Refined long time asymptotics for the Fisher-KPP equation

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Dedicated to H. Brezis, with admiration and respect

Abstract

We study the large time asymptotics of a solution of the Fisher-KPP reaction-diffusion equation, with an initial condition that is a compact perturbation of a step function. A well-known result of Bramson states that, in the reference frame moving as $2t-(3/2)\log t+x_\infty$, the solution of the equation converges as $t\to+\infty$ to a translate of the traveling wave corresponding to the minimal speed $c_*=2$. The constant x_∞ depends on the initial condition $u(0,x)$. The goal of this paper is to understand the convergence rate. We show that a level set $\{u(t,x)=s\}$ of the solution will move as $2t-(3/2)\log t+x_\infty+\bar{c}_s/\sqrt{t}$, with a universal constant $\bar{c}_s$ that depends on s but not on the initial condition. To this end, we construct a universal approximate solution, which solves the Fisher-KPP equation with an almost $O(t^{-1})$ precision. Then we prove the almost $O(t^{-1})$ convergence rate of the solution to the approximate solution.

1 Introduction

The goal of this paper is to prove sharp long time asymptotics of the solutions the Fisher-KPP equation:

$$u_t-u_{xx}=u-u^2,\quad t>0,\ x\in\mathbb{R}, \tag{1.1}$$

which have an initial condition of the form

$$u(0,x)=1-H(x)+v_0(x),\quad v_0 \text{ compactly supported.} \tag{1.2}$$

Here, $H(x)$ is the Heaviside function. The Fisher-KPP equation has a basic "minimal speed" traveling wave profile $u(t,x)=\phi(x-2t)$, connecting the stable eqilibrium $u\equiv 1$ to the unstable equilibrium $u\equiv 0$:

$$\begin{aligned}&-\phi''-2\phi'=\phi-\phi^2,\\&\phi(-\infty)=1,\quad \phi(+\infty)=0.\end{aligned} \tag{1.3}$$

Each solution $\phi(\xi)$ of (1.3) is a shift of a fixed profile $\phi_*(\xi)$: $\phi(\xi)=\phi_*(\xi+s)$, with some fixed $s\in\mathbb{R}$. The profile $\phi_*(\xi)$ satisfies the asymptotics

$$\phi_*(\xi)=(\xi+k)e^{-\xi}+O(e^{-(1+\omega_0)\xi}), \tag{1.4}$$

with two universal constants $\omega_0>0$, $k\in\mathbb{R}$.

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The large time behaviour of the solutions of this problem has a long history, starting with a striking paper of Fisher [13], which identifies the spreading velocity $c_* = 2$ via numerical computations and other arguments. In the same year, the pioneering KPP paper [18] proves that the solution of (1.1), starting from $1 - H(x)$, converges to ϕ_* in the following sense: there is a function

$$\sigma_\infty(t) = 2t + o(t),$$

such that

$$\lim_{t\to+\infty} u(t, x + \sigma_\infty(t)) = \phi_*(x). \tag{1.5}$$

Our result will build on two important series of works. The first is Bramson [6], [7], who proved the following

Theorem 1.1 *There is x_∞, depending on v_0, such that*

$$u(t,x) = \phi_*(x - 2t + \frac{3}{2}\log t - x_\infty) + o(1), \ \ as\ t \to +\infty.$$

This result is unusual, as the most common situation for reaction-diffusion equations of the type

$$u_t - u_{xx} = f(u), \quad t > 0, \ x \in \mathbb{R}$$

is exponential convergence to a constant shift of a traveling wave, see for instance Fife-McLeod [12]. Here, a more complicated phenomenon happens: there is a nontrivial logarithmic shift, which is the sign of a more intricate convergence mechanism. This raises the question of the convergence rate. One possible way to formulate the question is to set

$$\sigma(t) = \sup\{x : \ u(t,x) = 1/2\}, \quad \bar{\sigma}_\infty(t) := 2t - \frac{3}{2}\log t + x_\infty, \tag{1.6}$$

and estimate the error between $\sigma(t)$ and $\bar{\sigma}_\infty(t)$. The second key contribution is a very interesting paper of Ebert and Van Saarloos [10], completed by [26]. They perform a formal analysis of the convergence of $\sigma(t)$ to $\sigma_\infty(t)$, and state that

$$\sigma(t) = \bar{\sigma}_\infty(t) + \frac{c}{\sqrt{t}} + o(\frac{1}{\sqrt{t}}). \tag{1.7}$$

A striking feature is that the predicted constant c in (1.7) does not depend on the initial condition, unlike the zero order term x_∞. The predicted convergence rate is not exponential, which also signals the presence of a subtle convergence mechanism.

A natural issue is therefore to give a mathematically rigorous version of [10]. We will do this by constructing an approximate solution, which will be approached by the solutions of (1.1) at a rate almost equal to $O(t^{-1})$. Examination of the shift of the approximate solution will provide the asymptotics of $\sigma(t)$. As our procedure is different from that in [10], and the exact value of c is of much less interest than the existence of a universal approximate solution, we will not attempt to verify the value of c predicted in [10].

Main results

One of the main outcomes of this paper is the construction of an approximate solution which solves the equation up to suffciently high powers of $O(t^{-1})$. Here is the precise result.

Theorem 1.2 *For all $\gamma > 0$ there is a one-parameter family $(u_{app}(t, x+\lambda))_{\lambda\in\mathbb{R}}$ of approximate solutions to (1.1):*

$$u_{app}(t,x) = \phi_*(x - \widetilde{\sigma}(t)) + \frac{u_1(t, x - \widetilde{\sigma}(t))}{\sqrt{t}}, \quad \widetilde{\sigma}(t) = c_* t - \frac{3}{2}\log t + \frac{s_1(t)}{\sqrt{t}},$$

with $u_1(t,x)$ a bounded and continuous function that is C^1 everywhere except at $x = t^\gamma$, where it has a jump of the x-derivative, $s_1(t)$ a bounded differentiable function such that $\lim_{t\to+\infty} s_1(t) \in (-\infty, 0)$, and so that

$$\begin{aligned}\left|\left(\partial_t u_{app} - \partial^2_{xx} u_{app} - u_{app} + u^2_{app}\right)(t, x + \widetilde{\sigma}(t))\right| &\le C_\gamma t^{-1+2\gamma}\left(e^{-x}\mathbf{1}_{0<x<t^\gamma} + \mathbf{1}_{x<0}\right) \\ &+ C_\gamma t^{-3/2} e^{-x-x^2/((4+\gamma)t)}\mathbf{1}_{x>t^\gamma} + C_\gamma t^{-1+2\gamma}\delta(x - t^\gamma).\end{aligned} \tag{1.8}$$

