$\begin{aligned}\int W(t,x)dx &= -\int \omega_i\varepsilon_{ijk}(\nabla u)_{km}(\nabla v_\delta)_{mj}dx = -\int \varepsilon_{ijk}\omega_i\frac{\partial u_k}{\partial x_m}\frac{\partial v_{\delta,m}}{\partial x_j}dx \\ &= \int \varepsilon_{ijk}v_{\delta,m}\frac{\partial\omega_i}{\partial x_j}\frac{\partial u_k}{\partial x_m}dx + \int \varepsilon_{ijk}v_{\delta,m}\omega_i\frac{\partial^2 u_k}{\partial x_j\partial x_m}dx. \end{aligned} \tag{8.85}$$

The last integral above can be written as

$$\int \varepsilon_{ijk} v_{\delta,m} \omega_i \frac{\partial^2 u_k}{\partial x_j \partial x_m} dx = \int \omega_i v_{\delta,m} \frac{\partial}{\partial x_m} \Big(\varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \Big) dx = \int \omega_i v_{\delta,m} \frac{\partial \omega_i}{\partial x_m} dx = 0, \tag{8.86}$$

since v_δ is divergence-free. Therefore, we have a bound for W:

$$\begin{aligned} \Big| \int W(t,x) dx \Big| &\leq \frac{\nu}{16} \int |\nabla \omega(t,x)|^2 dx + \frac{C}{\nu} \int |v_\delta(t,x)|^2 |\nabla u(t,x)|^2 dx \\ &\leq \frac{\nu}{16} \int |\nabla \omega(t,x)|^2 dx + \frac{C}{\nu} \|v_\delta\|_{L^4}^2 \|\nabla u\|_{L^4}^2. \end{aligned} \tag{8.87}$$

The Gagliardo-Nirenberg inequality implies that

$$\|v_\delta\|_{L^4}^2 \leq C \|\nabla v_\delta\|_{L^2}^{3/2} \|v_\delta\|_{L^2}^{1/2}. \tag{8.88}$$

For the gradient term above we can simply bound

$$\|\nabla v_\delta\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla u_\delta\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 \leq C \|\omega\|_{L^2}^2, \tag{8.89}$$

and we may use the estimate (8.84) for $|v_\delta\|_{L^2}$. Therefore, we have

$$\|v_\delta\|_{L^4}^2 \leq C \delta^{1/2} \|\omega\|_{L^2}^2. \tag{8.90}$$

We may also use the same Gagliardo-Nirenberg inequality for $\|\nabla u\|_{L^4}$, leading to

$$\|\nabla u\|_{L^4}^2 \leq C \|\nabla \omega\|_{L^2}^{3/2} \|\omega\|_{L^2}^{1/2}. \tag{8.91}$$

Altogether, this gives

$$\begin{aligned} \frac{1}{\nu} \|v_\delta\|_{L^4}^2 \|\nabla u\|_{L^4}^2 &\leq \frac{C\delta^{1/2}}{\nu} \|\omega\|_{L^2}^2 \|\nabla \omega\|_{L^2}^{3/2} \|\omega\|_{L^2}^{1/2} = \frac{C\delta^{1/2}}{\nu} \|\omega\|_{L^2}^{5/2} \|\nabla \omega\|_{L^2}^{3/2} \\ &\leq \frac{\nu}{16} \|\nabla \omega\|_{L^2}^2 + \frac{C\delta^2}{\nu^7} \|\omega\|_{L^2}^{10}, \end{aligned} \tag{8.92}$$

thus

$$\Big| \int W(t,x) dx \Big| \leq \frac{\nu}{8} \int |\nabla \omega(t,x)|^2 dx + \frac{C\delta^2}{\nu^7} \|\omega\|_{L^2}^{10}. \tag{8.93}$$

Finally, we estimate the most dangerous term $Z(t,x)$,

$$Z = (S^{(2)} \omega^{(2)} \cdot \omega^{(2)}),$$

and this will be the only estimate that will involve the assumption that the direction $\xi(t,x)$ of the vorticity is Lipschitz:

$$\Big| P_{\xi(t,x)}^{\perp} (\xi(t, x+y)) \Big| \leq \frac{|y|}{\rho}, \tag{8.94}$$

We write

$$Z(t,x) = (S^{(2)} \omega^{(2)} \cdot \omega^{(2)}) = |\omega^{(2)}(t,x)|^2 (S^{(2)}(t,x) \xi^{(2)}(t,x) \cdot \xi^{(2)}(t,x)) = |\omega(t,x)|^2 \alpha^{(2)}(t,x), \tag{8.95}$$

with

$$\alpha^{(2)}(t,x) = \frac{3}{4\pi}\text{P.V.}\int D(\hat{y},\xi(x+y),\xi(x))|\omega^{(2)}(x+y)|\frac{dy}{|y|^3}, \tag{8.96}$$

where

$$D(e_1,e_2,e_3) = (e_1\cdot e_3)\text{Det}(e_1,e_2,e_3).$$

Assumption (8.94) means that

$$|D(\hat{y},\xi(x+y),\xi(x))| \le \frac{|y|}{\rho}, \tag{8.97}$$

so that

$$|Z(t,x)| \le \frac{3}{4\pi\rho}|\omega^{(2)}(t,x)|^2\int|\omega^{(2)}(t,x+y)|\frac{dy}{|y|^2} \le \frac{3}{4\pi\rho}|\omega(t,x)|^2\int|\omega(t,x+y)|\frac{dy}{|y|^2}. \tag{8.98}$$

Therefore, we have

$$\int|Z(t,x)|dx \le \frac{C}{\rho}\|\omega\|_{L^4}^2\left(\int|I(t,x)|^2dx\right)^{1/2}, \tag{8.99}$$

with

$$I(t,x) = \int|\omega(t,x+y)|\frac{dy}{|y|^2}.$$

In order to compute the L^2-norm of I, we proceed as in the proof of Nash inequality. Let us compute the Fourier transform of the function $\psi(y) = 1/|y|^2$:

$$\begin{aligned}\hat{\psi}(\xi) &= \int\frac{e^{2\pi i\xi\cdot y}dy}{|y|^2} = \int_0^\infty dr\int_{-\pi/2}^{\pi/2}d\theta\cos\theta\int_0^{2\pi}d\phi e^{2\pi i|\xi|r\sin\theta}\\ &= 2\pi\int_0^\infty dr\int_{-1}^1 due^{2\pi i|\xi|ru} = \frac{2}{|\xi|}\int_0^\infty\frac{\sin r dr}{r}.\end{aligned}$$

Hence, the L^2-norm of $I(t,x)$ can be bounded as (for any $R>0$)

$$\begin{aligned}\|I(t)\|_{L^2}^2 &= \int|\hat{I}(t,\xi)|^2d\xi \le C\int\frac{|\omega(\xi)|^2}{|\xi|^2}d\xi\\ &\le C\int_{|\xi|\le R}\frac{|\omega(\xi)|^2d\xi}{|\xi|^2} + C\int_{|\xi|\ge R}\frac{|\omega(\xi)|^2d\xi}{|\xi|^2} = A_R + B_R.\end{aligned}$$

Since

$$|\hat{\omega}(\xi)| \le \|\omega\|_{L^1},$$

the first term can be bounded as,

$$|A_R| \le C\int_0^R\|\omega\|_{L^1}^2d\xi \le CR\|\omega\|_{L^1}^2.$$

The second term can be simply bounded by

$$|B_R| \le \frac{C}{R^2}\int_{|\xi|\ge R}|\omega(\xi)|^2d\xi = \frac{C}{R^2}\|\omega\|_{L^2}^2.$$

It follows that for any $R > 0$ we have

$$\|I\|_{L^2}^2 \le CR\|\omega\|_{L^1}^2 + \frac{C}{R^2}\|\omega\|_{L^2}^2.$$

Choosing

$$R = \Big(\frac{\|\omega\|_{L^2}^2}{\|\omega\|_{L^1}^2}\Big)^{1/3},$$

we deduce that

$$\|I\|_{L^2}^2 \le C\|\omega\|_{L^1}^{4/3}\|\omega\|_{L^2}^{2/3}. \tag{8.100}$$

Returning to (8.99), we see that

$$\int |Z(t,x)|dx \le \frac{C}{\rho}\|\omega\|_{L^4}^2\|\omega\|_{L^1}^{2/3}\|\omega\|_{L^2}^{1/3}. \tag{8.101}$$

The L^4-norm of ω is estimated using the same Gagliardo-Nirenberg inequality:

$$\|\omega\|_{L^4}^2 \le C\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{1/2}, \tag{8.102}$$

so that

$$\int |Z(t,x)|dx \le C\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{5/6}\|\omega\|_{L^1}^{2/3} \le \frac{\nu}{15}\|\nabla\omega\|_{L^2}^2 + \frac{C}{\nu^3\rho^4}\|\omega\|_{L^2}^{20/6}\|\omega\|_{L^1}^{8/3}. \tag{8.103}$$

Recalling also the a priori bound (8.32) in Lemma 8.2:

$$\int_{\mathbb{R}^3} |\omega(t,x)|dx \le Q, \tag{8.104}$$

we see that Z is bounded as

$$\int |Z(t,x)|dx \le \frac{\nu}{15}\|\nabla\omega\|_{L^2}^2 + \frac{CQ^{8/3}}{\nu^3\rho^4}\|\omega\|_{L^2}^{10/3}. \tag{8.105}$$

Recollecting the starting point of our analysis (8.67)

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2dx + \nu\int |\nabla\omega|^2dx = \int (S\omega\cdot\omega)dx - \int \omega\cdot((\nabla u)\odot(\nabla v_\delta))dx, \tag{8.106}$$

and summarizing the bounds (8.76), (8.83), (8.93), (8.105) that we have obtained for the terms X, Y, W and Z, respectively, in the right side of the above identity, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\omega|^2dx + \nu\int |\nabla\omega|^2dx &\le C\Omega\|\omega\|_{L^2}^2 + \frac{\nu}{8}\|\nabla\omega\|_{L^2}^2 + \frac{C}{\nu^3}(\Omega Q)^2\|\omega\|_{L^2}^2 \\ &+\frac{\nu}{8}\|\nabla\omega\|^2 + \frac{C\delta^2}{\nu^7}\|\omega\|_{L^2}^{10} + \frac{\nu}{15}\|\nabla\omega\|_{L^2}^2 + \frac{CQ^{8/3}}{\nu^3\rho^4}\|\omega\|_{L^2}^{10/3}.\end{aligned} \tag{8.107}$$

Thus, the enstrophy

$$E(t) = \int |\omega(t,x)|^2dx,$$

satisfies a differential inequality

$$\frac{dE}{dt} \le C_1(1+E^{2/3})E + C_1\delta^2 E^5, \tag{8.108}$$

with a constant C_1 that depends on ν, ρ, Ω and Q. This is a nonlinear inequality and at the first glance it may seem useless as the solution of an ODE

$$\dot{z} = C_1(1+z^{2/3})z + C_1\delta^2 z^5, \quad z(0) = z_0 > 0, \tag{8.109}$$

blows up in a finite time. Here, however, we are only concerned with the solution being finite until time $t = T$, and, in addition, we have an extra piece of information: the function

$$k(t) = C_1(1+E^{2/3})$$

has a bounded integral:

$$\int_0^T k(t)dt \le CT + \int_0^T \|\omega(t)\|_{L^2}^{4/3} dt \le CT + CT^{1/3}\Big(\int_0^T \|\omega(t)\|_{L^2}^2 dt\Big)^{2/3} \le C(1+T) = D. \tag{8.110}$$

Crucially, the constant D does not depend on δ. Therefore, the solution of (8.109) with $\delta = 0$ does remain finite until the time T, and it is reasonable to expect that so does the solution with $\delta > 0$ but small. To formalize this observation, let

$$\bar{E}(t) = 2E(0)\exp\Big\{\int_0^t k(s)ds\Big\}.$$

Then $E(0) \le \bar{E}(0)$, and we may define τ as the first time such that $E(\tau) = \bar{E}(\tau)$. Until that time, the function $E(t)$ satisfies

$$\frac{dE}{dt} \le k(t)E + C_1\delta^2 \bar{E}^5, \quad 0 \le t \le \tau. \tag{8.111}$$

Therefore, as long as $E(t) \le \bar{E}(t)$, we have a bound for $E(t)$:

$$E(t) \le E(0)\exp\Big\{\int_0^t k(s)ds\Big\} + C_1\delta^2 \int_0^t \bar{E}^5(s)\exp\Big\{\int_s^t k(s')ds'\Big\}ds.$$

Thus, if δ is sufficiently small, we have $E(t) \le \bar{E}(t)$ for all $0 \le t \le T$. We conclude that there exists $\delta_0 > 0$ so that for all $0 < \delta < \delta_0$ the enstrophy is bounded:

$$\sup_{0\le t\le T} \int |\omega(t,x)|^2 dx < +\infty. \tag{8.112}$$

The last step is to observe that (8.107) together with (8.112) implies that

$$\nu \int_0^T \int |\nabla\omega|^2 dx \ < +\infty. \tag{8.113}$$

This completes the proof of Lemma 8.3, and thus that of Theorem 8.1. □

9 The Caffarelli-Kohn-Nirenberg theorem

In this section, we will describe the results of Caffarelli, Kohn and Nirenberg on the Hausdorff dimension of the set where the solution of the three-dimensional Navier-Stokes equations

$$u_t + u \cdot \nabla u + \nabla p = \Delta u + f, \tag{9.1}$$
$$\nabla \cdot u = 0, \tag{9.2}$$

can possibly be singular. We consider this problem in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, with the no-slip boundary condition

$$u(t,x) = 0 \text{ on } \partial\Omega. \tag{9.3}$$

The force $f(t,x)$ is assumed to satisfy the incompressibility condition $\nabla \cdot f = 0$ – this condition is not really necessary, as otherwise we would write $f = \nabla\Phi + g$, with $\nabla \cdot g = 0$, and absorb Φ into the pressure term.

Weak solutions

Let us recall the notion of a Leray weak solution of the Navier-Stokes equations: u is a weak solution if, first, it is a solution in the sense of distributions, that is, for any smooth compactly supported vector-valued function $\psi(t,x)$ we have

$$\begin{aligned}&\int_\Omega [u(t,x)\cdot\psi(t,x) - u_0(x)\cdot\psi(0,x)]dx - \int_0^t\int_\Omega (u\cdot\psi_s)dxds - \int_0^t\int_\Omega u_k u_j \frac{\partial\psi_j}{\partial x_k}dxds\\ &-\int_0^t\int_\Omega p(\nabla\cdot\psi)dxds = \int_0^t\int_\Omega (u\cdot\Delta\psi)dxds + \int_0^t\int_\Omega (f\cdot\psi)dfxds.\end{aligned} \tag{9.4}$$

The second condition is that u satisfies the energy inequality. Note that if u is a smooth solution of the Navier-Stokes equations, then for any smooth test function ϕ we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u(t,x)|^2\phi(t,x)dx + \int_0^t\int_\Omega |\nabla u(s,x)|^2\phi(s,x)dxds = \frac{1}{2}\int_\Omega |u_0(x)|^2\phi(0,x)dx \\ &+\frac{1}{2}\int_0^t\int_\Omega |u(s,x)|^2(\phi_s(s,x)+\Delta\phi(s,x))dxds\\ &+\int_0^t\int_\Omega \Big(\frac{|u(s,x)|^2}{2}+p(s,x)\Big)u\cdot\nabla\phi(s,x)dxds + \int_0^t\int_\Omega (f\cdot u)\phi(s,x)dxds.\end{aligned} \tag{9.5}$$

Taking, formally, $\phi \equiv 1$, the second condition for u to be a Leray weak solution is that it satisfies the energy inequality:

$$\frac{1}{2}\int_\Omega |u(t,x)|^2dx + \int_0^t\int_\Omega |\nabla u(s,x)|^2dxds \le \frac{1}{2}\int_\Omega |u_0(x)|^2dx + \int_0^t\int_\Omega (f\cdot u)dxds. \tag{9.6}$$

Suitable weak solutions

Caffarelli, Kohn and Nuremberg consider a slightly stronger class of solutions, which they call suitable weak solutions, defined on an open (time-space) set $D \in \mathbb{R}\times\mathbb{R}^3$. We will, obviously, require that u is a weak solution of the Navier-Stokes equations in the sense of distributions: (9.4) holds for any function ϕ supported in D. We will assume that $f \in L^q(D)$ with some $q > 5/2$ – this assumption is not very important, as the main result is interesting even for $f \in C^\infty(D)$. We will also assume that the pressure satisfies

$$p \in L^{5/4}(D), \tag{9.7}$$

and that there exist some constants E_0 and E_1 so that or any fixed time t we have

$$\int_{D_t} |u(t,x)|^2 dx \le E_0, \tag{9.8}$$

where $D_t = D \cap (\mathbb{R}^3 \times \{t\})$, and

$$\int_D |\nabla u(s,x)|^2 dx \le E_1. \tag{9.9}$$

In addition, we require that the generalized (or, localized) energy inequality holds: for any function $\phi \ge 0$ which is smooth and compactly supported in D, we have

$$\begin{aligned}\int_D |\nabla u(s,x)|^2 \phi(s,x) dx ds \le \frac{1}{2}\int_D |u(s,x)|^2(\phi_s(s,x) + \Delta\phi(s,x)) dx ds \\ + \int_D \Big(\frac{|u(s,x)|^2}{2} + p(s,x)\Big) u \cdot \nabla\phi(s,x) dx ds + \int_D (f\cdot u)\phi(s,x) dx ds.\end{aligned} \tag{9.10}$$

At the moment, it is not clear that a suitable weak solution exists – we will prove it below.

The parabolic Hausdorff measure

In order to formulate the main results, we need to define an analog of the Hausdorff measure $\mathcal{H}^1$ but suitable for the parabolic problems. For any set $X \subset \mathbb{R}\times\mathbb{R}^3$, $\delta > 0$ and $k \ge 0$ we define

$$\mathcal{P}^k_\delta(X) = \inf\Big\{\sum_{i=1}^\infty r_i^k:\ X \subset \bigcup_i Q_{r_i},\ r_i < \delta\Big\}. \tag{9.11}$$

Here, Q_r is a parabolic cylinder: it has the form

$$Q_r = [t - r^2, t] \times B_r(x),$$

where $B_r(x)$ is a ball of radius r centered at the point x. Then we set

$$\mathcal{P}^k(X) = \lim_{\delta\downarrow 0} \mathcal{P}^k_\delta(X). \tag{9.12}$$

The standard Hausdorff measure is defined in the same way but with Q_r replaced by an arbitrary closed subset of $\mathbb{R}\times\mathbb{R}^3$ of diameter at most r_i, thus we have

$$\mathcal{H}^1 \le C_k \mathcal{P}^k.$$

The main results

We may now describe the main results of the Caffarelli-Kohn-Nirenberg paper. We say that a point (t, x) is singular if u is not in L^∞_{loc} in any neighborhood of (t, x). Otherwise, we say that (t, x) is a singular point. We will denote by S the set of all singular points of $u(t, x)$. Their first result shows that the singularity set has zero Hausdorff measure $\mathcal{H}^1$.

Theorem 9.1. *Assume that either $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, and let $D = (0, T) \times \Omega$. Suppose that for some $q > 5/2$ we have*

$$f \in L^2(D) \cap L^q_{loc}(D) \quad \nabla \cdot f = 0$$

and

$$u_0 \in L^2(\Omega), \quad \nabla \cdot u_0 = 0, \quad u_0 \cdot \nu|_{\partial\Omega} = 0.$$

If Ω is bounded, we require, in addition, that $u_0 \in W^{2/5}_{5/4}(\Omega)$. Then the initial boundary value problem has a suitable weak solution in D whose singular set S satisfies $\mathcal{P}^1(S) = 0$.

Their second result concerns absence of singularities outside of a ball of radius $1/\sqrt{t}$.

Theorem 9.2. *Consider the Navier-Stokes equations in $\mathbb{R}^3$ with $f = 0$ and assume that the initial data satisfies $\nabla \cdot u_0 = 0$, and*

$$G = \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 |x| dx < +\infty. \tag{9.13}$$

Then there exists a weak solution of the initial value problem which is regular in the region $\{|x| \geq K_1/\sqrt{t}\}$, with the constant K_1 which depends only on G and E, where

$$E = \int_{\mathbb{R}^3} |u_0(x)|^2 |x| dx < +\infty.$$

Assumption (9.13) means that u is small at infinity, and this smallness, so to speak, invades the whole space as t grows. If we assume that u is "small near the origin", in the sense, that

$$L = \int_{\mathbb{R}^3} \frac{|u_0|^2}{|x|} dx = L < +\infty, \tag{9.14}$$

then we have the following result.

Theorem 9.3. *Consider the Navier-Stokes equations in $\mathbb{R}^3$ with $f = 0$ and assume that the initial data satisfies $\nabla \cdot u_0 = 0$, and (9.14) holds. There exists a universal constant L_0 so that if $L < L_0$, then u is regular in the region $\{|x| \leq \sqrt{(L_0 - L)t}\}$.*

The first key estimate: localizing "small data regularity"

We will denote the cylinders labeled by the top as

$$Q_r(t, x) = \{(s, y) : \ |y - x| < r, \ t - r^2 < s < t\},$$

and those labeled by a point slightly below the top as

$$Q_r^*(t,x) = \{(s,y): \ |y-x| < r, \ t - \frac{7}{8}r^2 < s < t + \frac{1}{8}r^2\}.$$

It is well known that if the initial condition u_0 and the force f are small in an appropriate norm, then the solution of the Navier-Stokes equations remains regular for a short time. The main issue in proving the partial regularity theorems is to localize this result. The first step in this direction is an estimate showing that if u, p and f are sufficiently small on the unit cylinder $Q_1 = Q_1(0,0)$, then u is regular in the smaller cylinder $Q_{1/2} = Q_{1/2}(0,0)$ – this is a very common theme in the parabolic regularity theory.