The estimate in the right side includes the spatial behavior of the error – this is needed in the region where the solution is small. The different error sizes in the regions $x < t^\gamma$ and $x > t^\gamma$ in (1.8) come about because we need less precision in approximating the solution to the left of $x = t^\gamma$, where u is either $O(1)$ or not too small, than to the right of $x = t^\gamma$, where u is "very small". The delta function in the last term in the right side is not an issue, and can be, in principle, eliminated by a modification of the approximate solution. With this result in hand, the next task is to prove that the solutions of (1.1) converge to a shift of u_{app} at a certain rate. We could take Theorem 1.1 as a starting point. We are, however, going to start from a little further and provide an alternative proof to it. There are two reasons for this: first, [6] and [7] use probabilistic arguments, and it is natural to use a PDE proof. Second, the proof of Theorem 1.1 will give the correct insight for the refined asymptotics.

Recall that the KPP result (1.5) means the following: if $\xi \in \mathbb{R}$ is the point where $\phi_*(\xi) = 1/2$, then, with definition (1.6), we have

$$|u(t, x + \sigma(t)) - \phi_*(x + \xi)| \to 0 \text{ as } t \to +\infty.$$

Our starting point is the main result of [14], which makes the KPP result precise up to $O(1)$ terms:

Theorem 1.3 ([14]) *We have $\sigma(t) = 2t - \frac{3}{2}\log t + O(1)$ as $t \to +\infty$.*

Let us say that a point x' is a "potential shift" if there exists a sequence $t_k \to +\infty$ such that

$$|u(t_k, x) - \phi_*(x - 2t_k + \frac{3}{2}\log t_k - x')| \to 0 \text{ as } k \to +\infty.$$

Theorem 1.3, together with Berestycki-Hamel [4], implies that that the set of all potential shifts is a bounded, closed interval I_∞ of $\mathbb{R}$. The first result that we are going to prove is

Theorem 1.4 *The set I_∞ is a point: $I_\infty = \{x_\infty\}$.*

This is exactly Theorem 1.1. Once we have the result, we will improve it to the following refined asymptotics, which is our second main result:

Theorem 1.5 *There exists $a > 0$ so that for all $\gamma > 0$, there is $C_\gamma > 0$ such that, for all $t \ge 0$ and all $x \in \mathbb{R}$, we have*

$$|u(t,x) - u_{app}(t, x + x_\infty)| \le \frac{C_\gamma}{t^{1-a\gamma}}. \tag{1.9}$$

We give two corollaries of this theorem. The first one is an improvement of Bramson's asymptotics in Theorem 1.1, which, in addition to the preceding theorem, simply relies on the fact that ϕ_* is a strictly decreasing function.

Corollary 1.6 *There is a shift x_∞, depending on the initial condition v_0, and a uniformly bounded function $\chi(x) > 0$ which does not depend on v_0, with the following property: set*

$$\widetilde{u}(t,x) = u\big(t, x + 2t - \frac{3}{2}\log t + x_\infty\big). \tag{1.10}$$

Then for any compact set K and any $\gamma > 0$ there exists $C_{K,\gamma} > 0$ so that

$$\Big|\widetilde{u}(t,x) - \phi_*\big(x + \frac{\chi(x)}{\sqrt{t}}\big)\Big| \le \frac{C_{K,\gamma}}{t^{1-\gamma}}. \tag{1.11}$$

The precise form of the function $\chi(x)$ is given in (5.5)-(5.6) below. Corollary 1.6 may be re-interpreted as follows:

Corollary 1.7 *If we fix $s \in (0,1)$ and define the front position as $\sigma_s(t) = \max\{x : \ u(t,x) = s\}$, then $\sigma_s(t)$ has an asymptotics of the form*

$$\sigma_s(t) = 2t - \frac{3}{2}\log t + x_\infty + \phi_*^{-1}(s) - \frac{\alpha_s}{\sqrt{t}} + O(\frac{1}{t^{1-\delta}}), \tag{1.12}$$

with a constant $\alpha_s > 0$ which depends on the level set s but not on the initial condition v_0.

Comparison with previous asymptotics

In [10], the authors use formal arguments to show that the velocity of the solution behaves as

$$v(t) := 2 - \frac{3}{2t} + \frac{3\sqrt{\pi}}{2t^{3/2}}.$$

In fact, they have a more general expression coming from the fact that their nonlinearity is more general than $u - u^2$, and their diffusion coefficient does not equal 1. This looks different from the expression (5.8) below that we obtain. However, the conclusion of their analysis is that $u(t,x)$ behaves, up to $O(t^{-1})$ error, like

$$\phi_{v(t)}\Big(x - \int_0^t v(t')dt'\Big),$$

where ϕ_v is a (not everywhere positive) traveling wave moving with the speed $v < 2$, v close to 2. With this choice of the normalization, their conclusion should coincide with ours.

The $3\sqrt{\pi}$ prediction has been verified by C. Henderson in [17], for a linearized moving boundary problem:

$$\begin{aligned} &U_t - U_{xx} = U, \quad t > 0, x > \sigma(t), \\ &U(t,\sigma(t)) = 0, \end{aligned} \tag{1.13}$$

with, for instance, an initial datum of the form $1 - H(x)$. The Dirichlet boundary condition serves the same purpose as the term $(-u^2)$ in the KPP equation – when the moving boundary is chosen "correctly", the solution of (1.13) does not grow or decay in time. Both solutions of (1.1) and (1.13) are governed by the "far ahead" tails where they are small – these are so called pulled fronts. The difference between (1.13) and the full KPP problem on the whole line is that (1.1) has an "inner"

layer where the solution transitions from $O(1)$ to very small values. The main result of [17] is as follows: assume $\sigma(t)$ has the form

$$\sigma(t) = 2t - \frac{3}{2}\log t - \frac{c}{\sqrt{t}}, \quad t \geq 1. \tag{1.14}$$

Then, if $c = 3\sqrt{\pi}$, there is $\alpha_0 > 0$ such that

$$\left| \int_{\sigma(t)}^{+\infty} U(t,x)dx - \alpha_0 \right| \leq \frac{C\log t}{t}.$$

On the other hand, if $c \neq 3\sqrt{\pi}$, the convergence rate is of the order $1/\sqrt{t}$. We refer to the recent preprint [5] for a very detailed study of the same problem, according to the behavior of the initial condition at infinity.

Probabilistic links and some related models

The time delay in models of the Fisher-KPP type has been the subject of various recent investigations, both from the PDE and probabilistic points of view. The Fisher-KPP equation appears in the theory of the branching Brownian motion (BBM) [21] as follows. Consider a BBM starting at $x = 0$ at time $t = 0$, with binary branching at rate 1. Let $X_1(t), \ldots, X_{N_t}(t)$ be the descendants of the original particle at time t, arranged in the increasing order: $X_1(t) \leq X_2(t) \leq \cdots \leq X_{N_t}(t)$. Then, the probability distribution function of the maximum:

$$v(t,x) = \mathbb{P}(X_{N_t}(t) > x),$$

satisfies the Fisher-KPP equation

$$v_t = \frac{1}{2}v_{xx} + v - v^2,$$

with the initial data $v_0(x) = \mathbb{1}_{x\leq 0}$. Therefore, the statement (1.7) and its refinement (1.12) are about the median location of the maximal particle X_{N_t}. Building on work of Lalley and Sellke [19], recent probabilistic analyses [9, 8, 2, 3, 1] of this particle system have identified a decorated Poisson-type point process which is the limit of the particle distribution from "seen from the tip": there is a random variable $Z > 0$ such that the point process defined by the shifted particles $\{X_1(t) - c(t), \ldots, X_{N_t}(t) - c(t)\}$, where $c(t) = 2t - \frac{3}{2}\log t + \log Z$, has a well-defined limit process as $t \to \infty$. Furthermore, Z is the limit of the martingale $Z_t = \sum_k (2t - X_k(t))e^{X_k(t)-2t}$, and $\phi_*(x) = 1 - \mathbb{E}[e^{-Ze^{-x}}]$ for all $x \in \mathbb{R}$ [19].

The logarithmic term in (1.7) arises in inhomogeneous variants of this model. For example, consider the Fisher-KPP equation in a periodic medium:

$$u_t - u_{xx} = \mu(x)(u - u^2) \tag{1.15}$$

where $\mu(x)$ is continuous and 1-periodic in $\mathbb{R}$, such that the first periodic eigenvalue of the operator $-\partial_{xx} - \mu(x)$ is negative. Then there is a minimal speed $c_* > 0$ such that for each $c \geq c_*$, there is a unique pulsating front $U_c(t,x)$, up to a time shift [4, 16]. It was shown in [15] that there is $s_0 > 0$ such that, if $u(t,x)$ solves (1.15) with a nonnegative, nonzero, compactly supported initial condition $u_0(x)$, and if, for $0 < s \leq s_0$, $\sigma_s(t)$ is defined as in Corollary 1.7, then

$$\sigma_s(t) = c_* t - \frac{3}{2\lambda_*}\log t + O(1).$$

This implies the convergence of $u(t, x-\sigma_s(t))$ to a closed subset of the family of minimal fronts. It is an open problem to determine whether convergence to a single front holds, not to mention the rate of this convergence. When $\mu(x) > 0$ everywhere, the solution u may be interpreted in terms of the extremal particle in a BBM with a spatially-varying branching rate [15]. We refer also to [11, 22, 20] for recent work on related models with temporal variation in the branching process, which leads to asymptotic questions about a FKPP-type equation with temporally varying diffusion coefficient.

Organization of the paper. In Section 2, we explain, in an informal way, why the results are likely to hold. We then prove Theorem 1.2 in Section 3, where we construct the approximate solution. In Section 4 we prove the convergence to a single wave. In Section 5, we use the approximate solution to prove Theorem 1.5 and its corollaries.

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2 Heuristics and strategy of the proofs

We first provide an informal argument for the convergence of the solution of the initial value problem to a traveling wave. Consider the Cauchy problem (1.1) starting at $t = 1$ for convenience of the notation:

$$\begin{aligned} &u_t - u_{xx} = u - u^2, \quad x \in \mathbb{R}, \quad t > 1, \\ &u(1, x) = u_1(x) = 1 - H(x) + v_0(x), \quad v_0 \text{ compactly supported}, \end{aligned} \tag{2.1}$$

and proceed with a standard sequence of changes of variables. The first change:

$$x \mapsto x - 2t + (3/2)\log t,$$

brings the problem into the reference frame moving as $2t - (3/2)\log t$:

$$u_t - u_{xx} - (2 - \frac{3}{2t})u_x = u - u^2. \tag{2.2}$$

Next, we take out the exponential factor: set

$$u(t, x) = e^{-x}v(t, x)$$

so that v solves

$$v_t - v_{xx} - \frac{3}{2t}(v - v_x) + e^{-x}v^2 = 0, \qquad x \in \mathbb{R}, \quad t > 1. \tag{2.3}$$