Proposition 9.4. *There exist absolute constants $C_1 > 0$ and $\varepsilon_1 > 0$ and a constant $\varepsilon_2(q) > 0$, which depends only on q with the following property. Suppose that (u,p) is a suitable weak solution of the Navier-Stokes system on Q_1 with $f \in L^q$, with $q > 5/2$. Assume also that*

$$\int_{Q_1} (|u|^3 + |u||p|) dx dt + \int_{-1}^{0} \Big(\int_{|x|<1} |p| dx \Big)^{5/4} dt \le \varepsilon_1, \tag{9.15}$$

and

$$\int_{Q_1} |f|^q dx dt \le \varepsilon_2. \tag{9.16}$$

Then we have $|u(t,x)| \le C_1$ for Lebesgue-almost every $(t,x) \in Q_{1/2}$. In particular, u is regular in $Q_{1/2}$.

In order to see how we may scale this result to a parabolic cylinder of length r, let us investigate the dimension of various terms in the Navier-Stokes equations

$$u_t + u \cdot \nabla u + \nabla p = \Delta u + f. \tag{9.17}$$

Let us assign dimension L to the spatial variable x. As all individual terms in (9.17) should have the same dimension, looking at the terms u_t and Δu we conclude that time should have dimension L^2. Comparing the terms u_t and $u \cdot \nabla u$ we see that u should have the dimension L^{-1}. Then, f should have the same dimension as u_t, which is L^{-3}. Finally, the dimension of the pressure term should be L^{-2}. Summarizing, we have

$$[x] = L, \ [t] = L^2, \ [u] = L^{-1}, \ [f] = L^{-3}, \ [p] = L^{-2}. \tag{9.18}$$

Let us look at the dimension of each term in the estimate (9.15): the term involving $|u|^3$ has the dimension

$$[x]^3 [t] [u]^3 = L^2,$$

the term involving $|u||p|$ has the same dimension:

$$[x]^3 [t] [u] [p] = L^2,$$

while the last term in the left side has the dimension

$$[t][x]^{15/4} [p]^{5/4} = L^{23/4} L^{-10/4} = L^{13/4}.$$

We also should note that the dimension of the L^q-norm of f (to the power q) is

$$[x]^3[t][f]^q = L^{5-3q}.$$

Accordingly, for a parabolic cylinder $Q_r(t,x)$ we set

$$M(r) = \frac{1}{r^2}\int_{Q_r}(|u|^3 + |u||p|)dxdt + \frac{1}{r^{13/4}}\int_{t-r^2}^{t}\Big(\int_{|y-x|<r}|p|dx\Big)^{5/4}dt, \tag{9.19}$$

and

$$F_q(r) = r^{3q-5}\int_{Q_r}|f|^q dyds. \tag{9.20}$$

Therefore, Proposition 9.4 has the following corollary.

Corollary 9.5. *Suppose hat (u,p) is a suitable weak solution of the Navier-Stokes system on a cylinder Q_r with $f \in L^q$, with $q > 5/2$. Assume also that*

$$M(r) \leq \varepsilon_1, \tag{9.21}$$

and

$$F_q(r) \leq \varepsilon_2. \tag{9.22}$$

Then we have $|u(t,x)| \leq C_1/r$ for Lebesgue-almost every $(t,x) \in Q_{r/2}$. In particular, u is regular in $Q_{r/2}$.

The second key estimate: the blow-up rate

One can deduce from Corollary 9.5 a heuristic estimate on the possible blow-up rate of the solution. Assume that (t_0,x_0) is a singular point. Then, (9.21) has to fail for all $Q_r(t,x)$ such that $(t_0,x_0) \in Q_{r/2}(t,x)$. Therefore, we must have

$$M(r) = M(r;t,x) > \varepsilon_1$$

for a family of parabolic cylinders shrinking to the point (t_0,x_0). Let us assume that

$$u(t,x) \sim r^{-m},$$

near x_0, with

$$r = (|x-x_0|^2 + |t-t_0|)^{1/2}.$$

Then we have

$$M(r) \sim \frac{1}{r^2}\frac{1}{r^{3m}}r^2r^3 = r^{3-3m}.$$

hence, a natural guess is $m = 1$, which translates into

$$|\nabla u| \geq \frac{C}{r^2}, \quad \text{as } (t,x) \to (t_0,x_0). \tag{9.23}$$

The next key estimate verifies that this is qualitatively correct.

Proposition 9.6. *There is an absolute constant $\varepsilon_3 > 0$ with the following property. If u is a suitable weak solution of the Navier-Stokes equations near (t, x), and if*

$$\limsup_{r\downarrow 0} \frac{1}{r}\int_{Q_r^*(t,x)} |\nabla u|^2 dyds \leq \varepsilon_3, \tag{9.24}$$

then (t, x) is a regular point.

Let us explain how Theorem 9.1 would follow. Take any (t, x) in the singular set, then, by Proposition 9.6 we have

$$\limsup_{r\downarrow 0} \frac{1}{r}\int_{Q_r^*(t,x)} |\nabla u|^2 dyds > \varepsilon_3. \tag{9.25}$$

Take a neighborhood V of the singular set S and $\delta > 0$. For each $(t, x) \in S$ we may choose a parabolic cylinder $Q_r^*(t, x)$ with $r < \delta$ and such that

$$\frac{1}{r}\int_{Q_r^*(t,x)} |\nabla u|^2 dyds > \varepsilon_3, \tag{9.26}$$

and $Q_r^*(t, x) \subset V$. We will make use of the following covering lemma.

Lemma 9.7. *Let $\mathcal{J}$ be a collection of parabolic cylinders $Q_r^*(t, x)$ contained in a bounded set V. Then there exists an at most countable sub-collection $\mathcal{J}' = \{Q_i^* = Q_{r_1}^*(t_i, x_i)\}$ of non-overlapping cylinders such that for any $Q^* \in \mathcal{J}$ there exists Q_i^* so that*

$$Q^* \subset Q_{5r_i}^*(t_i, x_i).$$

The proof is very similar to that of the classic Vitali lemma and we leave it to the reader as an exercise. Using this lemma, we obtain a disjoint collection of cylinders $Q_{r_i}^*(t_i, x_i)$ such that

$$S \subset \bigcup_i Q_{5r_i}^*(t_i, x_i),$$

and

$$\sum_i r_i \leq \frac{1}{\varepsilon_3}\int_{Q_{r_i}^*} |\nabla u|^2 dxdt \leq \frac{1}{\varepsilon_3}\int_V |\nabla u|^2 dxdt.$$

We deduce that

$$\mathcal{P}^1(S) \leq \frac{1}{\varepsilon_3}\int_V |\nabla u|^2 dxdt. \tag{9.27}$$

In particular, we deduce that the (three-dimensional) Lebesgue measure of S is zero. Then, as V is an arbitrary neighborhood of S, and the function $|\nabla u|^2$ is integrable, we can make the right side of (9.27) arbitrarily small. It follows that $\mathcal{P}^1(S) = 0$, proving Theorem 9.1. Thus, the crux of the matter is the proof of Propositions 9.4 and 9.6.

Serrin's interior regularity result

Before we proceed with the further discussion of the proofs of the theorems of Caffarelli, Kohn and Nirenberg, let us explain why we say a solution is regular if it is just bounded, and do not require further differentiability. The reason is a result of Serrin on the interior regularity of the weak solutions of the Navier-Stokes equations

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \Delta u + f, \\ &\nabla \cdot u = 0. \end{aligned} \tag{9.28}$$

We will assume for simplicity that $f = 0$ – the reader should consider the generalization to the case $f \neq 0$ as an exercise, or consult Serrin's original paper. Let us borrow the following very simple observation from Serrin's paper: if $\psi(x)$ is a harmonic function, then any function of the form

$$u(t,x) = a(t)\nabla\psi(x)$$

is a weak solution of the Navier-Stokes equations, as long as the function $a(t)$ is integrable. Therefore, boundedness of $u(t,x)$ can not, in general, imply any information on the time derivatives of u. On the other hand, this example does not rule out the hope that relatively weak assumptions on u would guarantee its spatial regularity.

Here is one version of Serrin's result, which says that bounded solutions of the force-less Navier-Stokes equations are essentially as good as the solutions of the heat equation.

Theorem 9.8. *Let u be a Leray weak solution of the Navier-Stokes equations in an open region $R = (t_1, t_2) \times \Omega$ of space-time, with $f = 0$, and such that*

$$\int_{t_1}^{t_2} \int_\Omega |\omega(t,x)|^2 dx dt < +\infty, \quad \sup_{t \in [t_1, t_2]} \int_\Omega |u(t,x)|^2 dx < +\infty, \tag{9.29}$$

where $\omega = \nabla \times u$ is the vorticity. Assume, in addition, that $u \in L^\infty(R)$. Then, u is of the C^∞ class in the space variables on every compact subset of R.

The full statement of the Serrin theorem says that if $u \in L^{s,s'}(R)$, with

$$\|u\|_{L^{s,s'}} = \Big(\int_{t_1}^{t_2} \|u\|_{L^s(\Omega)}^{s'} dt \Big)^{1/s'},$$

with (in three dimensions)

$$\frac{3}{s} + \frac{2}{s'} < 1, \tag{9.30}$$

then u is C^∞ in the spatial variables. If, in addition, we know that $u_t \in L^{2,p}$ with $p \geq 1$, then the spatial derivatives of u are absolutely continuous in time. We will not need these results for our purposes, so we will leave them out for now. Let us make one comment, however: if we take $s' = \infty$, then condition (9.30) is satisfied, as long as $s > 3$. That is, if we would have known a priori that

$$\int_{\mathbb{R}^3} |u(t,x)|^3 dx \leq \text{const},$$

then we could conclude that u is a smooth solution. Of course, we have this information only for the L^2-norm of the Leray weak solutions, and not for the L^3-norm.

For the proof of Theorem 9.8, let us recall the vorticity equation in three dimensions:

$$\omega_t + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u. \tag{9.31}$$

Written in the components, this equation is

$$\frac{\partial \omega_k}{\partial t} - \Delta \omega_k = \omega_j \frac{\partial u_k}{\partial x_j} - u_j \frac{\partial \omega_k}{\partial x_j}, \tag{9.32}$$

or

$$\frac{\partial \omega_k}{\partial t} - \Delta \omega_k = \frac{\partial}{\partial x_j}(\omega_j u_k - u_j\ \omega_k). \tag{9.33}$$

Let $\bar{\Omega}_1$ be a compact subset of Ω, and $t_1 < s_1 < s_2 < t_2$, so that $S = (s_1, s_2) \times \Omega_1$ is a proper subset of R, and define, for $s_1 \leq t \leq s_2$:

$$\begin{aligned}
\tilde{\omega}_k(t,x) &= \frac{\partial}{\partial x_j} \int_{s_1}^t \int_{\Omega_1} G(t-s, x-y)[\omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)]dyds \\
&= \int_{s_1}^t \int_{\Omega_1} \frac{\partial G(t-s,x-y)}{\partial x_j}[\omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)]dyds.
\end{aligned}$$

Here, $G(t,x)$ is the standard heat kernel. The functions

$$m_{kj}(t,x) = \int_{s_1}^t \int_{\Omega_1} G(t-s, x-y)[\omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)]dyds$$

satisfy

$$\frac{\partial m_{kj}}{\partial t} - \Delta m_{kj} = (\omega_j u_k - u_j\ \omega_k)\chi_{[s_1,s_2]}(t)\chi_{\bar{\Omega}_1}(x). \tag{9.34}$$

Thus, for $(t,x) \in S$, the function $\tilde{\omega}$ is the solution of

$$\frac{\partial \tilde{\omega}_k}{\partial t} - \Delta \tilde{\omega}_k = \frac{\partial}{\partial x_j}(\omega_j u_k - u_j\ \omega_k). \tag{9.35}$$

It follows that the difference

$$B(t,x) = \omega(t,x) - \tilde{\omega}(t,x)$$

satisfies the standard heat equation

$$B_t - \Delta B = 0,$$

on the set S.

We will now show that $\omega \in L^\infty(S)$, that is, if u is uniformly bounded on R, then the vorticity is uniformly bounded on any compact subset of R.

Exercise 9.9. *Use the convolution with the heat kernel to show that if $\phi(t,x)$ satisfies*

$$\phi_t - \Delta\phi = \frac{\partial g}{\partial x_j},$$

in the whole space $\mathbb{R}^n$, then

$$\|\phi\|_{L^r} \le C\|g\|_{L^q},$$

as long as

$$(n+2)\Big(\frac{1}{q} - \frac{1}{r}\Big) < 1.$$

The norms are take in space-time.

As u is a Leray weak solution, we know that $\omega \in L^2(R)$. As $u \in L^\infty(R)$, it follows that the functions

$$g_{jk}(s,y) = \omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)$$

are also in $L^2(R)$. The result of the above exercise says that then $\tilde\omega \in L^r$ with

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

But then $g \in L^6$, as well, and, as $1/6 < 1/3$, it follows that $\tilde\omega \in L^\infty(R)$. We also know that $B \in L^\infty(S)$ by the regularity estimates for the heat equation, as $B \in L^2(R)$ – it is the difference of two functions in $L^2(R)$. Moreover, we know that B is Hölder continuous.

Now that we know that $\omega \in L^\infty(R)$, we recall that the velocity and the vorticity are related by the stream vector ψ, defined as the solution of

$$-\Delta\psi = \omega, \quad \nabla\cdot\psi = 0,$$

and

$$u = -\nabla\times\psi.$$

Therefore, if $\omega \in L^\infty(R)$, then ψ is $C^{1,\alpha}$ in the spatial variable, hence u is Hölder in x, and, in particular, in L^∞. Then the functions m_{kj} are $C^{1,\alpha}$ in x, thus ω is Hölder in x. Then, the functions g_{kj} are Hölder in x, so ω_x is Hölder in x, continuing this argument we deduce that both ω and u are C^∞.

Existence of suitable weak solutions

We now prove the existence of suitable weak solutions, in the sense of Caffarelli, Kohn and Nirenberg. We will restrict ourselves to the whole space: $\Omega = \mathbb{R}^3$. Let us first define the appropriate function spaces. As usual, we will denote by $\mathcal{V}$ the space of smooth divergence-free vector fields u, by H the closure of $\mathcal{V}$ in $L^2(\mathbb{R}^3)$, by V the closure of $\mathcal{V}$ in $H^1(\mathbb{R}^3)$, and by V' the dual space of V. The Sobolev spaces $W^l_q(\mathbb{R}^3)$ with $q \ge 1$ and $0 < l < 1$ consists of functions with l derivatives in L^q, and with the norm

$$\|u\|_{W^l_q} = \|u\|_{L^q} + \|(-\Delta)^{l/2}u\|_{L^q}.$$

We will make the standard assumptions:

$$\Omega = \mathbb{R}^3,\ u_0 \in H,\ f \in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{9.36}$$

Theorem 9.10. *Assume that $\Omega = \mathbb{R}^3$, u_0 and f satisfy (9.36). Then there exists a suitable weak solution*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H),$$

of the Navier-Stokes equations with the force f and the initial condition u_0, in the sense that $u(t) \to u_0$ weakly in H as $t \to 0$. The pressure satisfies $p \in L^{5/3}((0,T) \times \mathbb{R}^3)$. In addition, if $\phi \in C^\infty([0,T] \times \mathbb{R}^3)$, $\phi \geq 0$ and is compactly supported, then

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} |u(t,x)|^2 \phi(t,x)dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(s,x)|^2 \phi(s,x) dxds \leq \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 \phi(0,x)dx \\
&+\frac{1}{2}\int_0^t \int_{\mathbb{R}^3} |u(s,x)|^2 (\phi_s(s,x) + \Delta\phi(s,x))dxds \qquad (9.37)\\
&+\int_0^t \int_{\mathbb{R}^3} \Big(\frac{|u(s,x)|^2}{2} + p(s,x)\Big) u \cdot \nabla \phi(s,x)dxds + \int_0^t \int_{\mathbb{R}^3} (f \cdot u)\phi(s,x)dxds.
\end{aligned}$$

The proof is done via a "retarded mollification". The (standard) idea is to take $\Psi_\delta(u)$ to be a mollifier of u such that $\Psi_\delta(u)$ is divergence-free and depends only on the values of $u(s,x)$ with $s \leq t - \delta$. The mollified system

$$u_t + \Psi_\delta(u) \cdot \nabla u + \nabla p = \Delta u + f \qquad (9.38)$$

is then linear on each time interval of the firm $(m\delta, (m+1)\delta)$. We will get uniform in δ a priori bounds on u, and then pass to the limit $\delta \to 0$.

Let us recall some basic facts about the linear Stokes equation, whose proof is very similar to what we have done on the torus previously.

$$u_t + \nabla p = \Delta u + f, \quad \nabla \cdot u = 0. \qquad (9.39)$$

Lemma 9.11. *Suppose that $f \in L^2(0,T;V')$, $u \in L^2(0,T;V)$, p is a distribution and (9.39) holds. Then $u_t \in L^2(0,T;V')$,*

$$\frac{d}{dt}\int_\Omega |u|^2 dx = 2\int_\Omega (u_t \cdot u)dx,$$

in the sense of distributions on $(0,T)$, and $u \in C([0,T],H)$, possibly after a modification on a set of measure zero.

Lemma 9.12. *Suppose that $f \in L^2(0,T;V')$, $u_0 \in H$, and $w \in C^\infty([0,T];\Omega)$ are prescribed, and $\nabla \cdot w = 0$. Then there exists a unique function $u \in L^2(0,T;V) \cap C([0,T];H)$, and a distribution p so that*

$$u_t + w \cdot \nabla u + \nabla p = \Delta u + f, \quad \nabla \cdot u = 0, \qquad (9.40)$$

in the sense of distributions, and $u(0) = u_0$.

Some pressure bounds and interpolation on the velocity

Note that if u solves (9.40) in the whole space, then the pressure satisfies the Poisson equation

$$\Delta p = -\sum_{i,j=1}^3 \partial_{ij}^2 (w_i u_j). \qquad (9.41)$$

The singular integral operator corresponding to the Fourier multiplier

$$\frac{\xi_i\xi_j}{|\xi|^2}$$

is bounded $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, thus, in particular, we have the bound

$$\int_0^T \int_{\mathbb{R}^3} |p|^{5/3} dxds \le C \int_0^T \int_{\mathbb{R}^3} |w|^{5/3}|u|^{5/3} dxds \tag{9.42}$$

$$\le C\Big(\int_0^T \int_{\mathbb{R}^3} |w|^{10/3} dxds\Big)^{1/2}\Big(\int_0^T \int_{\mathbb{R}^3} |u|^{10/3} dxds\Big)^{1/2}. \tag{9.43}$$

We will now use a Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^3} |u|^q dx \le C\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^a\Big(\int_{\mathbb{R}^3} |u|^2 dx\Big)^{q/2-a}, \tag{9.44}$$

with $2 \le q \le 6$ and $a = 3(q-2)/4$. Note that when $q = 2$, $a = 0$, this is a tautology, and when $q = 6$, $a = 3$, this is the familiar Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^3} |u|^6 dx \le C\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^3. \tag{9.45}$$

Taking $q = 10/3$, and $a = 1$ gives

$$\int_{\mathbb{R}^3} |u|^{10/3} dx \le C\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Big(\int_{\mathbb{R}^3} |u|^2 dx\Big)^{2/3} \tag{9.46}$$

Integrating in time and using the a priori assumptions (9.8) and (9.9) leads to

$$\int_0^T \int_{\mathbb{R}^3} |u|^{10/3} dxdt \le CE_1(u)E_0^{2/3}(u). \tag{9.47}$$

Another useful estimate, obtained, once again, by taking $q = 10/3$ and $a = 1$, is

$$\int_0^T \int_{\mathbb{R}^3} |w \cdot \nabla u|^{5/4} dxdt \le \Big(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 dxdt\Big)^{5/8}\Big(\int_0^T \int_{\mathbb{R}^3} |w|^{10/3} dxdt\Big)^{3/8} \tag{9.48}$$

$$\le CE_1(u)^{5/8}E_1(w)^{3/8}E_0(w)^{1/4}, \tag{9.49}$$

which can be restated as

$$\|w \cdot \nabla u\|_{L^{5/4}} \le CE_1(u)^{1/2}E_1(w)^{3/10}E_0(w)^{1/5}. \tag{9.50}$$

We will also use the following bound, which follows from (9.45) with $q = 5/2$ and $a = 3/8$:

$$\int_{\mathbb{R}^3} |u|^{5/2} dx \le CE_0^{7/8}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{3/8}. \tag{9.51}$$

As a consequence, we have

$$\int_0^T \Big(\int_{\mathbb{R}^3}|u|^{5/2}dx\Big)^2 dt \le CE_0(u)^{7/4}\int_0^T \Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^{3/4}dt \tag{9.52}$$
$$\le CE_0(u)^{7/4}T^{1/4}\Big(\int_0^T\int_{\mathbb{R}^3}|\nabla u|^2dxdt\Big)^{3/4} \le CT^{1/4}E_0^{7/4}E_1(u)^{3/4}.$$

This can be restated as

$$\|u\|_{L^5(0,T;L^{5/2})} \le CT^{1/20}E_0^{7/20}E_1(u)^{3/20}. \tag{9.53}$$

These bounds allow us to take a solution (in the sense of distributions) $u \in C([0,T];H)\cap L^2(0,T;V)$ of the Stokes advection equation

$$u_t + w\cdot u - \Delta u + \nabla p = f, \tag{9.54}$$

with $w \in C^\infty$, multiply by a test function ϕ and obtain

$$\int_{\mathbb{R}^3}|u|^2(T,x)\phi(T,x)dx + 2\int_0^T\int_{\mathbb{R}^3}|\nabla u(t,x)|^2\phi(t,x)dxdt = \int_{\mathbb{R}^3}|u_0(x)|^2\phi(0,x)dx \tag{9.55}$$
$$+\int_0^T\int_{\mathbb{R}^3}|u|^2(\phi_t+\Delta\phi)dxdt + \int_0^T\int_{\mathbb{R}^3}(|u|^2w+2pu)\cdot\nabla\phi dxdt + 2\int_0^T\int_{\mathbb{R}^3}(u\cdot f)dxdt.$$

Exercise 9.13. Justify the integration by parts above by mollifying (in time and space) each term in the Stokes equation, multiplying by ϕ, integrating by parts and then removing the mollification using the a priori bounds obtained above.