Along with the initial condition $v(1, x)$, this equation contains all the information on u. Observe that for any $x_\infty \in \mathbb{R}$, the function $V(x) = e^x\phi(x - x_\infty)$ satisfies

$$V_t - V_{xx} + e^{-x}V^2 = 0,$$

and we regard (2.3) as a perturbation of this equation. Although $v(1, x) = 0$ for $x > 0$ sufficiently large, we expect that $v(t, x) \to e^x\phi(x - x_\infty)$ as $t \to \infty$. In particular, we expect that v converges to an asymptotically linear profile for $x > 0$, since

$$e^x\phi(x - x_\infty) \sim xe^{x_\infty} \quad \text{as } x \to +\infty.$$

For $x \gg 1$, the term $e^{-x}v^2$ in (2.3) is negligible, while for $x \ll -1$ the same term $e^{-x}v^2$ will create a large absorption and force the solution to zero unless v is not very tiny already. For this reason, the linear Dirichlet problem

$$\begin{aligned} &z_t - z_{xx} - \frac{3}{2t}(z - z_x) = 0, \qquad x > 0 \\ &z(t,0) = 0 \end{aligned} \tag{2.4}$$

is a reasonable proxy for (2.3) for $x \gg 1$, and, as shown in [14, 15], it provides good sub- and super-solutions for $v(t,x)$. The main lessons of [14, 15] are that everything relevant to solutions of (2.4) happens at the spatial scale $x \sim \sqrt{t}$, and their asymptotics may be unraveled by a self-similar change of variables. Here, instead, we will accept the full nonlinear equation (2.3) and perform directly the self-similar change of variables

$$\tau = \log t, \quad \eta = \frac{x}{\sqrt{t}} \tag{2.5}$$

followed by the change of unknowns

$$v(\tau,\eta) = e^{\tau/2} w(\tau,\eta).$$

This transforms (2.3) into

$$w_\tau - \frac{\eta}{2} w_\eta - w_{\eta\eta} - w + \frac{3}{2} e^{-\tau/2} w_\eta + e^{3\tau/2 - \eta \exp(\tau/2)} w^2 = 0, \quad \eta \in \mathbb{R}, \quad \tau > 0. \tag{2.6}$$

This makes clear why the Dirichlet problem (2.4) appears naturally: for $\eta < 0$, the last term in the left side of (2.6) becomes very large, which forces w to be almost 0 in this region. On the other hand, for $\eta > 0$, this term is very small, so it should not play any role in the dynamics of w for $\eta > 0$.

Convergence to a single wave

Through this change of variables, we see why a particular translation of the wave will be chosen. Considering (2.4) in the self-similar variables, one can show – see [14, 17] – that $e^{-\tau/2} z(\tau,\eta)$ converges to a steady state of the form

$$e^{-\tau/2} z(\tau,\eta) \sim \alpha_\infty \eta e^{-\eta^2/4}, \quad \eta > 0. \tag{2.7}$$

Therefore, taking (2.4) as an approximation to (2.3), we should expect that

$$u(t,x) = e^{-x} v(t,x) \sim e^{-x} z(t,x) \sim e^{-x} e^{\tau/2} \alpha_\infty \eta e^{-\eta^2/4} = \alpha_\infty x e^{-x} e^{-x^2/(4t)}, \tag{2.8}$$

at least for x of order $O(\sqrt{t})$. This determines the unique translation: if we accept that u converges to a translate x_∞ of ϕ_*, then for large x (in the moving frame) we have

$$u(t,x) \sim \phi_*(x - x_\infty) \sim x e^{-x + x_\infty}. \tag{2.9}$$

Comparing this with (2.8), we infer that

$$x_\infty = \log \alpha_\infty.$$

The difficulty with this argument is that each asymptotics uses different ranges of x: (2.8) comes from the self-similar variables and uses $x \sim \sqrt{t}$, while (2.9) requires x to be large but finite – but the self-similar analysis does not tell us at this stage what happens on the scale $x \sim O(1)$. Indeed, it is clear from (2.6) that the error in the approximation (2.7) is at least of the order $O(e^{-\tau/2})$ –

note that the right side in (2.7) is a solution of (2.6) without the last two terms in the left side. On other hand, the scale $x \sim O(1)$ corresponds to $\eta \sim e^{-\tau/2}$. Thus, the leading order term and the error in (2.7) are of the same size for $x \sim O(1)$, which means that we can not extract information directly from (2.7) on that scale.

To overcome these issues, we proceed in two steps: first we use the self-similar variables to prove stabilization (i.e. that (2.8) holds) at the spatial scales $x \sim O(t^\gamma)$ with $\gamma > 0$, and not just at the diffusive scale $O(\sqrt{t})$. This boils down to the statement that $w(\tau, \eta) \sim \alpha_\infty \eta e^{-\eta^2/4}$ for the solution to (2.6), even when $\eta \sim e^{-(1/2-\gamma)\tau}$. Next, we show that stabilization at the scale $x \sim O(t^\gamma)$ is sufficient to ensure the stabilization on the scale $x \sim O(1)$ and convergence to a unique wave. This is the core of the argument, and it is logical: everything happening at $x \sim O(1)$ should be governed by the tail of the solution, the spatial scales $x \sim \sqrt{t}$. Once this analysis is done and the translate $x_\infty = \log \alpha_\infty$ is fixed, this determines the strategy for the more refined analysis in rest of the paper.

The approximate solution

The above argument gives the right insight for the construction of the approximate solution. The idea is to view $1/\sqrt{t}$ as a small parameter, in terms of which which one may expand the solution. In view of the above considerations, it is natural to identify two zones: the region near the front, that is, $x \sim O(1)$, and the diffusive region, where $x \sim \sqrt{t}$. The transition region is $x \sim t^\gamma$, $\gamma > 0$ small. We perform a classical asymptotic expansion of an inner solution (approximating u near the front) and an outer solution (approximating u in the leading edge, at distances $O(\sqrt{t})$ from the front). Matching between the inner and outer expansion is done in the intermediate region $x \sim t^\gamma$.

The sharper asymptotics

Once a translate x_∞ is selected, this also determines the translate of the approximate solution to which the solution is supposed to converge, with a rate faster than $t^{\gamma-1}$, for all small γ. Everything reduces to proving, from the equation, that the difference between the true solution and the approximate solution will not exceed $t^{\gamma-1}$. The argument is long and technical, and is carried out in the self-similar variables (2.5). However, it really relies on two simple ideas. The first is to transform the problem on the whole line into a Dirichlet problem on the half line, by a classical sequence of transformations and the final subtraction of the value of u at t^γ. Once this is done, we seek to apply a center manifold type argument. The trouble is that the nonlinear term u^2 in the original equation (1.1) provides, as is usual, a term which may grow like $e^{\tau/2}$. The difficulty is overcome by noticing that its support shrinks as $e^{-\tau/2}$. And so, a large part of the proof is devoted to estimating this term in the best way. For that, we first obtain weak estimates on $u - u_{app}$, which still yield an improvement of the nonlinear term. This improvement entails a better estimate on $u - u_{app}$, and so on. As we have mentioned, this is coupled to a center manifold type argument, and the technical details are nontrivial.

3 The approximate solution

The goal of this section is to construct an approximate solution to the KPP equation that we will use in the proof of Theorem 1.5. We will work with equation (2.3), in the moving frame:

$$v_t - v_{xx} - \frac{3}{2t}(v - v_x) + e^{-x}v^2 = 0. \tag{3.1}$$

Let us denote this nonlinear operator as

$$NL[v] = v_t - v_{xx} - \frac{3}{2t}(v - v_x) + e^{-x}v^2.$$

We will construct an approximate solution to (3.1) in the form

$$V_{app}(t,x) = V(t, x - s(t)).$$

If V_{app} were an exact solution to (3.1), then $V(t,x)$ would solve

$$V_t - V_{xx} - (\dot{s}(t) - \frac{3}{2t})V_x - \frac{3}{2t}V + e^{-(x+s(t))}V^2 = 0. \tag{3.2}$$

Our goal will be to ensure that the approximate solution satisfies

$$V_t - V_{xx} - (\dot{s}(t) - \frac{3}{2t})V_x - \frac{3}{2t}V + e^{-(x+s(t))}V^2 = O(t^{-1+2\gamma}). \tag{3.3}$$

Later we will assume without loss of generality that the Bramson shift x_∞ obtained in Theorem 1.4 vanishes: $x_\infty = 0$. The function $s(t)$ should be thought of as a refinement of the Bramson shift, and will be sought in the form

$$s(t) = \frac{s_1(t)}{\sqrt{t}} \tag{3.4}$$

with a slowly varying function $s_1(t)$ that will have a limit $s_1(\infty) \neq 0$ as $t \to +\infty$.

As we have mentioned, it is natural to consider an intermediate scale $x \sim O(t^\gamma)$, with some $\gamma > 0$, and construct two different approximate solutions to (3.2): one valid for $x \leq t^\gamma$, the other valid for $x \geq t^\gamma$:

$$V(t,x) = V^-(t,x) \text{ for } x < t^\gamma, \qquad V(t,x) = V^+(t,x) \text{ for } x > t^\gamma.$$

The functions V^- and V^+ will be matched at $x = t^\gamma$.

3.1 The inner approximate solution V^-

We will look for the function V^- in the form

$$V^-(t,x) = V_0^-(x) + \frac{1}{\sqrt{t}}V_1^-(t,x), \quad x < t^\gamma. \tag{3.5}$$

We will ask V_1^- to be slowly varying in time, so that $\partial_t V_1^-$ does not appear in the computations. Inserting expression (3.5) into (3.3), equating the terms of the orders $O(1)$ and $O(t^{-1/2})$, and dropping the higher order terms, yields, respectively:

$$-(V_0^-)_{xx} + e^{-x}(V_0^-)^2 = 0, \tag{3.6}$$

and

$$-(V_1^-)_{xx} + 2e^{-x}V_0^-V_1^- = s_1 e^{-x}(V_0^-)^2. \tag{3.7}$$

The condition $V_i^-(-\infty) = 0$ should also be satisfied.

With this choice, we define

$$V_{app}^-(t,x) = V^-(t, x - s(t)),$$

and we compute

$$
\begin{aligned}
&NL[V^-_{app}](t,x+s(t)) = V^-_t - V^-_{xx} - (\dot s(t) - \frac{3}{2t})V^-_x - \frac{3}{2t}V + e^{-(x+s(t))}(V^-)^2 \\
&= V^-_t(t,x) - \dot s(t)V^-_x(t,x) - (V^-_0)_{xx} - \frac{1}{\sqrt t}(V^-_1)_{xx} - \frac{3}{2t}(V^-(t,x) - V^-_x(t,x)) \\
&+e^{-x}(1-\frac{s_1(t)}{\sqrt t})\Big(V^-_0(x) + \frac{1}{\sqrt t}V^-_1(t,x)\Big)^2 + e^{-x}(e^{-s(t)} - 1 + \frac{s_1(t)}{\sqrt t})\Big(V^-_0(x) + \frac{1}{\sqrt t}V^-_1(t,x)\Big)^2 \\
&= V^-_t(t,x) - \dot s(t)V^-_x(t,x) - \frac{3}{2t}(V^-(t,x) - V^-_x(t,x)) + (1-\frac{s_1(t)}{\sqrt t})\frac{e^{-x}(V^-_1(t,x))^2}{t} \qquad (3.8) \\
&+e^{-x}(e^{-s(t)} - 1 + \frac{s_1(t)}{\sqrt t})\Big(V^-_0(x) + \frac{1}{\sqrt t}V^-_1(t,x)\Big)^2 - e^{-x}\frac{2s_1(t)}{t}V^-_0(x)V^-_1(x).
\end{aligned}
$$

Our task will be to choose solutions to (3.6) and (3.7) so that

$$NL[V^-_{app}] = O(t^{-1+2\gamma}), \qquad \text{for } x < O(t^\gamma). \tag{3.9}$$

Equation (3.6) implies that

$$V^-_0(x) = e^x\phi_*(x+x_m),$$

with an arbitrary x_m. Naturally, we will choose $x_m = x_\infty$, the value that we know exists from Theorem 1.2. We will also assume without loss of generality that

$$x_\infty = 0, \tag{3.10}$$

and thus

$$V^-_0(x) = e^x\phi_*(x). \tag{3.11}$$

As for (3.7), note that when $s_1 = 0$ it has a solution

$$\phi_1(x) = e^x\phi'_*(x).$$

Therefore, $V^-_1(t,x)$ has the form

$$V^-_1(t,x) = q^-_1(t)\phi_1(x) + v_1(t,x), \tag{3.12}$$

with $v_1(t,x)$ a particular solution of

$$-(v_1)_{xx} + 2e^{-x}V^-_0 v_1 = s_1(t)e^{-x}(V^-_0)^2, \tag{3.13}$$

solved "t by t". The time dependence of v_1 comes from the boundary conditions at $x = t^\gamma$ and the time-dependence of $s_1(t)$. We anticipate that the function

$$V^-_0(x) + t^{-1/2}q^-_1(t)\phi_1(x)$$

will almost match V^+ (still to be constructed) at $x = t^\gamma$, but that there will be an $O(1)$ discrepancy. As V^-_0 is growing on the right (see (3.16) below), this mismatch, to be corrected with the help of v_1, is already "relatively small". To account for it without polluting the matching of the derivatives (which we anticipate to be almost exact), we impose the following boundary conditions on v_1:

$$v_1(t,-\infty) = 0, \quad v_1(t,t^\gamma) = \zeta(t), \quad v_{1,x}(t,t^\gamma) = 0. \tag{3.14}$$