The retarded mollifier

We take a C^∞ function $\psi(t,x)\ge 0$ such that

$$\int \psi(t,x)dxdt = 1,$$

and

$$\mathrm{supp}\psi \subset \{(t,x):\ |x|^2<t,\ 1<t<2\}.$$

We also extend $u(t,x)$ by zero to $t<0$, and set

$$\Psi_\delta(u)(t,x) = \frac{1}{\delta^4}\int_{\mathbb{R}^4}\psi\big(\frac{s}{\delta},\frac{y}{\delta}\big)\tilde{u}(x-y,t-s)dyds. \tag{9.56}$$

The mollified u is divergence-free:

$$\nabla\cdot\Psi_\delta(u) = 0,$$

and it inherits the a priori bounds on u:

$$\sup_{0\le t\le t}\int_{\mathbb{R}^3}|\Psi_\delta(u)|^2(t,x)dx \le CE_0(u), \tag{9.57}$$

and

$$\int_0^T\int_{\mathbb{R}^3}|\Psi_\delta(u)|^2(t,x)dxdt \le CE_1(u). \tag{9.58}$$

The approximants

We will use the approximants

$$\begin{aligned}
&\frac{\partial u_N}{\partial t} + \Psi_\delta(u_N) \cdot \nabla u_N + \nabla p_N = \Delta u_N + f, \qquad (9.59)\\
&\nabla \cdot u_N = 0,\\
&u_N(0,x) = u_0(x),
\end{aligned}$$

with $\delta = T/N$. We may apply inductively the existence result for the Stokes equation with a prescribed advection, on the time intervals of the form $(m\delta,(m+1)\delta)$, $0 \le m \le N-1$. Then we have

$$\int_{\mathbb{R}^3} |u_N(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u_N(s,x)|^2 dx ds = \int_{\mathbb{R}^3} |u_0(x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} (f \cdot u_N) dx ds. \quad (9.60)$$

In particular, we have

$$\int_{\mathbb{R}^3} |u_N(t,x)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u_N(s,x)|^2 dx ds \le \int_{\mathbb{R}^3} |u_0(x)|^2 dx + \int_0^t \|f\|_{V'}^2 ds. \quad (9.61)$$

We conclude that u_N is uniformly bounded in $L^\infty(0,T;V) \cap L^\infty(0,T;H)$, the usual Leray bound. In addition, we know that p_N is bounded in $L^{5/3}([0,T]\times\mathbb{R}^3)$. It follows that, after an extraction of a sub-sequence, we have that $p_N \to p_*$ weakly in $L^{5/3}([0,T]\times\mathbb{R}^3)$, and $u_N \to u_*$, weak-star in $L^\infty(0,T;H)$, and weakly in $L^2(0,T;V)$.

Exercise 9.14. Show that if u_N is bounded in $L^\infty(0,T;V)\cap L^\infty(0,T;H)$, and $\frac{\partial u_N}{\partial t}$ is bounded in $L^2(0,T;H^{-2})$, then u_N has a convergent subsequence in $L^2([0,T]\times\mathbb{R}^3)$.

Exercise 9.15. *Show that if $u_N \to u_*$ strongly in L^q and u_N is bounded in L^r, $1 \le q < r$, then $u_N \to u_*$ strongly in L^s for all $q, s < r$.*

We may use this with $q = 2$ and $r = 10/3$ to conclude that $u_N \to u_*$ strongly in $L^s([0,T]\times\mathbb{R}^3)$ for all $2 \le s < 10/3$. Then one may easily check that (u_*, p_*) is the sought suitable weak solution of the Navier-Stokes equations.

The proof of Proposition 9.4

We now turn to the proof of the two main auxiliary results, and begin with Proposition 9.4. We recall its statement:

Proposition 9.16. *There exist two absolute constants $C_1 > 0$ and $\varepsilon_1 > 0$ and another constant $\varepsilon_2(q) > 0$, which depends only on q with the following property. Suppose that (u,p) is a suitable weak solution of the Navier-Stokes system on $Q_1(0,0)$ with $f \in L^q$, with $q > 5/2$. Assume also that*

$$\int_{Q_1} (|u|^3 + |u||p|) dx dt + \int_{-1}^0 \Big(\int_{|x|<1} |p| dx\Big)^{5/4} dt \le \varepsilon_1, \quad (9.62)$$

and

$$\int_{Q_1} |f|^q dx dt \le \varepsilon_2. \quad (9.63)$$

Then we have $|u(t,x)| \le C_1$ for Lebesgue-almost every $(t,x) \in Q_{1/2}(0,0)$. In particular, u is regular in $Q_{1/2}$.

Outline of the proof

Let us take an arbitrary point $(s, x_0) \in Q_{1/2}(0,0)$, where we want to show that $|u(s, x_0)| \le C_1$. As $Q_{1/2}(s, x_0) \subset Q_1(0,0)$, we have an integral estimate

$$\int_{Q_{1/2}(s,x_0)} (|u|^3 + |u||p|)dxdt + \int_{s-1/4}^{s} \Big(\int_{|x-x_0|<1/2} |p|dx \Big)^{5/4} dt \le \varepsilon_1. \tag{9.64}$$

We will consider a sequence of shrinking parabolic cylinders $Q_k = Q_{r_k}(s, x_0)$, "centered" at the point (s, x_0) with $r_k = 2^{-k}$. Our goal will be to show that for all $k \ge 2$ we have

$$⨍_{|x-x_0|<r_k} |u(s,x)|^2 dx \le C_0 \varepsilon_1^{2/3}, \tag{9.65}$$

where $⨍_S f$ denotes the average of a function f over the set S. Then, if (s, x_0) is a Lebesgue point for u, it follows that

$$|u(s,x_0)|^2 \le C_0 \varepsilon_1^{2/3}, \tag{9.66}$$

hence (9.66) holds for Lebesgue almost every point in $Q_{1/2}(0,0)$, which is exactly the claim of Proposition 9.16.

In order to prove (9.65) we will show that for all $k \ge 2$ we have a more general estimate

$$\sup_{s-r_k^2<t\le s} ⨍_{|x-x_0|\le r_k} |u(t,x)|^2 dx + \frac{1}{r_k^3} \int_{Q_k} |\nabla u(t,x)|^2 dxdt \le C_0 \varepsilon_1^{2/3}. \tag{9.67}$$

Note that (9.65) follows immediately from (9.67). Thus, the conclusion of Proposition 9.4 follows from (9.67).

The induction base. We will prove (9.67) by induction, starting with $k = 2$. For $k = 2$, we may use the localized energy inequality: for every smooth test function $\phi(t, x) \ge 0$, that vanishes near $|x| = 1$ and $t = -1$, we have, for $-1 < s < 0$, with $B_1 = B_1(0,0)$:

$$\begin{aligned} \int_{B_1} |u(s,x)|^2 \phi(s,x)dx + 2\int_{-1}^{s}\int_{B_1} |\nabla u(t,x)|^2 \phi(t,x) dxdt \le \int_{-1}^{s}\int_{B_1} |u(t,x)|^2 (\phi_t + \Delta\phi) dxdt \\ + \int_{-1}^{s}\int_{B_1} (|u|^2 + 2p) u \cdot \nabla\phi(t,x) dtdx + 2\int_{-1}^{s}\int_{B_1} (f\cdot u)\phi(t,x)dxdt. \end{aligned} \tag{9.68}$$

Taking ϕ such that $0 \le \phi \le 1$, $\phi \equiv 1$ on $Q_{1/2}(0,0)$ and ϕ is supported in $Q_1(0,0)$, we deduce that

$$\int_{|x-x_0|\le 1/4} |u(s,x)|^2 dx + \int_{Q_2} |\nabla u(t,x)|^2 dxdt \le C \int_{Q_1(0,0)} (|u|^2 + |u|^3 + |u||p| + |u||f|) dxdt. \tag{9.69}$$

Now, we may use Young's inequality on the term $|u||f|$, together with the L^q-bound on f, with $q > 5/2$, the Hölder inequality, as well as our assumption (9.64), to conclude that the left side of (9.69) is smaller than $C\varepsilon_1^{2/3}$, provided that ε_1 and ε_2 are both sufficiently small. Thus, (9.67) holds for $k = 2$.

The induction step. The induction step in the proof of (9.67) will be split into two sub-steps. First, we will show that if (9.67) holds for all $2 \le k \le n-1$, and $n \ge 3$, then we have

$$\frac{1}{|Q_n|}\int_{Q_n}|u|^3dxdt + \frac{r_n^{3/5}}{|Q_n|}\int_{Q_n}|u||p-\bar{p}_n|dxdt \le \varepsilon_1^{2/3}, \tag{9.70}$$

where

$$\bar{p}_n(t) = \fint_{|x-x_0|<r_n} p(t,x)dx. \tag{9.71}$$

Next, we will show that if (9.70) holds for all $3 \le k \le n$, then (9.67) holds for $k = n$. That is, we have the following two lemmas.

Lemma 9.17. *Assume that ε_1 and ε_2 are sufficiently small, and $n \ge 3$, and (9.67) holds for all $2 \le k \le n-1$, then (9.70) holds.*

Lemma 9.18. *Assume that (9.70) holds for all $3 \le k \le n$, and ε_1 and ε_2 are sufficiently small, then (9.67) holds for $k = n$.*

The proof of these lemmas is the heart of the argument.

The proof of Lemma 9.17

We set

$$A(r) = \sup_{s-r^2<t<s}\frac{1}{r}\int_{B_r(x_0)}|u(t,x)|^2dx, \qquad G(r) = \frac{1}{r^2}\int_{Q_r(s,x_0)}|u|^3dxdt,$$

and

$$\delta(r) = \frac{1}{r}\int_{Q_r(s,x_0)}|\nabla u(t,x)|^2dxdt.$$

Recalling that the dimension of u is $1/L$, and the dimension of t is L^2, while the dimension of p is $1/L^2$, we see that, $A(r)$, $G(r)$, and $\delta(r)$ are all dimensionless. The induction hypothesis is

$$A(r_k) + \delta(r_k) \le C\varepsilon_1^{2/3}r_k^2, \quad 2 \le k \le n-1. \tag{9.72}$$

In addition, we know that

$$G(r_1) + K(r_1) \le C\varepsilon_1, \tag{9.73}$$

which is part of (9.64).

Bound on the first term in (9.70). The two terms in the left side of (9.70) will be estimated separately. We will extensively use the Gagliardo-Nirenberg inequality in a ball

$$\int_{B_r}|u|^qdx \le C\Big(\int_{B_r}|\nabla u|^2dx\Big)^a\Big(\int_{B_r}|u|^2\Big)^{q/2-a} + \frac{C}{r^{2a}}\Big(\int_{B_r}|u|^2dx\Big)^{q/2}, \tag{9.74}$$

with $2 \le q \le 6$, and $a = 3(q-2)/4$ – this is the only choice of a which makes (9.74) dimensionally correct. Taking $q = 3$ and $a = 3/4$ gives a bound on the L^3-norm that appears in the left side of (9.70):

$$\int_{B_r}|u|^3dx \le C\Big(\int_{B_r}|\nabla u|^2dx\Big)^{3/4}\Big(\int_{B_r}|u|^2\Big)^{3/4} + \frac{C}{r^{3/2}}\Big(\int_{B_r}|u|^2dx\Big)^{3/2}. \tag{9.75}$$

Integrating in time and using Hölder's inequality leads to

$$
\begin{aligned}
&\int_{Q_r}|u|^3dxdt \le C\int_{s-r^2}^{s}\Big(\int_{B_r}|\nabla u|^2dx\Big)^{3/4}\Big(\int_{B_r}|u|^2dx\Big)^{3/4}dt+\frac{C}{r^{3/2}}\int_{s-r^2}^{s}\Big(\int_{B_r}|u|^2dx\Big)^{3/2}dt\\
&\le C\Big(\int_{Q_r}|\nabla u|^2dxdt\Big)^{3/4}\Big(\int_{s-r^2}^{s}\Big(\int_{B_r}|u|^2dx\Big)^{3}dt\Big)^{1/4}+\frac{C}{r^{3/2}}\int_{s-r^2}^{s}\Big(\int_{B_r}|u|^2dx\Big)^{3/2}dt\\
&\le C\Big(r\delta(r)\Big)^{3/4}r^{1/2}[rA(r)]^{3/4}+Cr^{1/2}[rA(r)]^{3/2}=Cr^2A(r)^{3/4}[\delta(r)^{3/4}+A(r)^{3/4}]. \qquad (9.76)
\end{aligned}
$$

Dividing by $|Q_r|$ gives

$$
\begin{aligned}
&\frac{1}{|Q_{r_{n-1}}|}\int_{Q_{r_{n-1}}}|u|^3dxdt\le\frac{C}{r_{n-1}^5}\int_{Q_{r_{n-1}}}|u|^3dxdt\le\frac{C}{r_{n-1}^3}A(r_{n-1})^{3/4}[\delta(r_{n-1})^{3/4}+A(r_{n-1})^{3/4}]\\
&\le\frac{C}{r_{n-1}^3}(A(r_{n-1})+\delta(r_{n-1}))^{3/2}\le C\varepsilon_1, \qquad (9.77)
\end{aligned}
$$

which, in turn, means that

$$
\frac{1}{|Q_{r_n}|}\int_{Q_{r_n}}|u|^3dxdt\le\frac{C'}{|Q_{r_{n-1}}|}\int_{Q_{r_{n-1}}}|u|^3dxdt\le C''\varepsilon_1. \qquad (9.78)
$$

Hence, if ε_1 is so small that

$$
C''\varepsilon^{1/3}\le\frac{1}{2},
$$

then

$$
\frac{1}{|Q_{r_n}|}\int_{Q_{r_n}}|u|^3dxdt\le\frac{1}{2}\varepsilon_1^{2/3}. \qquad (9.79)
$$

This is the estimate we need on the first term in the left side of (9.70). Note that (9.78) can be also restated as

$$
G(r_n)\le C\varepsilon_1 r_n^3. \qquad (9.80)
$$

Bound on the second term in (9.70). In order to get a bound on the second term in the left side of (9.70), we need to show that, under the assumption

$$
A(r_k)+\delta(r_k)\le C\varepsilon_1^{2/3}r_k^2,\quad 2\le k\le n-1, \qquad (9.81)
$$

we have

$$
\frac{r_n^{3/5}}{|Q_n|}\int_{Q_n}|u||p-\bar{p}_n|dxdt\le\frac{\varepsilon_1^{2/3}}{2}, \qquad (9.82)
$$

provided that ε_1 is sufficiently small. The main issue is bounding the pressure. Recall that p satisfies the Poisson equation (note that this is the first time in the proof of the current lemma that we use the Navier-Stokes equations)

$$
-\Delta p=\frac{\partial^2}{\partial x_i\partial x_j}(u_iu_j). \qquad (9.83)
$$

For any cut-off function ϕ we can write

$$\phi(x)p(t,x) = -\frac{3}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\Delta_y(\phi p)dy = -\frac{3}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}(p\Delta\phi + 2\nabla\phi\cdot\nabla p + \phi\Delta p)dy.$$

Using (9.83) and integrating by parts, we may write the above as

$$\phi p = p_1 + p_2 + p_3,$$

where

$$\begin{aligned}
p_1 &= \frac{3}{4\pi}\int_{\mathbb{R}^3}\frac{\partial^2}{\partial y_i\partial y_j}\Big[\frac{1}{|x-y|}\Big]\phi u_i u_j dy,\\
p_2 &= \frac{3}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\frac{\partial\phi}{\partial y_j}u_i u_j dy + \frac{3}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\frac{\partial^2\phi}{\partial y_i\partial y_j}u_i u_j dy,\\
p_3 &= \frac{3}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}p\Delta\phi dy + \frac{3}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}p\frac{\partial\phi}{\partial y_j}dy.
\end{aligned}$$

We will take a function ϕ so that $\phi(y)\equiv 1$ for $|y-x_0|\le 3/16$ and $\phi(y)=0$ if $|y-x_0|\ge 1/4$. Let us split p_1 as

$$p_1 = p_{11} + p_{12},$$

with

$$\begin{aligned}
p_{11} &= \frac{3}{4\pi}\int_{|y-x_0|<2r_n}\frac{\partial^2}{\partial y_i\partial y_j}\Big[\frac{1}{|x-y|}\Big]\phi u_i u_j dy,\\
p_{12} &= \frac{3}{4\pi}\int_{|y-x_0|>2r_n}\frac{\partial^2}{\partial y_i\partial y_j}\Big[\frac{1}{|x-y|}\Big]\phi u_i u_j dy.
\end{aligned}$$

We can write (dropping the subscript n for the moment)

$$|p-\bar p| \le |p_{11}-\bar p_{11}| + |p_{12}-\bar p_{12}| + |p_3-\bar p_3| + |p_4-\bar p_4|.$$

To estimate p_{11}, recall that the operators

$$T_{ij}(\psi) = \Big(\nabla^2_{ik}\frac{1}{|x|}\Big)\star\psi$$

are Calderon-Zygmund operators, hence they are uniformly bounded in L^q, $1<q<\infty$. It follows that (we denote $r=r_n$ and $B_r = B_{r_n}(x_0)$)

$$\|p_{11}\|_{L^{3/2}(B_r)} \le C\Big(\int_{B_{2r}}|u|^3 dx\Big)^{2/3},$$

and

$$\bar p_{11} \le \frac{1}{|B_r|}\int_{B_r}|p|dx \le \frac{1}{|B_r|^{2/3}}\Big(\int_{B_r}|p|^{3/2}dx\Big)^{2/3},$$

hence

$$\int_{B_r}|\bar p_{11}|^{3/2}dx \le \int_{B_r}|p|^{3/2}dx.$$

We conclude that

$$\int_{B_r} |u||p_{11} - \bar{p}_{11}|dx \le C\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\Big(\int_{B_{2r}} |u|^3 dx\Big)^{2/3}. \tag{9.84}$$

The terms $|p_i - \bar{p}_i|$ for p_{12}, p_2 and p_3 are estimated using the following bounds on the gradients ∇p_i for $|x - x_0| < r$ (recall that $\phi \equiv 1$ in the ball $B_{3/16}(x_0)$ so that $\nabla\phi = 0$ in that ball):

$$\begin{aligned}
&|\nabla p_{12}(x)| \le C\int_{2r<|y-x_0|<1/4} \frac{|u|^2}{|y-x|^3}dy \le C\int_{2r<|y-x_0|<1/4} \frac{|u|^2}{|y-x_0|^3}dy,\\
&|\nabla p_2(x)| \le C\int_{B_{1/4}(x_0)} |u|^2 dy,\\
&|\nabla p_3(x)| \le C\int_{B_{1/4}(x_0)} |p| dy.
\end{aligned}$$

This leads to

$$\begin{aligned}
\int_{B_r} |u||p_{12} - \bar{p}_{12}| &\le Cr\big[\sup_{x\in B_r} |\nabla p_{12}(x)|\big](r^3)^{2/3}\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\\
&\le Cr^3\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\int_{2r<|y-x_0|<1/4} \frac{|u|^2}{|y-x_0|^3}dy,
\end{aligned} \tag{9.85}$$

and

$$\begin{aligned}
&\int_{B_r} |u||p_2 - \bar{p}_2| \le Cr\big[\sup_{x\in B_r} |\nabla p_2(x)|\big](r^3)^{2/3}\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\\
&\le Cr^3\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\int_{B_{1/4}(x_0)} |u|^2 dy \le Cr^3\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\Big(\int_{B_{1/4}(x_0)} |u|^3 dy\Big)^{2/3}.
\end{aligned} \tag{9.86}$$

For p_3, we write

$$\begin{aligned}
\int_{B_r} |u||p_3 - \bar{p}_3| &\le Cr\Big(\int_{B_r} |u| dy\Big)\Big(\int_{B_{1/4}(x_0)} |p|\Big)\\
&\le Cr(r^3)^{3/5}\Big(\int_{B_r} |u|^2 dy\Big)^{1/5}\Big(\int_{B_r} |u|^3 dy\Big)^{1/5}\Big(\int_{B_{1/4}(x_0)} |p|\Big)\\
&\le Cr^3 A(r)^{1/5}\Big(\int_{B_r} |u|^3 dy\Big)^{1/5}\Big(\int_{B_{1/4}(x_0)} |p|\Big).
\end{aligned} \tag{9.87}$$