The nearly constant function $\zeta(t)$ will be fixed later. The boundary conditions on $v_1(-\infty)$ and $v_1(t^\gamma)$ yield (look for v_1 in the form $\phi_1 w_1$)

$$v_1(t,x) = \zeta(t)\frac{\phi_1(x)}{\phi_1(t^\gamma)} + s_1(t)\phi_1(x)\int_x^{t^\gamma} \frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy. \tag{3.15}$$

In order to estimate the remainder, recall that, as $x \to +\infty$, we have the asymptotics – see (1.4):

$$\begin{aligned} &V_0^-(x) = x + k + O(e^{-\omega_0 x}) \\ &\phi_1(x) = -x - k + 1 + O(e^{-\omega_0 x}) \\ &V_0^{-\prime}(x) = 1 + O(e^{-\omega_0 x}), \\ &\phi_1'(x) = -1 + O(e^{-\omega_0 x}). \end{aligned} \tag{3.16}$$

Note that, as $x \to -\infty$ we have, with some $\beta > 0$,

$$\phi_*(x) \sim 1 - Ce^{\beta x}, \quad V_0^-(x) \sim e^x, \quad \phi_1(x) \sim -Ce^{(1+\beta)x} \text{ as } x \to -\infty,$$

thus, for y negative we have

$$\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz \sim \int_{-\infty}^y e^{-z}e^{(1+\beta)z}e^{2z}dz \sim \frac{e^{(2+\beta)y}}{2+\beta}. \tag{3.17}$$

Then, for x positive we have

$$\phi_1(x)\int_x^{t^\gamma} \frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy \sim Cx\int_x^{t^\gamma}\frac{dy}{1+y^2} \sim Cx \tag{3.18}$$

and for negative x

$$\phi_1(x)\int_x^{t^\gamma} \frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy \sim e^{(1+\beta)x}\Big(1 + \int_x^0 e^{-(2+2\beta)y}e^{(2+\beta)y}\Big) \sim e^x. \tag{3.19}$$

We conclude that $v_1(t,x)$ grows linearly in x on the right, and decays as e^x on the left.

The function $s_1(t)$ will, at this stage, remain free. However, $\zeta(t)$ is determined by $s_1(t)$ via a compatibility condition. Indeed, the boundary condition $v_{1,x}(t^\gamma) = 0$ means that

$$\zeta(t)\phi_1'(t^\gamma) = s_1(t)\int_{-\infty}^{t^\gamma} e^{-z}\phi_1(z)(V_0^-(z))^2dz. \tag{3.20}$$

and thus

$$\frac{s_1(t)}{\zeta(t)} = \phi_1'(t^\gamma)\Big(\int_{-\infty}^{t^\gamma} e^{-z}\phi_1(z)(V_0^-(z))^2dz\Big)^{-1} = \frac{1}{I_1} + O(e^{-\omega_0 t^\gamma}), \tag{3.21}$$

with

$$I_1 = \int_{\mathbb{R}} e^{-z}|\phi_1(z)|(V_0^-(z))^2dz. \tag{3.22}$$

Recall that ϕ_1' is given by (3.16) and $\phi_1 < 0$. Therefore, we have

$$v_1(t,x) = \zeta(t)\Big[\frac{\phi_1(x)}{\phi_1(t^\gamma)} + \frac{\phi_1(x)}{I_1 + O(e^{-\omega_0 t^\gamma})}\int_x^{t^\gamma}\frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy\Big]. \tag{3.23}$$

From now on, we set $v_1^-(t,x) := v_1(t,x)$. The function V^- is thus chosen as

$$V^-(t,x) = V_0^-(x) + t^{-\frac{1}{2}}(q_1^-(t)\phi_1(x) + v_1^-(t,x)).$$

The values of $q_1^-(t)$ and $\zeta(t)$ remain to be chosen. Note that V^- grows linearly in x as x approaches t^γ from the left, and decays like e^x, as $x \to -\infty$, uniformly with respect to t. For the time derivative we have

$$\frac{\partial v_1(t,x)}{\partial t} = \dot\zeta(t)\frac{\phi_1(x)}{\phi_1(t^\gamma)} - \gamma t^{-1+\gamma}\zeta(t)\frac{\phi_1(x)\phi_1'(t^\gamma)}{\phi_1^2(t^\gamma)} + \frac{\gamma t^{-1+\gamma}s_1(t)\phi_1(x)}{\phi_1^2(t^\gamma)}\int_{-\infty}^{t^\gamma} e^{-z}\phi_1(z)(V_0^-)^2(z)dz$$
$$+\dot s_1(t)\phi_1(x)\int_x^{t^\gamma}\frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy \sim \dot\zeta(t)O\Big(\frac{x}{t^\gamma}\Big) + O\Big(\frac{x}{t^{1+\gamma}}\Big) + \dot s_1(t)O(x), \quad (3.24)$$

for large x.