Integrating the above estimates over the time interval $s - r^2 \le t \le s$, and collecting all the terms we get

$$\int_{Q_r} |u||p - \bar{p}_r| dxdt \le W_1 + W_2 + W_3 + W_4. \tag{9.88}$$

The term

$$W_1 = C\Big(\int_{Q_r} |u|^3 dxdt\Big)^{1/3}\Big(\int_{Q_{2r}} |u|^3 dxdt\Big)^{2/3} = Cr^2 G(r)^{1/3} G(2r)^{2/3} \tag{9.89}$$

comes from (9.84) and using Hölder's inequality. Using (9.80), W_1 can be bounded as

$$W_1 \leq C\varepsilon_1 r_n^2 r_n^3 = C\varepsilon_1 r_n^5. \tag{9.90}$$

The second term arises from (9.85) and also using Hölder's inequality (note that $13/3 = 3 + 2(2/3)$),

$$W_2 = Cr^{13/3}\Big(\int_{Q_r} |u|^3 dxdt\Big)^{1/3} \sup_{s-r^2<t<s} \int_{2r<|y-x_0|<1/4} \frac{|u(t,y)|^2}{|y-x_0|^3} dy. \tag{9.91}$$

Note that for $r = r_n = 2^{-n}$, the last factor in (9.91) can be estimated with the help of the induction hypothesis (9.81) as

$$\begin{aligned}
&\int_{2r_n<|y-x_0|<1/4} \frac{|u(t,y)|^2}{|y-x_0|^3} dy \leq \sum_{k=3}^{n-1} \int_{2^{-k}<|y-x_0|<2^{-(k-1)}} \frac{|u(t,y)|^2}{|y-x_0|^3} dy \\
&\leq \sum_{k=3}^{n-1} 2^{3k} \int_{2^{-k}<|y-x_0|<2^{-(k-1)}} |u(t,y)|^2 dy \leq \sum_{k=3}^{n-1} r_k^{-3} A(r_{k-1}) \leq C\varepsilon_1^{2/3} \sum_{k=3}^{n-1} r_k^{-1} \leq \frac{C\varepsilon_1^{2/3}}{r_n}.
\end{aligned}$$

Using this inequality, together with (9.80) in (9.91) gives

$$W_2 \leq Cr_n^{13/3}\big(r_n^2 G(r_n)\big)^{1/3} \frac{\varepsilon_1^{2/3}}{r_n} \leq Cr_n^4 G(r_n)^{1/3} \varepsilon_1^{2/3} \leq Cr_n^5 \varepsilon_1. \tag{9.92}$$

The third term

$$W_3 = Cr^3\Big(\int_{Q_r} |u|^3 dxdt\Big)^{1/3}\Big(\int_{Q_{1/4}} |u|^3 dxdt\Big)^{2/3} \tag{9.93}$$

comes from (9.86) and, of course, using Hölder's inequality once again, and can be bounded with the help of (9.80) as

$$W_3 \leq Cr_n^3\big(r_n^2 G(r_n)\big)\big)^{1/3} G(1/4)^{2/3} \leq Cr_n^{14/3} \varepsilon_1. \tag{9.94}$$

Finally, the last term in (9.88) comes from (9.87):

$$W_4 = Cr^3 A(r)^{1/5}\Big(\int_{Q_r} |u|^3 dxdt\Big)^{1/5}\Big(\int_{-1/16}^{0}\Big(\int_{B_{1/4}} |p| dx\Big)^{5/4} dt\Big)^{4/5}. \tag{9.95}$$

It can be bounded as (assuming that $\varepsilon_1 \leq 1$):

$$W_4 \leq Cr_n^3 A(r_n)^{1/5} (r_n^2 G(r_n))^{1/5} \varepsilon_1^{4/5} \leq Cr_n^3 (r_n^2 \varepsilon_1^{2/3})^{1/5} (r_n^5 \varepsilon_1)^{1/5} \varepsilon_1^{4/5} \leq Cr_n^{22/5} \varepsilon_1. \tag{9.96}$$

Altogether, we conclude that

$$\int_{Q_n} |u||p - \bar{p}_{r_n}| dxdt \leq Cr_n^{22/5} \varepsilon_1. \tag{9.97}$$

We conclude that

$$\frac{r_n^{3/5}}{|Q_n|} \int_{Q_n} |u||p - \bar{p}_{r_n}| dxdt \leq C\varepsilon_1 \leq \frac{\varepsilon_1^{2/3}}{2}, \tag{9.98}$$

provided that ε_1 is small enough. This bounds the second term in (9.70) and finishes the proof of Lemma 9.17.

Proof of Lemma 9.18

We now assume that

$$\frac{1}{|Q_k|}\int_{Q_k}|u|^3dxdt+\frac{r_k^{3/5}}{|Q_k|}\int_{Q_k}|u||p-\bar{p}_n|dxdt\le\varepsilon_1^{2/3},\tag{9.99}$$

for all $3\le k\le n$, and show that then

$$\sup_{s-r_n^2<t\le s}\fint_{|x-x_0|\le r_n}|u(t,x)|^2dx+\frac{1}{r_n^3}\int_{Q_n}|\nabla u(t,x)|^2dxdt\le C_0\varepsilon_1^{2/3}.\tag{9.100}$$

We will shift the origin so that $(s,x_0)=(0,0)$, to simplify the notation. The idea is to use the generalized energy inequality

$$\begin{aligned}&\int_{B_1}|u(s,x)|^2\phi(s,x)dx+2\int_{-1}^s\int_{B_1}|\nabla u(t,x)|^2\phi(t,x)dxdt\le\int_{-1}^s\int_{B_1}|u(t,x)|^2(\phi_t+\Delta\phi)dxdt\\&+\int_{-1}^s\int_{B_1}(|u|^2+2p)u\cdot\nabla\phi(t,x)dtdx+2\int_{-1}^s\int_{B_1}(f\cdot u)\phi(t,x)dxdt,\end{aligned}\tag{9.101}$$

with a suitable test function ϕ_n. We will set

$$\phi_n(t,x)=\chi(x)\psi_n(t,x),$$

with the backward heat kernel

$$\psi_n(t,x)=\frac{1}{(r_n^2-t)^{3/2}}\exp\Big\{-\frac{|x|^2}{4(r_n^2-t)}\Big\},$$

and a smooth function $\chi(x)\ge0$ so that $\chi(x)\equiv1$ on $Q_2=Q_{1/4}(0,0)$ and $\chi=0$ outside of $Q_{1/3}(0,0)$. Then we have

$$\frac{\partial\phi_n}{\partial t}+\Delta\phi_n=0,\quad\text{on }Q_2,$$

and

$$\Big|\frac{\partial\phi_n}{\partial t}+\Delta\phi_n\Big|\le C,\quad\text{everywhere},$$

and the following bounds hold:

$$\frac{1}{Cr_n^3}\le\phi_n\le\frac{C}{r_n^3},\quad|\nabla\phi_n|\le\frac{C}{r_n^4},\quad\text{on }Q_n,\ n\ge2\tag{9.102}$$

and

$$\frac{1}{Cr_k^3}\le\phi_n\le\frac{C}{r_k^3},\quad|\nabla\phi_n|\le\frac{C}{r_k^4},\quad\text{on }Q_{k-1}\setminus Q_k,\ n\ge2.\tag{9.103}$$

We may now insert this ϕ_n into (9.101), and use the lower bound for ϕ_n on Q_n to get

$$\begin{aligned}&\sup_{-r_n^2\le t\le0}\frac{1}{r_n^3}\int_{|x|<r_n}|u(t,x)|^2dx+\frac{1}{r_n^3}\int_{Q_n}|\nabla u|^2dxdt\le C\int_{Q_1}|u|^2|\frac{\partial\phi_n}{\partial t}+\Delta\phi_n|dxdt\\&+C\int_{Q_1}|u|^3|\nabla\phi_n|dtdx+C\Big|\int_{Q_1}p(u\cdot\nabla\phi_n)dtdx\Big|+C\int_{Q_1}^s|f||u||\phi|dxdt\\&=C(I_1+I_2+I_3+I_4).\end{aligned}\tag{9.104}$$

To estimate I_1 we simply use Hölder's inequality:

$$|I_1| \le C\int_{Q_1} |u|^2 dxdt \le C\Big(\int_{Q_1} |u|^3 dxdt\Big)^{2/3} \le C\varepsilon_1^{2/3}. \tag{9.105}$$

The second term is estimated as

$$|I_2| \le C\sum_{k=1}^{n} \frac{1}{r_k^4}\int_{Q_k} |u|^3 dxdt \le C\sum_{k=1}^{n} \frac{1}{r_k^4}\varepsilon_1^{2/3} r_k^5 \le C\varepsilon_1^{2/3}. \tag{9.106}$$

The last term in (9.104) is also easy:

$$\begin{aligned}|I_4| &\le C\sum_{k=1}^{n} \frac{1}{r_k^3}\int_{Q_k} |u||f| dxdt \le C\sum_{k=1}^{n} \frac{1}{r_k^3}\Big(\int_{Q_k} |u|^3\Big)^{1/3}\Big(\int_{Q_k} |f|^{3/2}\Big)^{2/3} \\ &\le C\sum_{k=1}^{n} \frac{1}{r_k^3}(\varepsilon_1^{2/3} r_k^5)^{1/3}\|f\|_{L^q(Q_1)} r_k^{10/3-5/q} \le C\varepsilon_2^{1/q}\varepsilon_1^{2/9}\sum_{k=1}^{n} r_k^{2-5/q} \le C\varepsilon_2^{1/q}\varepsilon_1^{2/9},\end{aligned} \tag{9.107}$$

as $q > 5/2$. Therefore, if ε_2 is sufficiently small, we have

$$|I_4| \le C\varepsilon_1^{2/3}. \tag{9.108}$$

Finally, we deal with I_3. Here, we will use the condition that u is a divergence-free flow. Let us take smooth functions $0 \le \chi_k \le 1$ such that $\chi_k \equiv 1$ on $Q_{7r_k/8}$, and $\chi_k \equiv 0$ outside of Q_{r_k}, and

$$|\nabla \chi_k| \le \frac{C}{r_k}.$$

Then, as $\chi_1\phi_n = \phi_n$, we can write I_3 as a telescoping sum:

$$I_3 = \int_{Q_1} p(u\cdot\nabla\phi_n) dtdx = \sum_{k=1}^{n-1}\int_{Q_1} pu\cdot\nabla((\chi_k - \chi_{k+1})\phi_n) + \int_{Q_1} pu\cdot(\chi_n\phi_n). \tag{9.109}$$

Since u is divergence-free, and $\chi_k - \chi_{+1}$ vanishes outside of Q_k, we can write for $k \ge 3$:

$$\int_{Q_1} pu\cdot\nabla((\chi_k - \chi_{k+1})\phi_n) = \int_{Q_k} pu\cdot\nabla((\chi_k - \chi_{k+1})\phi_n) = \int_{Q_k} (p - \bar{p}_k)u\cdot\nabla((\chi_k - \chi_{k+1})\phi_n).$$

For $k = 1, 2$ we simply have

$$\Big|\int_{Q_1} pu\cdot\nabla((\chi_k - \chi_{k+1})\phi_n)\Big| \le c\int_{Q_1} |p||u| \le C\varepsilon_1^{2/3},$$

while for the last term in (9.109) we have

$$\int_{Q_1} pu\cdot(\chi_n\phi_n) = \int_{Q_n} (p - \bar{p}_n)u\cdot\nabla(\chi_n\phi_n).$$

Putting these together, we have

$$I_3 \le C\varepsilon_1^{2/3} + C\sum_{k=3}^{n} \frac{1}{r_k^4}\int_{Q_k} |p - \bar{p}_k||u| \le C\varepsilon_1^{2/3} + C\sum_{k=3}^{n} \frac{1}{r_k^4}\varepsilon_1^{2/3} r_k^{5-3/5} \le C\varepsilon_1^{2/3}. \tag{9.110}$$

This finishes the proof of Lemma 9.18, and thus that of Proposition 9.4.

10 The weak solutions of the Euler equations

The goal of this section is to give a naive and simplistic glimpse of the recent results on the Onsager conjecture and the weak solutions of the Euler and Navier-Stokes equations that do not preserve the energy. This material is mostly based on a recent review by V. Vicol and T. Buckmaster, but also includes some material from the pioneering papers by C. De Lellis and L. Székelyhidi.

10.1 The statistical description of turbulence

The starting point of our discussion are the Navier-Stokes equations

$$\begin{aligned} &u^\nu_t + u^\nu \cdot \nabla u^\nu + \nabla p = \nu \Delta u^\nu + f, \\ &\nabla \cdot u^\nu = 0, \end{aligned} \tag{10.1}$$

with a small viscosity $\nu > 0$. Our favorite fundamental energy balance says that, as long as the solution $u^\nu(t,x)$ remains smooth, we have

$$\frac{1}{2}\frac{d}{dt}\int |u^\nu(t,x)|^2 dx = -\nu \int |\nabla u^\nu(t,x)|^2 dx + \int (f \cdot u^\nu) dx. \tag{10.2}$$

On the other hand, if we consider the Euler equations rather than the Navier-Stokes equations, with the same forcing

$$\begin{aligned} &v_t + v \cdot \nabla v + \nabla p = f, \\ &\nabla \cdot v = 0, \end{aligned} \tag{10.3}$$

and assume that $v(t,x)$ is also smooth, then the corresponding energy balance is simply

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}_L} |v(t,x)|^2 dx = \int_{\mathbb{T}_L} (f \cdot v) dx. \tag{10.4}$$

Our interest will be in two issues: first, should we think of the solutions to Euler equations as the solutions to the Navier-Stokes equations in the limit of a zero viscosity, and, second, how do the weak solutions to the Euler equations behave when the forcing f is, in some sense, small. In other words, can a small force f create a large (but oscillatory) solution to the Euler equations. These issues are quite closely related.

The answer to the first question depends, essentially, on what happens to the energy dissipation term in the right side of (10.2). Naively, one may expect that this term vanishes as $\nu \to 0$, so that for $\nu > 0$ small it is also small. This, of course, assumes that u^ν remains uniformly smooth as $\nu \to 0$. As we will see, this is not the case even in much simpler linear problems. In order to be more specific, we will assume, without any rigorous justification, that u^ν satisfies the following hypotheses that reflect the physical observations. First, $u^\nu(t,x)$ is a space-time stationary random process – its law is the same for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$, and for any a collection of space-time points $(t_1,x_1),\dots,(t_N,x_N)$ and any shifts $s \in \mathbb{R}$ and $y \in \mathbb{R}^3$, the joint law of

$$u^\nu(t_1+s, x_1+y), \dots, u^\nu(t_N+s, x_N+y)$$

does not depend on the "off-sets" $s \in \mathbb{R}$ and $y \in \mathbb{R}^3$ but only on the relative times and positions $t_1, \dots, t_N$ and $x_1, \dots, x_N$. Second, we assume that the field $u^\nu(t,x)$ is statistically isotropic: for any collection of points $x_1, \dots, x_N$, any $t \in \mathbb{R}$, and any orthogonal matrix R, the joint law of

$$u^\nu(t, Rx), \dots, u^\nu(t, Rx_N)$$

is the same as that of $u^\nu(t,x), \dots, u^\nu(t,x_N)$. For the final assumption, let us define the increments

$$\delta u^\nu(t,x,z) = u^\nu(t,x+z) - u^\nu(t,x).$$

We assume self-similarity of the increments: there is a range of scales ℓ, known as the inertial range, and a constant $\mu > 0$, so that the law of $\delta u^\nu(t,x,\lambda\ell\hat{z})$ is the same as that of $\lambda^\mu \delta u^\nu(t,x,\ell\hat{z})$ for all unit vectors $\hat{z}$ with $|\hat{z}| = 1$, and $\lambda > 0$ so that both ℓ and $\lambda\ell$ are in the inertial range.

A basic hypothesis of the theory of turbulence, together with the above space-time homogeneity, isotropy and self-similarity properties, is that the average energy dissipation rate

$$\varepsilon^\nu = \langle \nu |\nabla u^\nu(t,x)|^2 \rangle \to \varepsilon > 0 \text{ as } \nu \to 0, \tag{10.5}$$

does not vanish in the limit $\nu \to 0$. Here, $\langle \cdot \rangle$ denotes the statistical averaging. This, in a sense, defines, what it means for u^ν to be turbulent. This should, naturally, in the limit $\nu \to 0$, lead to the solutions to the Euler equations for which we have an inequality in (10.4) rather than an equality:

$$\frac{1}{2}\frac{d}{dt}\int |v(t,x)|^2 dx < \int (f \cdot v) dx, \tag{10.6}$$

and which are not smooth. This brings about two fundamental questions: first, how should we expect the energy dissipation rate to behave for ν small, and, second, for what kind of rough solutions to the Euler equations should we not expect energy conservation? The former is addressed by the Kolmogorov theory of turbulence, and the latter by the Onsager conjecture, though the two are closely related.

Let us define the mean energy per unit volume carried by wave numbers smaller than κ as $\langle |\mathbb{P}_{\le\kappa} u^\nu|^2 \rangle$. Here, $\mathbb{P}_{\le\kappa}$ denotes the projection on the wave numbers smaller than κ in the Fourier space. The energy spectrum of u^ν is then defined as

$$E(\kappa) = \frac{d}{d\kappa}\langle |\mathbb{P}_{\le\kappa} u^\nu|^2 \rangle. \tag{10.7}$$

The main hypothesis of the statistical turbulence theory is that in the inertial range the energy $E(\kappa)$ depends only on the limiting average energy density ε in (10.5) and the wave number κ but not on f or the viscosity ν. The dimensions of these objects are

$$\begin{aligned}
&[E(\kappa)] = \Big[\frac{d}{d\kappa}\langle |\mathbb{P}_{\le\kappa} u^\nu|^2 \rangle\Big] = \text{length}\frac{\text{length}^2}{\text{time}^2} = \frac{\text{length}^3}{\text{time}^2},\\
&[\varepsilon] = \frac{\text{length}^2}{\text{time}}\frac{1}{\text{time}^2} = \frac{\text{length}^2}{\text{time}^3},\\
&[\kappa] = \frac{1}{\text{length}}.
\end{aligned} \tag{10.8}$$

Therefore, the dimensional analysis implies that the ratio

$$\frac{E(\kappa)}{\varepsilon^a \kappa^b} \tag{10.9}$$

is non-dimensional, and thus should be a constant, if (and only if)

$$3 = 2a - b, \quad 2 = 3a, \tag{10.10}$$

so that $a = 2/3$, $b = -5/3$. We deduce that in a turbulent flow we should have

$$E(\kappa) = C_K \varepsilon^{2/3} \kappa^{-5/3}, \tag{10.11}$$

in the inertial range, with some constant $C_K > 0$, that should be determined from the physical considerations.

The self-similarity exponent μ can also be determined from purely dimensional considerations. Let us define the p-th order absolute structure function as

$$S_p(\ell) = \langle |u^\nu(t, x + \ell\hat{z}) - u^\nu(t, x)|^p \rangle, \quad |\hat{z}| = 1, \ \ell > 0.$$

In the inertial range we should have

$$S_p(\ell) = C_{dim} \ell^{p\mu}, \tag{10.12}$$

with a dimensional constant C_{dim}. The physical hypothesis is again that $S_p(\ell)$ depends only on ε and ℓ. Note that the corresponding dimensions are

$$[S_p(\ell)] = \frac{\text{length}^p}{\text{time}^p}, \quad [\varepsilon] = \frac{\text{length}^2}{\text{time}^3}, [\ell] = \text{length}.$$

We conclude that there exists a non-dimensional constant D_p so that

$$S_p(\ell) = D_p (\varepsilon \ell)^{p/3}.$$

Comparing to (10.12) we conclude that the self–similarity exponent $\mu = 1/3$.

The inertial range extends from the macroscopic scale of the forcing down to a small scale ℓ_K that should depend only on ε and the viscosity ν. Once again, looking at the dimensions

$$[\varepsilon] = \frac{\text{length}^2}{\text{time}^3}, [\nu] = \frac{\text{length}^2}{\text{time}},$$

we conclude that the Kolmogorov dissipation length is

$$\lambda_K = \frac{c_K \nu^{3/4}}{\varepsilon^{1/4}}, \tag{10.13}$$

with a constant c_K that comes from physical considerations. The constants c_K and C_K are not independent – they can be related using the hypothesis that the energy is concentrated in the inertial scale $\lambda_K \ll \ell \ll L$, together with (10.11) and the relation between ε and $E(\kappa)$.

10.2 The easy direction of Onsager's conjecture

Let us now turn to a more mathematical analysis. As usual, we work on the torus $\mathbb{T}^3$. We say that $v(t,x)$ is a weak solution to the Euler equations

$$\begin{aligned} & v_t + v\cdot\nabla v + \nabla p = 0,\ t>0,\ x\in\mathbb{T}^3,\\ & \nabla\cdot v = 0,\\ & v(0,x) = v_0(x), \end{aligned} \tag{10.14}$$

if $v \in C[0,T;L^2(\mathbb{R}^3)]$, for any $t>0$ the vector field $v(t,\cdot)$ is divergence-free in the sense of distributions, and for any divergence-free test function $\phi(t,x)$ we have

$$\int_0^\infty\int_{\mathbb{T}^3} v(t,x)\cdot[\partial_t\phi(t,x) + v(t,x)\cdot\nabla\phi(t,x)]dxdt + \int_{\mathbb{T}^3} v_0(x)\phi(0,x)dx = 0. \tag{10.15}$$

A smooth solution to Euler's equations conserves energy:

$$\int_{\mathbb{T}^3}|v(t,x)|^2dx = \int_{\mathbb{T}^3}|v_0(x)|^2dx. \tag{10.16}$$

For the weak solutions, Onsager's conjecture, directly related to the Kolmogorov self-similarity exponent $\mu = 1/3$, says that (i) a weak solution to the Euler equations that belongs to the Hölder space $C^\alpha_{t,x}$ with $\alpha > 1/3$ conserves energy, and (ii) for any $\alpha < 1/3$ there exists a weak solution to the Euler equations in the Hölder space $C^\alpha_{t,x}$ that does not conserve energy.

The first part of this conjecture is much easier to prove. Let us assume that $v(t,x)$ is C^α in the x-variable, with $\alpha > 1/3$. Let $\phi \geq 0$ be a smooth test function in $C_c^\infty(\mathbb{R}^3)$ such that $\|\phi\|_{L^1} = 1$ and set $\phi_\ell = \ell^{-3}\phi(x/\ell)$, a standard mollifier. Given a function f we will use the notation

$$f_\ell = \phi_\ell \star f. \tag{10.17}$$

The mollified vector field $v_\ell = v\star\phi_\ell$ satisfies

$$\begin{aligned} & \partial_t v_\ell + (v_\ell\cdot\nabla v_\ell) + ([v\cdot\nabla v]_\ell - v_\ell\cdot\nabla v_\ell) + \nabla p_\ell = 0,\ t>0,\ x\in\mathbb{T}^3,\\ & \nabla\cdot v_\ell = 0. \end{aligned} \tag{10.18}$$

We can write, using the divergence-free property of v:

$$[v\cdot\nabla v]_{\ell,j} = \phi_\ell\star[v_k\partial_k v_j] = \phi_\ell\star[\partial_k(v_kv_j)] = \partial_k[\phi_\ell\star(v_kv_j)] = \partial_k[(v_kv_j)_\ell]. \tag{10.19}$$

Then, multiplying (10.18) by v_ℓ and integrating by parts gives, as v_ℓ is also divergence-free:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{T}^3}|v_\ell(t,x)|^2dx - \frac{1}{2}\int_{\mathbb{T}^3}|v_\ell(0,x)|^2dx = -\int_0^t\int_{\mathbb{T}^3}\big(v_{\ell,j}[v_k\partial_k v_j]_\ell - v_{\ell,j}[v_{\ell,k}\partial_k v_{\ell,j}\big)dxds\\ &= -\int_0^t\int_{\mathbb{T}^3}\big(v_{\ell,j}\partial_k[(v_kv_j)_\ell] - v_{\ell,j}[v_{\ell,k}\partial_k v_{\ell,j}\big)dxds = \int_0^t\int_{\mathbb{T}^3}[(v_kv_j)_\ell - v_{\ell,k}v_{\ell,j}]\partial_k v_{\ell,j}dxds. \end{aligned} \tag{10.20}$$

We have the following lemma.

Lemma 10.1. *Let $\phi \geq 0$ be in $C_c^\infty(\mathbb{R}^d)$ and such that $\|\phi\|_{L^1} = 1$, and set $\phi_\ell(x) = \ell^{-d}\phi(x/\ell)$. Then, for any $\alpha \in (0,1)$ we have*

$$\|f \star \phi_\ell\|_{C^1} \leq C\ell^{-(1-\alpha)}\|f\|_{C^\alpha}, \tag{10.21}$$

and

$$\|(fg) \star \phi_\ell - (f \star \phi_\ell)(g \star \phi_\ell)\|_{C^0} \leq C\ell^{2\alpha}\|f\|_{C^\alpha}\|g\|_{C^\alpha}, \tag{10.22}$$

with a constant C that depends on ϕ.

With this lemma in hand, and assuming that $v \in C^\alpha(\mathbb{R}^3)$, we may estimate the integral in the right side of (10.20) as

$$\begin{aligned}\Big|\int_{\mathbb{T}^3}[(v_kv_j)_\ell - v_{\ell,k}v_{\ell,j}]\partial_k v_{\ell,j}dx\Big| &\leq C\|(v_kv_j)_\ell - v_{\ell,k}v_{\ell,j}\|_{C^0}\|v_\ell\|_{C^1}\\ &\leq C\ell^{2\alpha}\|v\|_{C^\alpha}^2\ell^{-(1-\alpha)}\|v\|_{C^\alpha} = C\ell^{3\alpha-1}\|v\|_{C^\alpha}^3 \to 0, \end{aligned}\tag{10.23}$$

if $\alpha > 1/3$. Therefore, passing to the limit $\ell \to 0$ in (10.20), we obtain

$$\int_{\mathbb{T}^3}|v(t,x)|^2dx = \int_{\mathbb{T}^3}|v(0,x)|^2dx, \tag{10.24}$$

thus the energy is conserved.

Let us now prove Lemma 10.1. To prove the first bound in this lemma, we write

$$\begin{aligned}\partial_k(f\star\phi_\ell)(x) &= \lim_{h\to 0}\int\frac{\phi_\ell(x+he_k-y)-\phi_\ell(x-y)}{h}f(y)dy\\ &= \lim_{h\to 0}\int\frac{\phi_\ell(x+he_k-y)-\phi_\ell(x-y)}{h}(f(y)-f(x))dy\\ &= \lim_{h\to 0}\int\frac{\phi((x+he_k-y)/\ell)-\phi((x-y)/\ell)}{h}(f(y)-f(x))\frac{dy}{\ell^n}\\ &= \lim_{h\to 0}\int\frac{\phi(z+h\ell^{-1}e_k)-\phi(z)}{h}(f(x-\ell z)-f(x))dz,\end{aligned}$$

so that

$$\begin{aligned}|\partial_k(f\star\phi_\ell)(x)| &\leq \lim_{h\to 0}\int\frac{|\phi(z+h\ell^{-1}e_k)-\phi(z)|}{h}|f(x-\ell z)-f(x)|dy\\ &\leq \ell^{\alpha-1}\|f\|_{C^\alpha}\lim_{h\to 0}\int\frac{|\phi(z+he_k)-\phi(z)|}{h}|z|^\alpha dz \leq C\ell^{\alpha-1}\|f\|_{C^\alpha}.\end{aligned}\tag{10.25}$$

For the second bound, we note that

$$\begin{aligned}&(fg)\star\phi_\ell(x) - (f\star\phi_\ell)(x)(g\star\phi_\ell)(x) = \int f(y)g(y)\phi_\ell(x-y)\phi_\ell(x-z)dydz\\ &-\int f(y)\phi_\ell(x-y)g(z)\phi_\ell(x-z)dydz\\ &= \int\phi(y)\phi(z)[f(x-\ell y)g(x-\ell y) - f(x-\ell y)g(x-\ell z)]dydz\\ &= \int\phi(y)\phi(z)[f(x-\ell z)-f(x-\ell y)]g(x-\ell z)dydz\\ &= \int\phi(y)\phi(z)[f(x-\ell z)-f(x-\ell y)][g(x-\ell z)-g(x)]dydz,\end{aligned}\tag{10.26}$$

so that

$$
\begin{aligned}
|(fg)\star\phi_\ell(x)-(f\star\phi_\ell)(x)(g\star\phi_\ell)(x)| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10.27)\\
\le \int \phi(y)\phi(z)|f(x-\ell z)-f(x-\ell y)||g(x-\ell z)-g(x)|dydz\\
\le \ell^{2\alpha}\|f\|_{C^\alpha}\|g\|_{C^\alpha}\int \phi(y)\phi(z)|z-y|^\alpha|z|^\alpha dydz = C\ell^{2\alpha}\|f\|_{C^\alpha}\|g\|_{C^\alpha},
\end{aligned}
$$

finishing the proof of Lemma 10.1.

10.3 The wild continuous weak solutions of the Euler equations

In this section, we prove existence of a Hölder continuous solution of the Euler equations, with a sufficiently small Hödler exponent $\beta > 0$. The Euler equations written in the divergence form are

$$
\begin{aligned}
&v_t+\nabla\cdot(v\otimes v)+\nabla p=0,\ t>0,\ x\in\mathbb{T}^3,\\
&\nabla\cdot v=0,
\end{aligned}
\tag{10.28}
$$

that does not conserve the energy. Here, the torus is normalized as $\mathbb{T}^3=[0,1]^3$, with the periodic boundary conditions. We use here and below the notation

$$
[a\otimes b]_{ij}=a_ib_j \tag{10.29}
$$

for the standard tensor product of two vectors and

$$
[a\otimes_{tr} b]_{ij}=a_ib_j-\frac{1}{n}(a\cdot b)\delta_{ij}, \tag{10.30}
$$

for a traceless tensor product of a pair of vectors a and b in $\mathbb{R}^n$. Only the divergence of the traceless tensor products will appear below in various equations that also have the pressure terms, and the trace part can be always added to the gradient of the pressure.

Theorem 10.2. *There exists $\beta > 0$ and a weak solution $v\in C([0,T];C^\beta(\mathbb{T}^3))$ to the Euler equations such that*

$$
\int_{\mathbb{T}^3}|v(1,x)|^2dx \ge 2\int_{\mathbb{T}^3}|v(0,x)|^2dx. \tag{10.31}
$$

The proof proceeds by an induction. We will construct a sequence v_q, $q=0,1,2,\dots$ of solutions to the forced Euler equations

$$
\begin{aligned}
&\partial v_q+\nabla\cdot(v_q\otimes v_q)+\nabla p_q=\nabla\cdot R_q,\ t>0,\ x\in\mathbb{T}^3,\\
&\nabla\cdot v_q=0,
\end{aligned}
\tag{10.32}
$$

with a Reynolds stress R_q that goes uniformly to zero as $q\to+\infty$, and v_q converges uniformly to a weak solution to the Euler equations satisfying the "reverse" energy inequality (10.31). At each induction step, we do not design v_{q+1} directly but rather use v_q to construct the increment $w_{q+1}=v_{q+1}-v_q$, in such a way that v_{q+1} satisfies

$$
\begin{aligned}
&\partial_t v_{q+1}+\nabla\cdot(v_{q+1}\otimes v_{q+1})+\nabla p_{q+1}=\nabla\cdot R_{q+1},\ t>0,\ x\in\mathbb{T}^3,\\
&\nabla\cdot v_{q+1}=0,
\end{aligned}
\tag{10.33}
$$

with a smaller Reynolds stress R_{q+1}. Given w_{q+1}, the Reynolds stress R_{q+1} is determined as the trace-less symmetric matrix satisfying

$$\nabla \cdot R_{q+1} = E_{osc} + E_{tr} + E_{Nash}, \tag{10.34}$$

with the "error" terms in the right side depending on v_q and w_{q+1}:

$$E_{osc} = \nabla \cdot (w_{q+1} \otimes w_{q+1}) - \nabla \cdot R_q + \nabla(p_{q+1} - p_q), \tag{10.35}$$
$$E_{tr} = \partial_t w_{q+1} + v_q \cdot \nabla w_{q+1}, \tag{10.36}$$
$$E_{Nash} = w_{q+1} \cdot \nabla v_q. \tag{10.37}$$

These terms are known as the oscillation error, the transport error and the Nash error, respectively. Given the iterate v_q, the goal will be to choose w_{q+1} so that R_{q+1}, the symmetric trace-less solution to (10.34) with a given right side, is small, and, in addition, the series

$$\sum_q w_q \tag{10.38}$$

converges. In order to make sure that the reverse energy inequality (10.31) holds, we will choose the first iterate v_0 so that $v_0(0,x) \equiv 0$, and $v_0(1,x)$ does not vanish. This means that $v_0(t,x)$ satisfies (10.31) trivially. The induction construction will ensure that actually all $v_q(t,x)$ stay sufficiently close to $v_0(t,x)$ for all $0 \le t \le 1$, so that in the limit $q \to +\infty$ the inequality (10.31) will still hold.

The correction w_{q+1} will consist of two parts:

$$w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)}. \tag{10.39}$$

Here, $w_{q+1}^{(p)}$ is the principal part of the perturbation, chosen so that the low frequency terms in the trace-less product $w_{q+1} \otimes_{tr} w_{q+1}$ essentially cancel those in R_q, so that these contributions to the oscillation error cancel each other. Roughly speaking, it is of the form

$$w_{q+1}^{(p)} \sim \sum_\xi a_\xi(R_q) W_\xi. \tag{10.40}$$

Here, W_ξ are "building blocks" oscillating at a high frequency λ_{q+1}, and the coefficients $a_\xi(R_q)$ are chosen so that the aforementioned cancellation of the lower frequencies takes place. As an additional minor complication, $w_{q+1}^{(p)}$ will need to be corrected to decrease the transport error. The correction $w_{q+1}^{(c)}$ is chosen to ensure that w_{q+1} is divergence-free.

In order to see yet another way the threshold 1/3 for the Hödler regularity comes up, let us assume that the frequencies are chosen so that

$$\lambda_q = \lambda^q, \tag{10.41}$$

with some $\lambda \in \mathbb{N}$. Then, in order for the series in (10.38) to converge to a C^β function v, we should have, at least,

$$\|w_q\|_{C^0} \le \lambda_q^{-\beta}. \tag{10.42}$$

The Reynolds stress should then satisfy, roughly

$$\|R_q\|_{C^0} \leq \lambda_{q+1}^{-2\beta}, \tag{10.43}$$

because it is related quadratically to w_{q+1}. The contribution of the Nash error to the Reynolds stress R_{q+1} is one derivative smoother that E_{Nash}, and oscillates at frequency λ_{q+1}. In the uniform norm, it should be of the order

$$\begin{aligned}\|R_{q+1}^{Nash}\|_{C^0} &\leq \frac{C\|w_{q+1}\|_{C^0}\|v_q\|_{C^1}}{\lambda_{q+1}} \leq \frac{C\lambda_{q+1}^{-\beta}}{\lambda_{q+1}}\sum_{m\leq q}\lambda_m\lambda_m^{-\beta} \leq C\lambda_{q+1}^{-\beta-1}\lambda_q^{1-\beta} \\ &= C\lambda_{q+2}^{-\beta-1}\lambda^{\beta+1}\lambda_{q+2}^{1-\beta}\lambda^{2(\beta-1)} \leq C\lambda_{q+2}^{-2\beta}\lambda^{3\beta-1}. \end{aligned}\tag{10.44}$$

In other words, for the bound (10.43) to be "iteratable" we need to have $\beta < 1/3$, another indication for why Onsager's conjecture holds. In reality, we will take the frequencies growing much faster than in (10.41), and we will also take β to be very small.

10.4 The iterative estimate

We now turn to an implementation of the above scheme. We will take the frequencies

$$\lambda_q = a^{2^q}, \tag{10.45}$$

with $a \in \mathbb{N}$ sufficiently large, to be specified later, so that

$$\lambda_{q+1} = \lambda_q^2. \tag{10.46}$$

We also set

$$\delta_q = \lambda_q^{-2\beta}, \tag{10.47}$$

with $\beta > 0$ sufficiently small, also to be specified later. We will assume the following inductive bounds on v_q and R_q:

$$\|v_q\|_{C^0} \leq 1 - \delta_q^{1/2}, \tag{10.48}$$

$$\|v_q\|_{C^1_{t,x}} \leq C_R\delta_q^{1/2}\lambda_q, \tag{10.49}$$

$$\|R_q\|_{C^0} \leq c_R\delta_{q+1}, \tag{10.50}$$

with a pair of universal constants C_R and c_R, to be specified below. Let us explain the choices here. As we have mentioned above, the basic premise is that the increment $w_{q+1} = v_{q+1} - v_q$ is of the size $\delta_{q+1}^{1/2}$ in the uniform norm – see (10.52) below, and oscillates at frequency λ_{q+1}. Then the Reynolds stress R_q should be of the size δ_{q+1} in the uniform norm, simply because it is quadratic in w_{q+1}, which gives the induction assumption (10.50). The uniform bound (10.48) is a convenient induction assumption since

$$v_{q+1} = v_q + w_{q+1},$$

so that if (10.48) holds at level q, and we have (10.52) below, then

$$\|v_{q+1}\| \leq 1 - \delta_q^{1/2} + \delta_{q+1}^{1/2} \leq 1 - \delta_{q+1}^{1/2}.$$

Finally, assumption (10.49) on the C^1-norm of v_q comes about because the frequencies λ_q grows sufficiently fast, so that even though $\|w_{q+1}\|_{C^0} \ll \|w_q\|_{C^0}$, we still have $\|w_{q+1}\|_{C^1} \gg \|w_q\|_{C^1}$, so that the main contribution to $\|v_q\|_{C^1}$ comes from $\|w_q\|_{C^1}$, which is of the size

$$\|w_q\|_{C^1} \sim \lambda_{q+1}\|w_q\|_{C^0} = \lambda_{q+1}\delta^{1/2}.$$

Note that $\delta_q^{1/2}\lambda_q \to +\infty$ as $q \to +\infty$ in (10.49), since $\beta > 0$ is small – because of the easy part of Onsager's conjecture, we do not expect v_q to converge in a Hölder space $C^\alpha_{t,x}$ with $\alpha > 1/3$, let alone in $C^1_{t,x}$. The induction step is described in the following.

Proposition 10.3. *There exists $\beta > 0$ sufficiently small and a_0 sufficiently large, so that for any $a \geq a_0$ there exist v_q and R_q, $q \geq 0$, that satisfy*

$$\begin{aligned}
&\partial v_q + \nabla\cdot(v_q \otimes v_q) + \nabla p_q = \nabla \cdot R_q,\ \ t > 0,\ \ x \in \mathbb{T}^3,\\
&\nabla \cdot v_q = 0,
\end{aligned} \tag{10.51}$$

and obey (10.48)-(10.50), and such that

$$\|v_{q+1} - v_q\|_{C^0} \leq \delta_{q+1}^{1/2}. \tag{10.52}$$

Let us explain how Proposition 10.3 implies the conclusion of Theorem 10.2. We take the first iterate to be an oscillatory shear flow

$$v_0(t,x) = \frac{t}{2}(\sin(\lambda_0^{1/2}x_3), 0, 0). \tag{10.53}$$

Then we have

$$\|v_0(t,\cdot)\|_{C^0} \leq \frac{1}{2} \leq 1 - \delta_0^{1/2}, \tag{10.54}$$

so that (10.48) is satisfied, for a large enough. We also have

$$\|v_0(t,\cdot)\|_{C^1_{t,x}} \leq \lambda_0^{1/2} \leq \lambda_0\delta_0^{1/2}, \tag{10.55}$$

as long as $\delta_0^{-1} \leq \lambda_0$, which is true as long as $\beta < 1/2$, and a is sufficiently large. Hence, (10.49) also holds for $q = 0$. To find R_0 we note that, as v_0 is a shear flow, we have $v_0 \cdot \nabla v_0 = 0$, hence

$$R_0 = \frac{1}{2\lambda_0^{1/2}}\begin{pmatrix} 0 & 0 & -\cos(\lambda_0^{1/2}x_3)\\ 0 & 0 & 0\\ -\cos(\lambda_0^{1/2}x_3) & 0 & 0\end{pmatrix}, \tag{10.56}$$

so that

$$\nabla\cdot R_0 = \frac{1}{2}(\sin(\lambda_0^{1/2}x_3), 0, 0) = \frac{\partial v_0}{\partial t}.$$

It follows that

$$\|R_0\|_{C^0} = \frac{1}{2\lambda_0^{1/2}} \leq a^{-1/2} \leq c_R\delta_1 = c_R a^{-4\beta}, \tag{10.57}$$

provided that $\beta < 1/8$ and a is sufficiently large. Hence, condition (10.50) also holds at $q = 0$. A key consequence of (10.57) is that v_0 is a solution of the forced Euler equations with a

Reynolds stress that is already very small in the uniform norm, provided that we take a sufficiently large. In addition, the L^2-norm of v_0 vanishes at $t = 0$ but is not zero at $t = 1$. We will now construct a rough weak solution to the unforced Euler equations that will be close to $v_0(t, x)$ in the uniform norm for all $0 \le t \le 1$, and this will force it to violate the energy inequality.

We start the iteration as in Proposition 10.3, with the initialization (v_0, R_0), and obtain a sequence (v_q, R_q). Let us take $\beta > 0$ as in the definition (10.47) of δ_q. Then, for any $\alpha < \beta$ the bounds (10.48), (10.49) and (10.52), together with an interpolation inequality between the Hölder norms, and the fact that the sequence $\delta_q^{1/2}\lambda_q$ in the right side of (10.49) is monotonically increasing, imply that

$$\|v_{q+1}-v_q\|_{C^\alpha_x} \le c_\alpha \|v_{q+1}-v_q\|^{1-\alpha}_{C^0_x}\|v_{q+1}-v_q\|^{\alpha}_{C^1_x} \le c_\alpha \delta_{q+1}^{(1-\alpha)/2}\delta_{q+1}^{\alpha/2}\lambda_{q+1}^{\alpha} = c_\alpha \delta_{q+1}^{1/2}\lambda_{q+1}^{\alpha} = c_\alpha \lambda_{q+1}^{-(\beta-\alpha)}. \tag{10.58}$$

Thus, the limit

$$v = \lim_{q\to+\infty} v_q$$

exists in $C([0,1], C^\alpha(\mathbb{T}^3))$ for any $\alpha < \beta$. Furthermore, (10.50) implies that

$$R_q \to 0 \text{ in } C^0([0,1]\times\mathbb{T}^3).$$

It follows that $v(t, x)$ is a weak solution to the Euler equations that lies in $C([0, 1], C^\alpha(\mathbb{T}^3))$ for any $\alpha < \beta$.