Estimating $NL[V_{app}^-]$

Let us now go back to expression (3.8) for $NL[V_{app}^-]$, and estimate all its terms. With our choice of V^- we know from (3.24) that

$$\frac{\partial V^-}{\partial t} = \frac{\partial}{\partial t}\Big(\frac{1}{\sqrt{t}}(q_1^-\phi_1(x) + v_1^-(t,x))\Big) = O(t^{-1}), \quad (3.25)$$

as long as we keep $\dot\zeta(t)$ and $\dot s_1(t)$ sufficiently small. The second term in (3.8) is of the order of $\dot s(t)$, which will make it $O(t^{-1})$. The third term is of the order $O(x/t) \sim O(t^{-1+\gamma})$ for $x \le O(t^\gamma)$. The fourth term is exponentially small for $x \gg 1$, and $x \ll -1$, and is of the order $O(1/t)$ for $x \sim O(1)$. The fifth term is of the order $O(x^2s^2) \sim O(x^2/t) \sim O(t^{-1+2\gamma})$ for $x \le O(t^\gamma)$, while the last term is of the order $O(1/t)$. In addition, all terms decay as e^x for $x < 0$. Thus, we have

$$NL[V_{app}^-](t, x+s(t)) = n_1(t,x)(\mathbf{1}_{0<x<2t^\gamma}(x) + \mathbf{1}_{\mathbb{R}_-}(x)e^x), \qquad x \le 2t^\gamma, \quad (3.26)$$

with

$$|n_1(t,x)| \le Ct^{-1+2\gamma}. \quad (3.27)$$

3.2 The outer approximate solution V^+

In the outer region $x > t^\gamma$, we pass to the self-similar variables

$$\tau = \log t, \quad \eta = \frac{x+x_0}{\sqrt{t}}, \quad (3.28)$$

the point x_0 being kept free for the moment. Our starting point is (3.2):

$$V_t - V_{xx} - (\dot s(t) - \frac{3}{2t})V_x - \frac{3}{2t}V + e^{-(x+s(t))}V^2 = 0. \quad (3.29)$$

In the self-similar variables, the equation for V^+ is

$$v_\tau + (L - \frac{1}{2})v + (\frac{3}{2} - \dot s(t)e^\tau)e^{-\tau/2}v_\eta + e^{\tau - \eta e^{\tau/2} + x_0 - s(t)}v^2 = 0. \quad (3.30)$$

We have set

$$Lv = -v_{\eta\eta} - \frac{\eta}{2}v_\eta - v.$$

Note that

$$\dot s(t) \sim -\frac{s_1}{2t^{3/2}} = \frac{s_1}{2}e^{-3\tau/2},$$

thus the term $\dot{s}(t)e^{\tau}$ in (3.30) is of the order $e^{-\tau/2}$.

As in the construction of V^-_{app}, we are not going to solve (3.30) exactly, but find an asymptotic expansion for an approximate solution. Strictly speaking, we only need V^+ defined for $x > t^\gamma$, that is, for $\eta > e^{-(1/2-\gamma)\tau}$ but we will define it for $\eta \geq 0$. We impose the boundary condition

$$V^+(\tau, 0) = 0, \tag{3.31}$$

which is consistent with the presence of the absorption term $e^{\tau-\eta e^{\tau/2}}v^2$ in the left side of (3.30), which is huge as soon as η is just a little negative. As $V^-(t,x)$ is of the order $O(t^\gamma)$ at $x = t^\gamma$, to have a hope of a good matching we need

$$V^+(\tau, e^{-(1/2-\gamma)\tau}) \sim e^{\gamma\tau}.$$

On the other hand, the boundary condition (3.31) means that

$$V^+(\tau, e^{-(1/2-\gamma)\tau}) \sim \frac{\partial V^+(\tau,0)}{\partial \eta} e^{-(1/2-\gamma)\tau},$$

thus we need

$$\frac{\partial V^+(\tau,0)}{\partial \eta} \sim e^{\tau/2}.$$

Hence, it is natural to look for V^+ in the form

$$V^+(\tau,\eta) = e^{\tau/2}V_0^+(\eta) + V_1^+(\eta).$$

Inserting this ansatz into (3.30) and collecting the leading order terms gives

$$LV_0^+ = 0, \tag{3.32}$$

and

$$(L - \frac{1}{2})V_1^+ + \frac{3}{2}(V_0^+)_\eta = 0, \tag{3.33}$$

with the boundary conditions

$$V_i^+(0) = V_i^+(+\infty) = 0, \quad i = 0, 1. \tag{3.34}$$

Setting

$$e_0(\eta) = \eta e^{-\eta^2/4} \text{ for } \eta > 0,$$

we have

$$V_0^+(\eta) = q_0^+ e_0(\eta), \tag{3.35}$$

the constant q_0^+ being for the moment free. Once V_0^+ is fixed, there is a unique solution V_1^+ to (3.33), with $e^{\eta^2/(4+\gamma)}V_1 \in L^2(\mathbb{R}_+)$, because the spectrum of L is $\{0, 1, 2, \dots\}$.

We will need the derivative $(V_1^+)_\eta(0)$ for the matching procedure. Integrating (3.33) gives

$$(V_1^+)_\eta(0) = \int_0^\infty V_1^+(\eta)d\eta, \tag{3.36}$$

while multiplying (3.33) by $\eta^2/2$ and integrating gives

$$\int_0^\infty V_1^+(\eta)d\eta = \frac{3}{4}\int_0^\infty \eta^2(V_0^+)_\eta(\eta)d\eta = -\frac{3q_0^+}{2}\int_0^\infty \eta^2 e^{-\eta^2/4}d\eta = -3q_0^+\sqrt{\pi}. \tag{3.37}$$

Therefore, we have

$$(V_1^+)_\eta(0) = -3q_0^+\sqrt{\pi}. \tag{3.38}$$

Estimating $NL[V^+]$

Let us denote by $\mathcal{NL}[v]$ the nonlinear operator in the left side of (3.30). Then we have

$$|\mathcal{NL}[V^+]| \le Ce^{-\tau/2}\mathbf{1}_{\mathbb{R}_+}(\eta)e^{-\eta^2/(4+\gamma)}. \tag{3.39}$$

In the original variables the function V^+ has the form

$$V^+(t,x) = q_0^+(x+x_0)e^{-(x+x_0)^2/(4t)} + V_1^+\Big(\frac{x+x_0}{\sqrt{t}}\Big), \tag{3.40}$$

and (3.39) implies that

$$|NL[V^+](t,x+s(t))| \le Ct^{-3/2}\mathbf{1}_{\{x+x_0>0\}}e^{-(x+x_0)^2/((4+\gamma)t)}, \qquad \text{for } x \ge -x_0. \tag{3.41}$$