To finish the proof of Theorem 10.2, it remains to show that the reverse energy inequality

$$\|v(1,\cdot)\|_{L^2} \ge 2\|v(0,\cdot)\|_{L^2} \tag{10.59}$$

holds. The point is that, if a is sufficiently large, then, on one hand, $v(t, x)$ is close in the uniform norm to $v_0(t, x)$ for all $0 \le t \le 1$, and on the other $v_0(0, x) = 0$ while $v_0(1, x)$ has a fixed non-zero L^2-norm that is independent of λ_0. Indeed, we have using (10.52):

$$\begin{aligned}\|v-v_0\|_{C^0} &\le \sum_{q=0}^{\infty}\|v_{q+1}-v_q\|_{C^0} \le \sum_{q=0}^{\infty}\delta_{q+1}^{1/2} = \sum_{q=0}^{\infty}\lambda_q^{-\beta} = \sum_{q=0}^{\infty}a^{-\beta\cdot 2^q}\\ &\le \sum_{q=0}^{\infty}a^{-\beta(q+1)} \le \frac{1}{10000},\end{aligned} \tag{10.60}$$

if a is sufficiently large, so that v and v_0 are close. It follows that

$$2\|v(0,\cdot)\|_{L^2} \le 2\|v_0(0,\cdot)\|_{L^2} + \frac{1}{100} = \frac{1}{100} \le \|v_0(1,\cdot)\|_{L^2} - \|v_0(1,\cdot)-v(1,\cdot)\|_{L^2} \le \|v(1,\cdot)\|_{L^2},$$

finishing the proof of Theorem 10.2.

10.5 Proof of Proposition 10.3

We now prove Proposition 10.3. We only need to prove the inductive step as we have already constructed the pair (v_0, R_0). It will be more convenient to work with the mollified versions of v_q and R_q defined as

$$\begin{aligned} v_\ell &= (v_q \star_x \phi_\ell)\star_t \varphi_\ell,\\ R_\ell &= (R_q \star_x \phi_\ell)\star_t \varphi_\ell. \end{aligned} \tag{10.61}$$

We dropped the subscript q above in v_ℓ and R_ℓ to simplify the notation. Here, $\phi_\ell(x)$ and $\varphi_\ell(t)$ are standard scalar-valued mollifiers of compact support in x and t, respectively. As we are not aiming to prove an optimal result, we choose the mollification scales in x and t to be the same, and take ℓ as an intermediate scale between λ_q^{-1} and λ_{q+1}^{-1}:

$$\ell = \lambda_q^{-3/2}, \tag{10.62}$$

so that

$$\lambda_{q+1}^{-1} = \lambda_q^{-2} \leq \ell \leq \lambda_q^{-1}. \tag{10.63}$$

Note that, by the induction hypothesis (10.48), we have

$$\|v_\ell\|_{C^0} \leq \|v_q\|_{C^0} \leq 1 - \delta_q^{1/2}, \tag{10.64}$$

and for any $N \geq 1$ we have, because of the way ℓ was chosen and the second induction hypothesis (10.49):

$$\|v_\ell\|_{C^N} \leq C\ell^{-N+1}\|v_q\|_{C^1} \leq C\ell^{-N+1}\lambda_q\delta_q^{1/2} \leq C\ell^{-N}, \tag{10.65}$$

while

$$\|v_q - v_\ell\|_{C^0} \leq \ell\|v_q\|_{C^1} \leq C\ell\lambda_q\delta_q^{1/2} \leq C\lambda_q^{-1/2}\delta_q^{1/2} \ll \delta_{q+1}^{1/2}, \tag{10.66}$$

as long as $\beta > 0$ is sufficiently small.

As in (10.18), we obtain

$$\begin{aligned} &\partial_t v_\ell + \nabla \cdot [v_\ell \otimes v_\ell] + \nabla p_\ell = \nabla \cdot (R_\ell + R_{comm}), \\ &\nabla \cdot v_\ell = 0, \end{aligned} \tag{10.67}$$

with

$$R_{comm} = v_\ell \otimes_{tr} v_\ell - [(v \otimes_{tr} v) \star_x \phi_\ell] \star_t \varphi_\ell. \tag{10.68}$$

Recall that the traceless tensor product $\otimes_{tr}$ is defined in (10.30). In (10.67), with a slight abuse of notation, the pressure p_ℓ includes both the convolution of p_q with the mollifiers and what should have been the trace part of R_{comm}. Note that, as in (10.27), we have, using (10.49) and (10.30):

$$\|R_{comm}\|_{C^0_{t,x}} \leq C\ell\|v\|_{C^1_{t,x}}\|v\|_{C^0_{t,x}} \leq C\ell\delta_q^{1/2}\lambda_q = C\lambda_q^{-3/2}\lambda_q^{-\beta}\lambda_q = \lambda_q^{-\beta-1/2} \ll \delta_{q+2}, \tag{10.69}$$

provided that β is sufficiently small and a is sufficiently large.

10.5.1 The Reynolds stress equation

Let us first address the equation for the Reynolds stress:

$$\nabla \cdot R = E(x), \quad x \in \mathbb{T}^3, \tag{10.70}$$

with the condition that $R(x)$ is a symmetric trace-free matrix. Here, E is a mean-zero vector-field on $\mathbb{T}^3$:

$$\int_{\mathbb{T}^3} E(x)dx = 0. \tag{10.71}$$

We claim that a trace-less symmetric solution to (10.70) is given by

$$R_{km}(x) = (\partial_k\Delta^{-1}E_m + \partial_m\Delta^{-1}E_k) - \frac{1}{2}(\delta_{km} + \partial_k\partial_m\Delta^{-1})\Delta^{-1}(\nabla\cdot E). \tag{10.72}$$

The symmetry and mean-zero properties of R are obvious from (10.72). Its trace vanishes because

$$\mathrm{Tr}R = 2\Delta^{-1}(\nabla\cdot E) - \frac{n+1}{2}\Delta^{-1}(\nabla\cdot E) = 0$$

in dimension $n = 3$. To check (10.70) we write

$$\begin{aligned}(\nabla\cdot R)_m &= \partial_k R_{km} = \partial_k(\partial_k\Delta^{-1}E_m + \partial_m\Delta^{-1}E_k) - \frac{1}{2}\partial_k(\delta_{km} + \partial_k\partial_m\Delta^{-1})\Delta^{-1}(\nabla\cdot E)\\ &= E_m + \partial_m\Delta^{-1}(\nabla\cdot E) - \frac{1}{2}\partial_m\Delta^{-1}(\nabla\cdot E) - \frac{1}{2}\partial_m\Delta^{-1}(\nabla\cdot E) = E_m. \end{aligned}\tag{10.73}$$

The next lemma says that R is similar to $(-\Delta)^{-1/2}E$ when E is oscillatory.

Lemma 10.4. *Assume that $a(x) \in C^{m,\alpha}(\mathbb{T}^3)$ and $\Phi \in C^{m,\alpha}(\mathbb{T}^3)$ be smooth $\mathbb{R}^3$-valued functions, let C be such that*

$$C^{-1} \le |\nabla\Phi(x)| \le C \text{ for all } x \in \mathbb{T}^3.$$

Let $\omega \in \mathbb{Z}^3$, $\alpha \in (0,1)$ and $m \ge 1$, and $R(x)$ be the solution to (10.70) with

$$E(x) = a(x)e^{i\omega\cdot\Phi(x)} - \int_{\mathbb{T}^3} a(y)e^{i\omega\cdot\Phi(y)}dy, \tag{10.74}$$

given by (10.72). There exists a constant K that depends on C, α and m but not on $|\omega|$ such that

$$\|R\|_{C^\alpha} \le C\Big(\frac{\|a\|_{C^0}}{|\omega|^{1-\alpha}} + \frac{1}{|\omega|^{m-\alpha}}\big(\|a\|_{C^{m,\alpha}} + \|a\|_{C^0}\|\nabla\Phi\|_{C^{m,\alpha}}\big)\Big). \tag{10.75}$$

Proof. To be filled in.

The reason we allow a phase factor $\Phi(t,x)$ in Lemma 10.4 is that we will need to modify the phase to decrease the transport error, as discussed in Section 10.5.3 below. Our strategy will be to construct w_{q+1} so that R_q satisfies (10.70) with a right side that is as in Lemma 10.4: mean-zero and oscillatory, "essentially" at a single, sufficiently high frequency: in particular, the terms

$$\frac{\|a\|_{C^0}}{|\omega|^{1-\alpha}},\ \frac{\|a\|_{C^{m,\alpha}}}{|\omega|^{m-\alpha}}$$

in the right side of (10.75) should be small.

10.5.2 The Beltrami flows

The building blocks we will use to construct the principal part of the perturbation w_{q+1} as in (10.40):

$$w^{(p)}_{q+1} \sim \sum_\xi a_\xi(R_q)W_\xi \tag{10.76}$$

are the Beltrami waves W_ξ. They are defined as follows. Recall that the set $\mathbb{Q}^3 \cap \mathbb{S}^2$ of rational points is dense on the unit sphere $\mathbb{S}^2$. To see that, consider the inverse map of the stereographic projection $s(x,y) : \mathbb{R}^2 \to \mathbb{S}^2$

$$s(x,y) = \Big(\frac{2y}{x^2+y^2+1}, \frac{2x}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\Big).$$

It is clear that s maps $\mathbb{Q}^2$ to $\mathbb{Q}^3 \cap \mathbb{S}^2$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ and s is a diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{S}^2 \setminus (0,0,1)$, the density of $\mathbb{Q}^3 \cap \mathbb{S}^2$ in $\mathbb{S}^2$ follows.

Next, given $\xi \in \mathbb{Q}^3 \cap \mathbb{S}^2$, we take $A_\xi \in \mathbb{Q}^3 \cap \mathbb{S}^2$ so that

$$A_\xi \cdot \xi = 0, \quad A_{-\xi} = A_\xi. \tag{10.77}$$

The choice of A_ξ is not unique: for instance, we can take $A_\xi = (-\xi_2, \xi_1, 0)$ for $\xi = (\xi_1, \xi_2, \xi_3)$ with $\xi_1 \geq 0$ and extend it to ξ with $\xi_1 < 0$ using the even symmetry in (10.77). We also define the complex vector

$$B_\xi = \frac{1}{\sqrt{2}}(A_\xi + i\xi \times A_\xi). \tag{10.78}$$

By construction, the vector B_ξ satisfies

$$|B_\xi| = 1, \quad B_\xi \cdot \xi = 0, \quad i\xi \times B_\xi = B_\xi, \quad B_{-\xi} = \overline{B}_\xi, \tag{10.79}$$

with $\cdot$ denoting the standard real inner product, without the complex conjugation, and the bar denoting the complex conjugation. The third identity above relies on the formula

$$\begin{aligned}[]
[\xi \times (\xi \times A_\xi)]_k &= \varepsilon_{kmj}\xi_m\varepsilon_{jrs}\xi_r(A_\xi)_s = [\delta_{rk}\delta_{ms} - \delta_{ks}\delta_{mr}]\xi_m\xi_r(A_\xi)_s \\
&= \xi_k(\xi \cdot A_\xi) - |\xi|^2(A_\xi)_k = -(A_\xi)_k.
\end{aligned}$$

It follows that for any $\lambda \in \mathbb{Z}$ such that $\lambda\xi \in \mathbb{Z}^3$, the function

$$W_{\xi,\lambda}(x) = B_\xi e^{2\pi i\lambda\xi\cdot x} \tag{10.80}$$

satisfies

$$[\nabla \times W_{\xi,\lambda}]_j = \varepsilon_{jkm} 2\pi i\lambda\xi_k B_{\xi,m} e^{2\pi i\lambda\xi\cdot x} = 2\pi\lambda B_{\xi,j} e^{2\pi i\lambda\xi\cdot x}, \tag{10.81}$$

and is therefore a periodic eigenfunction of the curl operator corresponding to the eigenvalue $2\pi\lambda$:

$$\nabla \times W_{\xi,\lambda} = 2\pi\lambda W_{\xi,\lambda}. \tag{10.82}$$

We can now fix $\lambda \in \mathbb{Z}$ and take any finite set $\Gamma \subset \mathbb{Q}^3 \cap \mathbb{S}^2$ such that $-\Gamma = \Gamma$ and $\lambda\xi \in \mathbb{Z}^3$ for any $\xi \in \Gamma$. Then for any collection of coefficients $a_\xi \in \mathbb{C}$ such that $a_{-\xi} = \bar{a}_\xi$, the vector field

$$W(x) = \sum_{\xi\in\Gamma} a_\xi B_\xi e^{2\pi i\lambda\xi\cdot x} \tag{10.83}$$

is a real-valued divergence free vector field on $\mathbb{T}^3$ such that

$$\nabla \times W(x) = 2\pi\lambda W(x). \tag{10.84}$$

Note that for any vector W we have

$$(W \times [\nabla \times W])_i = \varepsilon_{ijk} W_j \varepsilon_{kmn} \partial_m W_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) W_j \partial_m W_n = W_j \partial_i W_j - W_j \partial_j W_i,$$

which gives the vector identity

$$W \times [\nabla \times W] = \nabla\Big(\frac{|W|^2}{2}\Big) - W \cdot \nabla W. \tag{10.85}$$

It follows from (10.84) and (10.85) that

$$W \cdot \nabla W = \nabla\Big(\frac{|W|^2}{2}\Big). \tag{10.86}$$

In other words, any W of the form constructed above is a solution of the Euler equations, with zero pressure.

Observe also that given any $\xi \in \mathbb{Q}^3 \cap \mathbb{S}^2$, the vectors ξ, $A(\xi)$ and $\xi \times A(\xi)$ form an orthonormal basis, so that

$$\xi \otimes \xi + A_\xi \otimes A_\xi + (\xi \times A_\xi) \otimes (\xi \times A_\xi) = \mathrm{Id}, \tag{10.87}$$

which implies

$$\begin{aligned} B_\xi \otimes B_{-\xi} &= \frac{1}{2}(A_\xi + i\xi \times A_\xi) \otimes (A_\xi - i\xi \times A_\xi) \qquad (10.88)\\ &= \frac{1}{2}(A_\xi \otimes A_\xi + (\xi \times A_\xi) \otimes (\xi \times A_\xi)) + \frac{i}{2}[(\xi \times A_\xi) \otimes A_\xi - A_\xi \otimes (\xi \times A_\xi)]\\ &= \frac{1}{2}(\mathrm{Id} - \xi \otimes \xi) + \frac{i}{2}[(\xi \times A_\xi) \otimes A_\xi - A_\xi \otimes (\xi \times A_\xi)]. \end{aligned}$$

It follows that for W of the form (10.83) we have

$$\begin{aligned} \int_{\mathbb{T}^3} (W \otimes W) dx &= \sum_{\xi,\xi' \in \Gamma} \int_{\mathbb{T}^3} a_\xi a_{\xi'} e^{2\pi i \mu (\xi + \xi')\cdot x} (B_\xi \otimes B_{\xi'}) dx = \sum_{\xi \in \Gamma} a_\xi a_{-\xi} (B_\xi \otimes B_{-\xi})\\ &= \frac{1}{2} \sum_{\xi \in \Gamma} |a_\xi|^2 (\mathrm{Id} - \xi \otimes \xi), \qquad (10.89) \end{aligned}$$

because

$$\sum_{\xi \in \Gamma} |a_\xi|^2 ((\xi \times A_\xi) \otimes A_\xi - A_\xi \otimes (\xi \times A_\xi) = 0, \tag{10.90}$$

as the individual terms inside the sum are odd in ξ and the set Γ is symmetric: $-\Gamma = \Gamma$.

We will use the Beltrami flows as building blocks in the decomposition (10.40) for the principal part of the perturbation w_{q+1}:

$$w_{q+1}^{(p)} \sim \sum_\xi a_\xi(R_q) W_{\xi,\lambda_{q+1}}. \tag{10.91}$$

The goal will be to cancel out the average of the R_q term in the oscillation error (10.35)

$$E_{osc} = \nabla \cdot (w_{q+1} \otimes w_{q+1}) - \nabla \cdot R_q + \nabla(p_{q+1} - p_q), \tag{10.92}$$

so that E_{osc} has the form (10.74) in Lemma 10.4. To this end, we need to know that the family of the Beltrami flows is rich enough so that the cancellation is achievable for a large class of given matrices R_q. Keeping in mind expression (10.89), we will now prove the following. We denote by $B_r(\mathrm{Id})$ the closed ball of 3×3 symmetric matrices centered at Id, of radius r.

Lemma 10.5. *There exist two disjoint finite subsets $\Lambda_0, \Lambda_1 \subset \mathbb{Q}^3 \cap \mathbb{S}^2$ such that if $\xi \in \Lambda_j$ then $-\xi \in \Lambda_j$, and $r_0 > 0$, so that for each matrix $M \in B_{r_0}(Id)$ and $j = 0, 1$, we have a decomposition*

$$M = \frac{1}{2} \sum_{\xi \in \Lambda_j} (\gamma_\xi^{(j)}(M))^2 (Id - \xi \otimes \xi). \tag{10.93}$$

Moreover, for each $\xi \in \Lambda_j$ and $j = 0, 1$, the coefficients $\gamma_\xi^{(j)}(R)$ are C^∞-functions on $B_{r_0}(Id)$.

Proof. To be filled in.

10.5.3 The principal part of the perturbation

We would like to take the principal part of the perturbation as a sum of the Beltrami waves. At the same time, we need to make sure that we have a small transport error in (10.36)

$$E_{tr} = \partial_t w_{q+1} + v_q \cdot \nabla w_{q+1}. \tag{10.94}$$

To this end, we will replace the phase $\xi \cdot x$ in the definition of the Beltrami wave by a phase $\Phi(t, x)$ that is transported by the vector field v_q. We divide the interval $0 \le t \le 1$ into intervals of length ℓ, and for $j = 0, \dots, [\ell^{-1}]$, we define $\Phi_j(t, x)$ as the $\mathbb{T}^3$-periodic solution to

$$\begin{aligned} &\partial_t \Phi_j + v_\ell \cdot \nabla \Phi_j = 0, \\ &\Phi_j(j\ell, x) = x. \end{aligned} \tag{10.95}$$

We have the following standard estimates for Φ_j: first, differentiating (10.95) in x, and using Gronwall's inequality and the inductive assumption (10.49) gives

$$\|\nabla \Phi_j(t) - \mathrm{Id}\|_{C^0} \le C\ell \|v_\ell\|_{C^1} \le C\ell \lambda_q \delta_q^{1/2} = C\lambda_q^{-1/2} \delta_q^{1/2} \ll 1, \text{ for all } (j-1)\ell \le t \le (j+1)\ell. \tag{10.96}$$

Differentiating (10.95) once again gives

$$\|\nabla \Phi_j(t)\|_{C^1_{t,x}} \le C\lambda_q \delta_q^{1/2}, \text{ for all } (j-1)\ell \le t \le (j+1)\ell, \tag{10.97}$$

and, more generally,

$$\|\nabla \Phi_j(t)\|_{C^n} \le C\ell^{1-n} \lambda_q \delta_q^{1/2} \ll \ell^{-n}, \text{ for all } (j-1)\ell \le t \le (j+1)\ell. \tag{10.98}$$

Each $\Phi_j(t, x)$ will play a role only on the time interval $[(j-2)\ell, (j+2)\ell]$. For this, we will make use of time-cutoffs: take a non-negative bump function $\chi(t)$ supported in $[-1, 1]$ so that $\chi(t) \equiv 1$ on $[-1/2, 1/2]$ and such that the shifts

$$\chi_j(t) = \chi(\ell^{-1} t - j)$$

satisfy

$$\sum_j \chi_j^2(t) \equiv 1 \text{ for } 0 \le t \le 1. \tag{10.99}$$

Note that each time t at most two of $\chi_j(t)$ are non-zero.

Let us recall the sets Λ_0 and Λ_1 from Lemma 10.5. For a general j we will set $\Lambda_j = \Lambda_0$ if j is even and $\Lambda_j = \Lambda_1$ if j is odd. We do the same for the functions $\gamma_\xi^{(j)}$ appearing in that lemma. With this notation, we define for the principal part of the perturbation $w_{q+1}^{(p)}(t,x)$ as

$$w_{q+1}^{(p)}(t,x) = \sum_j \sum_{\xi\in\Lambda_j} w_{(\xi)}(t,x), \tag{10.100}$$

with each individual wave $w_{(\xi)}(t,x)$ in the form of a modulated Beltrami wave

$$w_{(\xi)}(t,x) = a_{q+1,j,\xi}(t,x) W_{\xi,\lambda_{q+1}}(\Phi_j(t,x)) = a_{q+1,j,\xi}(t,x) B_\xi \exp\left\{2\pi i \lambda_{q+1}\xi\cdot\Phi_j(t,x)\right\}. \tag{10.101}$$

Note that

$$(\partial_t + v_q\cdot\nabla)(\exp\{2\pi i\lambda_{q+1}\xi\cdot\Phi_j(t,x)\}) = 0, \tag{10.102}$$

so that

$$(\partial_t + v_q\cdot\nabla) w_{(\xi)} = (\partial_t + v_q\cdot\nabla)[a_{q+1,j,\xi}](t,x) B_\xi(\exp\{2\pi i\lambda_{q+1}\xi\cdot\Phi_j(t,x)\}), \tag{10.103}$$

and the potentially dangerous term of the size λ_{q+1} coming from the differentiation of the exponent vanishes. This is why we use the phases $\Phi_j(t,x)$ rather than simply x. The amplitudes $a_{q+1,j,\xi}(t,x)$ are chosen as

$$a_{q+1,j,\xi}(t,x) = c_R^{1/4}\delta_{q+1}^{1/2}\chi_j(t)\gamma_\xi^{(j)}(M_\ell(t,x)), \tag{10.104}$$

with the matrix

$$M_\ell(t,x) = \mathrm{Id} - c_R^{-1/2}\delta_{q+1}^{-1} R_\ell(t,x). \tag{10.105}$$

As the functions $\gamma_\xi^{(j)}$ are defined only in the ball $B_{r_0}(\mathrm{Id})$, we need to check that the matrices $M_\ell(t,x)$ are in that ball for all $0 \le t \le 1$ and $x \in \mathbb{T}^3$. Recalling the inductive assumption (10.50), we see that

$$c_R^{-1/2}\delta_{q+1}^{-1}\|R_\ell\|_{C^0} \le c_R^{-1/2}\delta_{q+1}^{-1} c_R\delta_{q+1} \le c_R^{1/2} \le r_0, \tag{10.106}$$

with r_0 as in Lemma 10.5, provided we take

$$c_R \le r_0^2. \tag{10.107}$$

It follows that the matrix $M_\ell(t,x)$ is, indeed, in the domain of definition of the functions $\gamma_\xi^{(j)}$ for all j, all $t\in[0,1]$ and $x\in\mathbb{T}^3$. As at most two of the functions χ_j do not vanish for any given $t\in[0,1]$, and they satisfy $0\le\chi_j(t)\le 1$, we have a uniform estimate

$$\|w_{q+1}^{(p)}(t,x)\|_{C^0} \le K_0 c_R^{1/4}\delta_{q+1}^{1/2} \le \frac{\delta_{q+1}^{1/2}}{2}, \tag{10.108}$$

provided that we choose c_R sufficiently small, depending only on a universal constant K_0 that itself depends only on the uniform norm of the functions $\gamma_\xi^{(j)}(M)$ on $B_{r_0}(\mathrm{Id})$ and on the number of elements in the finite sets Λ_0 and Λ_1. The above estimate accounts for the contribution of $w_{q+1}^{(p)}$ to the error bound (10.52).

10.5.4 The incompressibility correction

Let us write the individual terms $w_{(\xi)}(t,x)$ that appear in (10.100) as

$$w_{(\xi)}(t,x) = a_{q+1,j,\xi}(t,x)\exp\big\{2\pi i\lambda_{q+1}\xi\cdot\phi_j(t,x)\big\}B_\xi \exp\big\{2\pi i\lambda_{q+1}\xi\cdot x\big\} \\ = a_{q+1,j,\xi}(t,x)\exp\big\{2\pi i\lambda_{q+1}\xi\cdot\phi_j(t,x)\big\}W_{\xi,\lambda_{q+1}}(x) = b_{q+1,j,\xi}(t,x)W_{\xi,\lambda_{q+1}}(x), \tag{10.109}$$

with

$$\phi_j(t,x) = \Phi_j(t,x) - x, \quad b_{q+1,j,\xi}(t,x) = a_{q+1,j,\xi}(t,x)\exp\big\{2\pi i\lambda_{q+1}\xi\cdot\phi_j(t,x)\big\}. \tag{10.110}$$

Recalling (10.96), we can think of $\phi_j(t,x)$ as small, so the largest contribution to $\nabla w_{(\xi)}(t,x)$ should come from the Beltrami wave $W_{\xi,\lambda_{q+1}}(x)$. However, the latter is incompressible so one can think of $w_{(\xi)}$ as incompressible to the leading order. To be more precise, let us use (10.84) to write

$$b_{q+1,j,\xi}(t,x)W_{\xi,\lambda_{q+1}}(x) = \frac{1}{2\pi\lambda_{q+1}}\Big[\nabla\times(b_{q+1,j,\xi}(t,x)W_{\xi,\lambda_{q+1}}(x)) - (\nabla b_{q+1,j,\xi}(t,x))\times W_{\xi,\lambda_{q+1}}(x)\Big].$$

While the first term above is incompressible, the second is not. Accordingly, to compensate for the second term, we define

$$\begin{aligned} w^{(c)}_{(\xi)}(t,x) &= \frac{1}{2\pi\lambda_{q+1}}(\nabla b_{q+1,j,\xi}(t,x))\times W_{\xi,\lambda_{q+1}}(x) \\ &= \frac{1}{2\pi\lambda_{q+1}}\Big(\nabla a_{q+1,j,\xi} + 2\pi i\lambda_{q+1}a_{q+1,j,\xi}(\nabla\Phi_j(t,x)-\mathrm{Id})\xi\Big)\times B_\xi\exp\{2\pi i\lambda_{q+1}\xi\cdot\Phi_j(t,x)\} \\ &= \Big(\frac{\nabla a_{q+1,j,\xi}}{2\pi\lambda_{q+1}} + i a_{q+1,j,\xi}(\nabla\Phi_j(t,x)-\mathrm{Id})\xi\Big)\times W_{\xi,\lambda_{q+1}}(\Phi_j(t,x)). \end{aligned} \tag{10.111}$$

The full incompressibility correction is then

$$w^{(c)}_{q+1}(t,x) = \sum_j\sum_{\xi\in\Lambda_j} w^{(c)}_{(\xi)}(t,x), \tag{10.112}$$

and the full perturbation is

$$w_{q+1}(t,x) = w^{(p)}_{q+1}(t,x) + w^{(c)}_{q+1}(t,x) = \frac{1}{2\pi\lambda_{q+1}}\sum_j\sum_{\xi\in\Lambda_j}\nabla\times[b_{q+1,j,\xi}(t,x)W_{\xi,\lambda_{q+1}}(x)], \tag{10.113}$$

so that

$$\nabla\cdot w_{q+1} = 0, \tag{10.114}$$

and $w_{q+1}(t,x)$ is mean-zero. We may also estimate the incompressible correction, starting with the right side of (10.111), and once again using the fact that $\chi_j(t)$ satisfy $0\le\chi_j(t)\le 1$, and only two of $\chi_j(t)$ do not vanish for any $t\in[0,1]$ as

$$\|w^{(c)}_{q+1}\|_{C^0} \le K\sup_j\sup_{\xi\in\Lambda_j}\Big[\frac{\|\nabla a_{q+1,j,\xi}\|_{C^0}}{\lambda_{q+1}} + \|a_{q+1,j,\xi}\|_{C^0}\|\nabla\Phi_j - \mathrm{Id}\|_{C^0}\Big], \tag{10.115}$$

with a universal constant K. At the moment, we do not have a good bound on $\|\nabla a_{q+1,j,\xi}\|_{C^0}$ as that would require a bound on $\|\nabla R_\ell\|_{C^0}$, since R_ℓ enters the definition (10.104)-(10.105) of $a_{q+1,j,\xi}$. However, a standard mollification estimate, together with (10.104)-(10.105) and the induction assumption (10.50), show that the first term above can be bounded as

$$\frac{\|\nabla a_{q+1,j,\xi}\|_{C^0}}{\lambda_{q+1}} \leq K\delta_{q+1}^{1/2}\delta_{q+1}^{-1}\lambda_{q+1}^{-1}\frac{\|R_\ell\|_{C^0}}{\ell} \leq K\frac{\delta_{q+1}^{1/2}}{\ell\lambda_{q+1}} \leq \frac{\delta_{q+1}^{1/2}}{100}, \tag{10.116}$$

because

$$\ell\lambda_{q+1} = \lambda_q^{-3/2}\lambda_q^2 \gg 1.$$

Here we see that it is important that $a_{q+1,j,\xi}$ oscillate on scales much larger than λ_{q+1}^{-1}. The second term in the right side of (10.115) can be estimated with the help of (10.96) as

$$\|a_{q+1,j,\xi}\|_{C^0}\|\nabla\Phi_j - \mathrm{Id}\|_{C^0} \leq K\delta_{q+1}^{1/2}\lambda_q^{-1/2}\delta_q^{1/2} \leq \frac{\delta_{q+1}^{1/2}}{100}, \tag{10.117}$$

provided that a is sufficiently large and β is sufficiently small. It follows that

$$\|w_{q+1}^{(c)}\|_{C^0} \leq \frac{\delta_{q+1}^{1/2}}{10}. \tag{10.118}$$

Together with (10.108), this finishes the proof of the error bound (10.52):

$$\|w_{q+1}\|_{C^0} \leq \frac{3}{4}\delta_{q+1}^{1/2}. \tag{10.119}$$

However, we still need to verify that the induction bounds (10.48)-(10.50) hold for v_{q+1} and R_{q+1}.

10.5.5 The induction estimates on the velocity

We first prove the inductive estimates (10.48)-(10.49) on the velocity v_{q+1}, as they follow directly from the construction of the perturbation w_{q+1}. It is convenient to define v_{q+1} not as $v_q + w_{q+1}$ but as

$$v_{q+1} = v_\ell + w_{q+1}. \tag{10.120}$$

The uniform bound in (10.48) for $q+1$ follows simply from this estimate at level q and (10.119), together with (10.66):

$$\|v_q - v_\ell\|_{C^0} \ll \delta_{q+1}^{1/2}, \tag{10.121}$$

which gives

$$\|v_{q+1}\|_{C^0} \leq \|v_q\|_{C^0} + \|v_q - v_\ell\|_{C^0} + \|w_{q+1}\|_{C^0} \leq 1 - \delta_q^{1/2} + \frac{1}{10}\delta_{q+1}^{1/2} + \frac{3}{4}\delta_{q+1}^{1/2} \leq 1 - \delta_{q+1}^{1/2}, \tag{10.122}$$

since we have $\delta_{q+1} \leq 4\delta_q$ if we choose a sufficiently large, for a given fixed small $\beta > 0$.

To get the gradient bound (10.49) at the level $q+1$ we first recall that for the spatial derivatives we have (10.96), (10.97) and (10.116):

$$\|\nabla\Phi_j(t) - \mathrm{Id}\|_{C^0} \le C\lambda_q^{-1/2}\delta_q^{1/2}, \tag{10.123}$$

$$\|\nabla\Phi_j(t)\|_{C^1_{t,x}} \le C\lambda_q\delta_q^{1/2}, \tag{10.124}$$

$$\|\nabla a_{q+1,j,\xi}\|_{C^0} \le C\frac{\delta_{q+1}^{1/2}}{\ell}. \tag{10.125}$$

Once again, as at most two of $\chi_j(t)$ do not vanish for any $t>0$, it follows from (10.101)-(10.105) that the principal part of the perturbation satisfies

$$\|\nabla w_{q+1}^{(p)}\|_{C^0} \le K\sup_j\sup_{j\in\Lambda_j}\Big(\|\nabla a_{q+1,j,\xi}\|_{C^0} + \|a_{q+1,j,\xi}\|_{C^0}\lambda_{q+1}\|\nabla\Phi_j\|_{C_0}\Big), \tag{10.126}$$

with a constant K that depends only on the number of the elements of the sets Λ_0 and Λ_1. The first term above we estimate by (10.125), and the second by (10.123), which gives

$$\|\nabla w_{q+1}^{(p)}\|_{C^0} \le C\frac{\delta_{q+1}^{1/2}}{\ell} + C\delta_{q+1}^{1/2}\lambda_{q+1} \le C\delta_{q+1}^{1/2}\lambda_{q+1}. \tag{10.127}$$

For the spatial derivative of $w_{q+1}^{(c)}$, we note that

$$\begin{aligned}\|\nabla w_{q+1}^{(c)}\|_{C^0} \le K\sup_j\sup_{j\in\Lambda_j}\Big(&\frac{\|a_{q+1,j,\xi}\|_{C^2}}{\lambda_{q+1}} + \|\nabla a_{q+1,j,\xi}\|_{C^0}\|\nabla\Phi_j - \mathrm{Id}\|_{C^0}\\ &+\|a_{q+1,j,\xi}\|_{C^0}\|\nabla\Phi_j\|_{C^1} + \lambda_{q+1}\|w_{q+1}^{(c)}\|_{C^0}\Big).\end{aligned} \tag{10.128}$$

The first term above, once again, can be bounded using the basic mollification estimate as

$$\frac{\|a_{q+1,j,\xi}\|_{C^2}}{\lambda_{q+1}} \le \frac{K\delta_{q+1}^{1/2}}{\lambda_{q+1}}\Big[\frac{\|R_\ell\|_{C^0}}{\delta_{q+1}\ell^2} + \frac{\|R_\ell\|_{C^0}^2}{\delta_{q+1}^2\ell^2}\Big] \le \frac{K\delta_{q+1}^{1/2}\lambda_{q+1}^{3/2}}{\lambda_{q+1}} \le \frac{\delta_{q+1}^{1/2}\lambda_{q+1}}{100}. \tag{10.129}$$

The second term in the right side of (10.128) is estimated using (10.116) and (10.123) as

$$\|\nabla a_{q+1,j,\xi}\|_{C^0}\|\nabla\Phi_j - \mathrm{Id}\|_{C^0} \le \frac{K\delta_{q+1}^{1/2}}{\ell}\delta_q^{1/2}\lambda_q^{-1/2} = K\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q \le \frac{\delta_{q+1}^{1/2}\lambda_{q+1}}{100}. \tag{10.130}$$

The third and the fourth terms in right side of (10.128) satisfy

$$\|a_{q+1,j,\xi}\|_{C^0}\|\nabla\Phi_j\|_{C^1} + \lambda_{q+1}\|w_{q+1}^{(c)}\|_{C^0} \le K\delta_{q+1}^{1/2}\lambda_q\delta_q^{1/2} + \lambda_{q+1}\delta_{q+1}^{1/2} \le 2\lambda_{q+1}\delta_{q+1}^{1/2}. \tag{10.131}$$

Putting together the above estimates, we see that

$$\|\nabla w_{q+1}\|_{C^0} \le C_R\lambda_{q+1}\delta_{q+1}^{1/2}, \tag{10.132}$$

with a universal constant C_R. In particular, we have not used the estimate (10.49) at level q in deriving (10.131), hence there is no danger that C_R may change from step q to step $q+1$.

For the time derivative we have (10.95), which, together with (10.96) shows that

$$\|\partial_t \Phi_j(t)\|_{C^0} \leq C, \tag{10.133}$$

while

$$\|\partial_t a_{q+1,j,\xi}\|_{C^0} \leq C\delta_{q+1}^{1/2}(\ell^{-1} + \ell^{-1}) = C\delta_{q+1}^{1/2}\ell^{-1}, \tag{10.134}$$

as in the estimate (10.129) for the gradient of $a_{q+1,\xi,j}$. These two bounds give

$$\|\partial_t w_{q+1}^{(p)}\|_{C^0} \leq C\delta_{q+1}^{1/2}\ell^{-1}, \tag{10.135}$$

in the same way as (10.125) and (10.126) lead to (10.127). For the time derivative of $w_{q+1}^{(c)}$, we note that

$$\begin{aligned}\|\partial_t w_{q+1}^{(c)}\|_{C^0} \leq K \sup_j \sup_{j\in\Lambda_j} \Big(&\frac{\|\partial_t \nabla a_{q+1,j,\xi}\|_{C^0}}{\lambda_{q+1}} + \|\partial_t a_{q+1,j,\xi}\|_{C^0}\|\nabla\Phi_j - \mathrm{Id}\|_{C^0} \\ &+ \|a_{q+1,j,\xi}\|_{C^0}\|\partial_t\nabla\Phi_j\|_{C^0} + \lambda_{q+1}\|w_{q+1}^{(c)}\|_{C^0}\Big).\end{aligned} \tag{10.136}$$

The first term above is estimated exactly as in (10.129), the second as in (10.130), the third and the fourth as in (10.131), which gives us

$$\|\partial_t w_{q+1}^{(c)}\|_{C^0} \leq C\delta_{q+1}^{1/2}\ell^{-1}, \tag{10.137}$$

finishing the proof of (10.49) at level $q+1$.

10.5.6 The new Reynolds stress

We finally come to the key estimate in the proof of Proposition 10.3: the proof of the inductive estimate (10.50) at level $q+1$ that shows that the Reynolds stress decreases at each inductive step and tends to zero in the uniform norm as $q \to +\infty$. The analysis is based on Lemma 10.4 that we state again here.

Lemma 10.6. *Assume that $a(x) \in C^{m,\alpha}(\mathbb{T}^3)$ and $\Phi \in C^{m,\alpha}(\mathbb{T}^3)$ be smooth $\mathbb{R}^3$-valued functions, and let C be such that*

$$C^{-1} \leq |\nabla\Phi(x)| \leq C \text{ for all } x \in \mathbb{T}^3.$$

Let $\omega \in \mathbb{Z}^3$, $\alpha \in (0,1)$ and $m \geq 1$, and $R(x)$ be the solution to

$$\nabla \cdot R = E(x), \quad TrR(x) = 0, \quad R(x) \text{ is a symmetric matrix,} \tag{10.138}$$

with

$$E(x) = a(x)e^{i\omega\cdot\Phi(x)} - \int_{\mathbb{T}^3} a(y)e^{i\omega\cdot\Phi(y)}dy, \tag{10.139}$$

given by

$$R_{km}(x) = (\partial_k\Delta^{-1}E_m + \partial_m\Delta^{-1}E_k) - \frac{1}{2}(\delta_{km} + \partial_k\partial_m\Delta^{-1})\Delta^{-1}(\nabla\cdot E). \tag{10.140}$$

There exists a constant K that depends on C, α and m but not on $|\omega|$ such that

$$\|R\|_{C^\alpha} \leq C\Big(\frac{\|a\|_{C^0}}{|\omega|^{1-\alpha}} + \frac{1}{|\omega|^{m-\alpha}}\big(\|a\|_{C^{m,\alpha}} + \|a\|_{C^0}\|\nabla\Phi\|_{C^{m,\alpha}}\big)\Big). \tag{10.141}$$

In particular, if $E(x)$ is of the form

$$E(x) = a(x)W_{(\xi)}(\Phi_j(x)), \quad W_{(\xi)}(x) = B_\xi e^{2\pi i\lambda_{q+1}\xi\cdot x}, \tag{10.142}$$

with

$$\|a\|_{C^n} \le C\ell^{-n}, \quad \|\nabla\Phi_j\|_{C^n} \le C\ell^{-n}, \tag{10.143}$$

then $R(x)$ satisfies

$$\|R\|_{C^0} \le \|R\|_{C^\alpha} \le \frac{C}{\lambda_{q+1}^{1-\alpha}}\Big(1+\frac{\ell^{-m-\alpha}}{\lambda_{q+1}^{m-\alpha}}\Big) \le \frac{C}{\lambda_{q+1}^{1-\alpha}}\Big(1+\frac{\ell^{-m-1}}{\lambda_{q+1}^{m-1}}\Big) = \frac{C}{\lambda_{q+1}^{1-\alpha}}\Big(1+\lambda_{q+1}^{3/4(m+1)-m+1}\Big) \le \frac{C}{\lambda_{q+1}^{1-\alpha}}, \tag{10.144}$$

as long as we take $m \ge 8$.

To get an equation for R_{q+1} we recall that v_ℓ satisfies (10.67):

$$\begin{aligned} &\partial_t v_\ell + \nabla\cdot[v_\ell\otimes v_\ell] + \nabla p_\ell = \nabla\cdot(R_\ell + R_{comm}),\\ &\nabla\cdot v_\ell = 0, \end{aligned} \tag{10.145}$$

with

$$R_{comm} = v_\ell \otimes_{tr} v_\ell - [(v\otimes_{tr} v)\star_x \phi_\ell]\star_t \varphi_\ell. \tag{10.146}$$

Hence, $v_{q+1} = v_\ell + w_{q+1}$ satisfies

$$\begin{aligned} &\partial_t v_{q+1} + \nabla\cdot(v_{q+1}\otimes v_{q+1}) = \partial_t v_\ell + \nabla\cdot(v_\ell\otimes v_\ell) + \partial_t w_{q+1} + \nabla\cdot(w_{q+1}\otimes w_{q+1})\\ &+\nabla\cdot(v_\ell\otimes w_{q+1}) + \nabla\cdot(w_{q+1}\otimes v_\ell) = \nabla\cdot(R_\ell + R_{comm}) - \nabla p_\ell + \partial_t w_{q+1} + v_\ell\cdot\nabla w_{q+1}\\ &+w_{q+1}\cdot\nabla v_\ell + \nabla\cdot(w_{q+1}\otimes w_{q+1}), \end{aligned} \tag{10.147}$$

so that the Reynolds stress R_{q+1} and pressure p_{q+1} satisfy, after absorbing p_ℓ into p_{q+1}

$$\nabla\cdot R_{q+1} = \nabla p_{q+1} + \nabla\cdot(R_\ell + R_{comm}) + \partial_t w_{q+1} + v_\ell\cdot\nabla w_{q+1} + w_{q+1}\cdot\nabla v_\ell + \nabla\cdot(w_{q+1}\otimes w_{q+1}). \tag{10.148}$$

We write

$$w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)},$$

and represent the right side of (10.148) as

$$\nabla\cdot R_{q+1} = E_{tr} + E_{osc} + E_{Nash} + E_{corr} + \nabla\cdot R_{comm} + \nabla\cdot R_{corr}^{(c)} + \nabla p_{q+1}, \tag{10.149}$$

with the transport error

$$E_{tr} = \partial_t w_{q+1}^{(p)} + v_\ell\cdot\nabla w_{q+1}^{(p)} - \langle\partial_t w_{q+1}^{(p)}\rangle, \tag{10.150}$$

the oscillation error

$$E_{osc} = \nabla\cdot(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} + R_q), \tag{10.151}$$

and the Nash error

$$E_{Nash} = w_{q+1}^{(p)}\cdot\nabla v_\ell, \tag{10.152}$$

coming from the principal part of the perturbation, and the corrector error

$$E_{corr} = \partial_t w_{q+1}^{(c)} + v_\ell\cdot\nabla w_{q+1}^{(c)} - \langle\partial_t w_{q+1}^{(c)}\rangle, \tag{10.153}$$

and the tensor

$$R_{corr}^{(c)} = w_{q+1}^{(c)} \otimes_{tr} w_{q+1}^{(c)} + w_{q+1}^{(p)} \otimes_{tr} w_{q+1}^{(c)} + w_{q+1}^{(c)} \otimes_{tr} w_{q+1}^{(p)} \tag{10.154}$$

coming from the incompressibility correction to the perturbation. Note that in the definition of $R_{corr}^{(c)}$ we have replaced the tensor products $\otimes$ by the trace-free tensor products $\otimes_{tr}$, with the difference going into the pressure p_{q+1}. The notation $\langle \cdot \rangle$ refers to the spatial average, as before:

$$\langle f \rangle = \int_{\mathbb{T}^3} f(y)dy.$$

As w_{q+1} is a curl, its spatial average vanishes, hence

$$\langle \partial_t w_{q+1}^{(p)} \rangle + \langle \partial_t w_{q+1}^{(c)} \rangle = 0, \tag{10.155}$$

so that the addition of these two terms to (10.150) and (10.153) does not change anything. Then we can write

$$R_{q+1} = R_{tr} + R_{Nash} + R_{comm} + R_{corr} + R_{corr}^{(c)} + R_{osc}, \tag{10.156}$$

with R_{comm} and $R_{corr}^{(c)}$ defined in (10.146) and (10.154), respectively, and R_{tr}, R_{Nash}, R_{comm} and R_{corr} given by (10.140) with the corresponding E in the right side.

The term R_{osc} in (10.156) should be a trace-less symmetric solution to

$$\nabla \cdot R_{osc} = E_{osc} + \nabla p_{osc}, \tag{10.157}$$

with E_{osc} given by (10.151) and some pressure p_{osc} that we will absorb into p_{q+1}. We can re-write E_{osc} as

$$E_{osc} = \nabla \cdot (w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + R_\ell) = \nabla \cdot \Big(\sum_{j,j'} \sum_{\xi \in \Lambda_j, \xi' \in \Lambda_{j'}} w_{(\xi)} \otimes w_{\xi'} + R_\ell \Big). \tag{10.158}$$

Note that $w_{(\xi)}$ and $w_{(\xi')}$ have disjoint support in time if $\xi \in \Lambda_j$ and $\xi' \in \Lambda_{j'}$ with $|j - j'| > 1$. In addition, if $|j - j'| = 1$, then Λ_j and $\Lambda_{j'}$ are disjoint sets so that $\xi + \xi' \neq 0$ – this is why we took Λ_0 and Λ_1 as two different sets. Hence, the only terms in the sum in (10.158) that satisfy $\xi + \xi' = 0$ are those with $j = j'$. Thus, we have

$$E_{osc} = \nabla \cdot \Big(\sum_j \sum_{\xi \in \Lambda_j} w_{(\xi)} \otimes w_{(-\xi)} + R_\ell \Big) + \nabla \cdot \Big(\sum_{j,j'} \sum_{\xi \in \Lambda_j, \xi' \in \Lambda_{j'}, \xi + \xi' \neq 0} w_{(\xi)} \otimes w_{\xi'} \Big). \tag{10.159}$$

We claim that the divergence of the first sum in (10.159) actually vanishes – and that is the reason we have chosen the coefficients $a_{q+1,j,\xi}$ in the way we did. Indeed, recall that

$$a_{q+1,j,\xi}(t,x) = c_R^{1/4} \delta_{q+1}^{1/2} \chi_j(t) \gamma_\xi^{(j)}(M_\ell(t,x)), \tag{10.160}$$

with the coefficients $\gamma_\xi^{(j)}$ defined so that

$$M_\ell(t,x) = \frac{1}{2} \sum_{\xi \in \Lambda_j} (\gamma_\xi^{(j)}(M_\ell(t,x)))^2 (\mathrm{Id} - \xi \otimes \xi), \tag{10.161}$$

for each j, where

$$M_\ell(t,x) = \mathrm{Id} - c_R^{-1/2}\delta_{q+1}^{-1} R_\ell(t,x), \tag{10.162}$$

which implies

$$c_R^{1/2}\delta_{q+1}\mathrm{Id} - R_\ell(t,x) = \frac{1}{2}\sum_{\xi\in\Lambda_j} c_R^{1/2}\delta_{q+1}(\gamma_\xi^{(j)}(M_q(t,x)))^2(\mathrm{Id} - \xi\otimes\xi), \tag{10.163}$$

again for each j. Multiplying (10.163) by $\chi_j^2(t)$ and summing over j, using (10.99), we arrive at

$$\begin{aligned} c_R^{1/2}\delta_{q+1}\mathrm{Id} - R_\ell(t,x) &= \frac{1}{2}\sum_j\sum_{\xi\in\Lambda_j} \chi_j^2(t) c_R^{1/2}\delta_{q+1}(\gamma_\xi^{(j)}(M_q(t,x)))^2(\mathrm{Id} - \xi\otimes\xi) \\ &= \frac{1}{2}\sum_j\sum_{\xi\in\Lambda_j} |a_{q+1,j,\xi}(t,x)|^2(\mathrm{Id} - \xi\otimes\xi). \end{aligned} \tag{10.164}$$

On the other hand, as in (10.89), we have, since $a_{q+1,j,-\xi} = a_{q+1,j,\xi}$, that

$$\begin{aligned} &\sum_{\xi\in\Lambda_j} w_{(\xi)}\otimes w_{(-\xi)} = \sum_{\xi\in\Lambda_j} |a_{q+1,j,\xi}|^2 B_\xi\otimes B_{-\xi} \\ &= \frac{1}{2}\sum_{\xi\in\Lambda_j} |a_{q+1,j,\xi}|^2 (A_\xi + i\xi\times A_\xi)\otimes(A_\xi - i\xi\times A_\xi) \\ &= \frac{1}{2}\sum_{\xi\in\Lambda_j} |a_{q+1,j,\xi}|^2 (A_\xi\otimes A_\xi + (\xi\times A_\xi)\otimes(\xi\times A_\xi)) + \frac{i}{2}[(\xi\times A_\xi)\otimes A_\xi - A_\xi\otimes(\xi\times A_\xi)] \\ &= \frac{1}{2}\sum_{\xi\in\Lambda_j} |a_{q+1,j,\xi}|^2(\mathrm{Id} - \xi\otimes\xi). \end{aligned} \tag{10.165}$$

Since the set Λ_j is symmetric: $\Lambda_j = -\Lambda_j$, the second term in the third line above vanishes after summation over $\xi\in\Lambda_j$, and for the first term in that line we used (10.87):

$$\xi\otimes\xi + A_\xi\otimes A_\xi + (\xi\times A_\xi)\otimes(\xi\times A_\xi) = \mathrm{Id}. \tag{10.166}$$

We deduce from (10.164) and (10.165) that

$$\nabla\cdot\Big(\sum_{\xi\in\Lambda_j} w_{(\xi)}\otimes w_{(-\xi)} + R_\ell\Big) = 0, \tag{10.167}$$

as we have claimed. Recall also that for a scalar-valued function $g(x)$ and a matrix-valued function $F(x)$ we have

$$\begin{aligned} [\nabla\cdot(g(x)F(x))]_i = \partial_j(g(x)F_{ji}(x)) &= (\partial_j g(x))F_{ji}(x) + g(x)(\partial_j(F_{ij}(x)) \\ &= (F^t(x)\nabla g(x))_i + g(x)(\nabla\cdot F(x))_i, \end{aligned}$$

so that

$$\nabla\cdot(gF) = F^t\nabla g + g\nabla\cdot F.$$

Hence, E_{osc} has the form

$$
\begin{aligned}
E_{osc} &= \nabla\cdot\Big(\sum_{j,j'}\sum_{\xi\in\Lambda_j,\xi'\in\Lambda_{j'},\xi+\xi'\neq 0} w_{(\xi)}\otimes w_{\xi'}\Big) \qquad (10.168)\\
&= \frac{1}{2}\sum_{j,j'}\sum_{\xi\in\Lambda_j,\xi'\in\Lambda_{j'},\xi+\xi'\neq 0} b_{q+1,j,\xi}b_{q+1,j,\xi'}\nabla\cdot(W_{\xi,\lambda_{q+1}}\otimes W_{\xi',\lambda_{q+1}}+W_{\xi,\lambda_{q+1}}\otimes W_{\xi',\lambda_{q+1}})\\
&+\sum_{j,j'}\sum_{\xi\in\Lambda_j,\xi'\in\Lambda_{j'},\xi+\xi'\neq 0}(W_{\xi',\lambda_{q+1}}\otimes W_{\xi,\lambda_{q+1}})\nabla(b_{q+1,j,\xi}b_{q+1,j,\xi'}),
\end{aligned}
$$

with $b_{q+1,j,\xi}$ as in (10.109) and (10.110). In addition, as in the derivation of the Euler equation

$$\nabla\cdot(W_{\xi,\lambda}\otimes W_{\xi,\lambda})=\nabla\Big(\frac{|W_{\xi,\lambda}|^2}{2}\Big),$$

we also have **fill this in**

$$\nabla\cdot(W_{\xi,\lambda}\otimes W_{\xi',\lambda}+W_{\xi,\lambda}\otimes W_{\xi',\lambda})=\nabla(W_{\xi,\lambda}\cdot W_{\xi',\lambda}). \qquad (10.169)$$

Therefore, (10.168) becomes

$$
\begin{aligned}
E_{osc} &= \frac{1}{2}\sum_{j,j'}\sum_{\xi\in\Lambda_j,\xi'\in\Lambda_{j'},\xi+\xi'\neq 0} b_{q+1,j,\xi}b_{q+1,j,\xi'}\nabla(W_{\xi,\lambda_{q+1}}\cdot W_{\xi',\lambda_{q+1}})\\
&+\sum_{j,j'}\sum_{\xi\in\Lambda_j,\xi'\in\Lambda_{j'},\xi+\xi'\neq 0}(W_{\xi',\lambda_{q+1}}\otimes W_{\xi,\lambda_{q+1}})\nabla(b_{q+1,j,\xi}b_{q+1,j,\xi'})\\
&= \frac{1}{2}\sum_{j,j'}\sum_{\xi\in\Lambda_j,\xi'\in\Lambda_{j'},\xi+\xi'\neq 0}\nabla\Big[b_{q+1,j,\xi}b_{q+1,j,\xi'}(W_{\xi,\lambda_{q+1}}\cdot W_{\xi',\lambda_{q+1}})\Big] \qquad (10.170)\\
&+\sum_{j,j'}\sum_{\xi\in\Lambda_j,\xi'\in\Lambda_{j'},\xi+\xi'\neq 0}\Big[W_{\xi',\lambda_{q+1}}\otimes W_{\xi,\lambda_{q+1}}-\frac{1}{2}(W_{\xi,\lambda_{q+1}}\cdot W_{\xi',\lambda_{q+1}})\mathrm{Id}\Big]\nabla(b_{q+1,j,\xi}b_{q+1,j,\xi'})\\
&= \nabla p_{osc}+\tilde{E}_{osc} \qquad (10.171)
\end{aligned}
$$

The first term in the right side can be incorporated into pressure, so that we can define R_{osc} as the solution to

$$\nabla\cdot R_{osc}=\tilde{E}_{osc}, \qquad (10.172)$$

given by (10.140) with $E=\tilde{E}_{osc}$. Summarizing, and recalling (10.156), we have the following expression for R_{q+1}:

$$R_{q+1}=R_{tr}+R_{Nash}+R_{comm}+R_{corr}+R_{corr}^{(c)}+R_{osc}, \qquad (10.173)$$

with R_{comm} and $R_{corr}^{(c)}$ defined in (10.146) and (10.154), respectively, and the individual contributions R_{tr}, R_{Nash}, R_{comm}, R_{corr} and R_{osc} given by (10.140) with the corresponding E in the right side.

10.5.7 The inductive estimates on the new Reynolds stress

Now we estimate each individual term in the right side of (10.173).

The transport error

Recall that the transport error is given by (10.150):

$$E_{tr} = \partial_t w_{q+1}^{(p)} + v_\ell \cdot \nabla w_{q+1}^{(p)} - \langle \partial_t w_{q+1}^{(p)} \rangle. \tag{10.174}$$

The last term in the right side does not contribute to (10.140) and only serve to ensure that $\langle E_{tr} \rangle = 0$. In addition, we have

$$\partial_t W_{\xi,\lambda}(\Phi_j) + v_q \cdot \nabla W_{\xi,\lambda}(\Phi_j) = 0, \tag{10.175}$$

because Φ_j is advected by v_q: it satisfies (10.95). It follows that

$$E_{tr} = \sum_j \sum_{\xi\in\Lambda_j} (\partial_t a_{q+1,\xi,j}(t,x) + v_q \cdot \nabla a_{q+1,\xi,j}(t,x)) W_{\xi,\lambda q+1}(\Phi_j(t,x)) \tag{10.176}$$

As we have seen many times, the standard mollification estimates on the derivatives of R_ℓ in terms of $\|R_\ell\|_{C^0}$, imply the bounds

$$\|a_{q+1,\lambda_{q+1},\xi}\|_{C^m} \le C\delta_{q+1}^{1/2}\ell^{-m}, \quad \|\partial_t a_{q+1,\xi,j} + v_q \cdot \nabla a_{q+1,\xi,j}\|_{C^m} \le C\delta_{q+1}^{1/2}\ell^{-m-1}. \tag{10.177}$$

Thus, we are in the situation as in (10.142)-(10.144), with $C = C'\delta^{1/2}\ell^{-1}$ in (10.144), which gives

$$\|R_{tr}\|_{C^0} \le \|R_{tr}\|_{C^\alpha} \le C\delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{\alpha-1} = C\delta_{q+1}^{1/2}\lambda_{q+1}^{\alpha-1/4} \le \frac{c_R\delta_{q+2}}{100}, \tag{10.178}$$

provided that α and β are sufficiently small.

The oscillation error

The estimate for the oscillation error is similar. First, we note that

$$\begin{aligned} |\nabla(b_{q+1,j,\xi}b_{q+1,j,\xi'})| &\le |\nabla(a_{q+1,j,\xi}a_{q+1,j,\xi'})| + \lambda_{q+1}\delta_{q+1}\Big(|\nabla\Phi_j - \mathrm{Id}| + |\nabla\Phi_{j'} - \mathrm{Id}|\Big) \\ &\le C\delta_{q+1}\ell^{-1} + C\lambda_{q+1}\delta_{q+1}\ell\lambda_q\delta_q^{1/2}. \end{aligned} \tag{10.179}$$

A very similar argument, using (10.98) yields

$$\|\nabla(b_{q+1,j,\xi}b_{q+1,j,\xi'})\|_{C^m} \le C\delta_{q+1}\ell^{-m-1} + C\lambda_{q+1}\delta_{q+1}\ell\lambda_q\delta_q^{1/2}\ell^{-m}. \tag{10.180}$$

Hence, we can use (10.144) (strictly speaking, we are using its analog for the case when the right side of (10.138) has the form of a tensor product of two right sides as in (10.142) but the same argument applies) with

$$C = C'[\delta_{q+1}\ell^{-1} + \lambda_{q+1}\delta_{q+1}\ell\lambda_q\delta_q^{1/2}],$$

which gives

$$\|R_{osc}\|_{C^0} \le \|R_{tr}\|_{C^\alpha} \le C[\delta_{q+1}\ell^{-1} + \lambda_{q+1}\delta_{q+1}\ell\lambda_q\delta_q^{1/2}]\lambda_{q+1}^{\alpha-1} \le \frac{c_R\delta_{q+2}}{100}, \tag{10.181}$$

provided that α and β are sufficiently small.

The Nash error

The Nash error comes from (10.152):

$$E_{Nash} = w^{(p)}_{q+1} \cdot \nabla v_\ell = \sum_j \sum_{\xi \in \Lambda_j} a_{q+1,j,\xi} W_{\xi,\lambda_{q+1}}(\Phi_j) \cdot \nabla v_\ell, \tag{10.182}$$

so it is again of the form (10.142) and we can appeal to (10.144). The estimate

$$\|a_{q+1,j,\xi} \nabla v_\ell\|_{C^n} \le C \delta_{q+1}^{1/2} \lambda_q \delta_q^{1/2} \ell^{-n} \tag{10.183}$$

then leads to

$$\|R_{Nash}\|_{C^0} \le \|R_{Nash}\|_{C^\alpha} \le C \delta_{q+1}^{1/2} \lambda_q \delta_q^{1/2} \lambda_{q+1}^{\alpha-1} \le \frac{c_R \delta_{q+2}}{100}. \tag{10.184}$$

The corrector error

The corrector error has two components:

$$R_{corr} + R^{(c)}_{corr}. \tag{10.185}$$

Here, $R^{(c)}_{corr}$ is given by (10.154):

$$R^{(c)}_{corr} = w^{(c)}_{q+1} \otimes_{tr} w^{(c)}_{q+1} + w^{(p)}_{q+1} \otimes_{tr} w^{(c)}_{q+1} + w^{(c)}_{q+1} \otimes_{tr} w^{(p)}_{q+1}, \tag{10.186}$$

and R_{corr} is given by (10.140) with $E = E_{corr}$, which is defined in (10.153)

$$E_{corr} = \partial_t w^{(c)}_{q+1} + v_\ell \cdot \nabla w^{(c)}_{q+1} - \langle \partial_t w^{(c)}_{q+1} \rangle. \tag{10.187}$$

The first term is estimated using the estimates (10.108)

$$\|w^{(p)}_{q+1}(t,x)\|_{C^0} \le \frac{\delta_{q+1}^{1/2}}{2}, \tag{10.188}$$

and (10.115)-(10.117) which say that

$$\|w^{(c)}_{q+1}\|_{C^0} \le K \sup_j \sup_{\xi \in \Lambda_j} \Big[\frac{\|\nabla a_{q+1,j,\xi}\|_{C^0}}{\lambda_{q+1}} + \|a_{q+1,j,\xi}\|_{C^0} \|\nabla \Phi_j - \mathrm{Id}\|_{C^0} \Big] \tag{10.189}$$

$$\le K \frac{\delta_{q+1}^{1/2}}{\ell \lambda_{q+1}} + K \delta_{q+1}^{1/2} \lambda_q^{-1/2} \delta_q^{1/2} \le K' \frac{\delta_{q+1}^{1/2}}{\ell \lambda_{q+1}}. \tag{10.190}$$

This allows us to estimate $R^{(c)}_{corr}$ simply as

$$\begin{aligned} \|R^{(c)}_{corr}\|_{C^0} &= \|w^{(c)}_{q+1} \otimes_{tr} w^{(c)}_{q+1} + w^{(p)}_{q+1} \otimes_{tr} w^{(c)}_{q+1} + w^{(c)}_{q+1} \otimes_{tr} w^{(p)}_{q+1}\|_{C^0} \\ &\le C \delta_{q+1} \ell^{-1} \lambda_{q+1}^{-1} \le \frac{c_R \delta_{q+2}}{100}. \end{aligned} \tag{10.191}$$

As for R_{corr}, note that, once again, because $W_{\xi,\lambda_{q+1}}(\Phi_j)$ solves the transport equation (10.175), we have

$$E_{corr} = \sum_j \sum_{\xi \in \Lambda_j} \Big((\partial_t + v_\ell \cdot \nabla) \Big(\frac{\nabla a_{q+1,j,\xi}}{\lambda_{q+1}} + i a_{q+1,j,\xi} (\nabla \Phi_j - \mathrm{Id}) \xi \Big) \Big) W_{\xi,\lambda_{q+1}}(\Phi_j(t,x)). \tag{10.192}$$

We know that

$$\frac{1}{\lambda_{q+1}}\|(\partial_t + v_\ell \cdot \nabla)\nabla a_{q+1,j,\xi}\|_{C^n} \leq K\delta_{q+1}^{1/2}\lambda_{q+1}^{-1}\ell^{-2-n} = K\delta_{q+1}^{1/2}\lambda_q\ell^{-n}, \tag{10.193}$$

and

$$\|(\partial_t + v_\ell \cdot \nabla)[a_{q+1,j,\xi}(\nabla\Phi_j - \mathrm{Id})\|_{C^n} \leq K\delta_{q+1}^{1/2}[\ell^{-n-1}\ell\lambda_q\delta_q^{1/2} + \ell^{1-n}\lambda_q\delta_q^{1/2}] \leq K'\delta_{q+1}^{1/2}\lambda_q\delta_q^{1/2}\ell^{-n}. \tag{10.194}$$

Appealing to (10.144) one more time, we obtain

$$\|R_{corr}\|_{C^0} \leq C\delta_{q+1}^{1/2}\lambda_q\lambda_{q+1}^{\alpha-1} \leq \frac{c_R\delta_{q+2}}{100}, \tag{10.195}$$

if α and β are sufficiently small. This was the last estimate we needed to prove that

$$\|R_{corr}\|_{C^0} \leq \frac{c_R\delta_{q+2}}{2}, \tag{10.196}$$

and we are done. This completes the proof of Proposition 10.3 and hence that of Theorem 10.2 as well.