3.3 Matching the inner and outer approximate solutions

Our next task is to choose the parameters so that the inner and outer approximate solutions match at $x = t^\gamma$. Ideally, we would like to match both V^- and V^+ and their derivatives at this point. Because V^- and V^+ are of the size $O(t^\gamma)$ in this region, the key is to match V^- and V^+. Recall that we have

$$\begin{aligned} V^-(t,t^\gamma) &= t^\gamma + k + t^{-\frac{1}{2}}\big(q_1^-(-t^\gamma - k + 1) + \zeta(t)\big) + O(e^{-\omega_0 t^\gamma}) \\ &= t^\gamma + k - \frac{q_1^-}{t^{1/2-\gamma}} + \frac{q_1^-(1-k)+\zeta(t)}{t^{1/2}} + O(e^{-\omega_0 t^\gamma}), \end{aligned} \tag{3.42}$$

while for $V^+(t,t^\gamma)$, using expression (3.38) we get

$$\begin{aligned} V^+(t,t^\gamma) &= t^{1/2}V_0^+\Big(\frac{t^\gamma+x_0}{\sqrt{t}}\Big) + V_1^+\Big(\frac{t^\gamma+x_0}{\sqrt{t}}\Big) \\ &= q_0^+\Big((t^\gamma+x_0)(1+O(t^{2\gamma-1})) - 3\sqrt{\pi}t^{-1/2}(t^\gamma+x_0)\Big) + O(\frac{1}{t^{1-2\gamma}}) \\ &= q_0^+t^\gamma + q_0^+x_0 - \frac{3\sqrt{\pi}q_0^+}{t^{1/2-\gamma}} - \frac{3\sqrt{\pi}q_0^+x_0}{\sqrt{t}} + O(\frac{1}{t^{1-3\gamma}}). \end{aligned} \tag{3.43}$$

Equating the terms of the order $O(t^\gamma)$ and $O(1)$ gives

$$q_0^+ = 1, \quad x_0 = k. \tag{3.44}$$

Equating the terms of the order $O(t^{-1/2+\gamma})$ gives then

$$q_1^- = 3\sqrt{\pi}. \tag{3.45}$$

Finally, we choose $\zeta(t)$ to eliminate the terms of the order $O(t^{-1/2})$ together with the remainder, which means that

$$\zeta(t) = q_1^-(k-1) - 3\sqrt{\pi}q_0^+x_0 + O(\frac{1}{t^{1/2-3\gamma}}) = -3\sqrt{\pi} + O(\frac{1}{t^{1/2-3\gamma}}). \tag{3.46}$$

It follows from (3.21) that $s_1(t)$ is given by

$$s_1(t) = -\frac{3\sqrt{\pi}}{I_1} + O(\frac{1}{t^{1/2-3\gamma}}). \tag{3.47}$$

Choosing the parameters in this way, we have matched V^+ and V^- at $x = t^\gamma$:

$$V^+(t, t^\gamma) = V^-(t, t^\gamma),$$

but we have no freedom left in terms of the parameters to match their derivatives at this point. This is a relatively minor inconvenience as $NL[V^{app}]$ would then have a Dirac mass, of the size proportional to the jump in the derivatives. Taking into account (3.44)-(3.46), we see that these derivatives are given by:

$$V_x^+(t, t^\gamma) = e^{-(t^\gamma+k)^2/(4t)} - \frac{(t^\gamma + k)^2}{2t} e^{-(t^\gamma+k)^2/(4t)} + \frac{1}{\sqrt{t}}(V_1^+)_x\Big(\frac{t^\gamma + k}{\sqrt{t}}\Big) = 1 - \frac{3\sqrt{\pi}}{\sqrt{t}} + O(\frac{1}{t^{1-2\gamma}}), \tag{3.48}$$

and,

$$V_x^-(t, t^\gamma) = (V_0^-)'(t^\gamma) + \frac{1}{\sqrt{t}}(3\sqrt{\pi}\phi_1'(t^\gamma) + v_{1,x}^-(t, t^\gamma)) = 1 - \frac{3\sqrt{\pi}}{\sqrt{t}} + O(e^{-\omega_0 t^\gamma}). \tag{3.49}$$

Here, we used, in addition, the asymptotics (3.16) and the boundary condition (3.14):

$$v_{1,x}(t, t^\gamma) = 0. \tag{3.50}$$

We conclude that with our choice of V^+ and V^- the jump in the derivatives is very small:

$$V_x^+(t, t^\gamma) - V_x^-(t, t^\gamma) \sim O(\frac{1}{t^{1-2\gamma}}). \tag{3.51}$$

We could have avoided this jump by modifying slightly the boundary condition (3.50) that would force the exact matching of the derivatives, at the expense of even longer formulas for $v_1^-(t, x)$.

Summary: The full approximate solution $V^{app}(t, x)$ is defined by

$$V_{app}(t, x + s(t)) = V^-(t, x)\mathbf{1}_{x<t^\gamma} + V^+(t, x)\mathbf{1}_{x\geq t^\gamma} \tag{3.52}$$

with $s(t) \sim -\frac{3\sqrt{\pi}}{I_1\sqrt{t}}$. The inner and outer pieces have the form:

$$V^-(t, x) = e^x\phi_*(x) + t^{-1}3\sqrt{\pi}\left(e^x\phi_*'(x) + v_1^-(t, x)\right)$$

and

$$V^+(t, x) = (x + k)e^{-(x+k)^2/(4t)} + V_1^+\Big(\frac{x + k}{\sqrt{t}}\Big),$$

The function V^+ does not depend on the choice of γ; the function v_1^- does depend on γ, through the boundary condition at t^γ.

Inserting this into equation (3.1) yields, in view of (3.26)-(3.27) and (3.41), and taking into account that we use V^- for $x < t^\gamma$ and V^+ for $x > t^\gamma$:

$$\begin{aligned} |NL[V_{app}](t, x + s(t))| &\leq Ct^{-1+2\gamma}(\mathbf{1}_{0<x<t^\gamma} + e^x\mathbf{1}_{x<0})) + Ct^{-3/2}e^{-x^2/((4+\gamma)t)}\mathbf{1}_{x>t^\gamma} \\ &+ Ct^{-1+2\gamma}\delta(x - t^\gamma). \end{aligned} \tag{3.53}$$

The first two terms come from $NL[V^-]$ and $NL[V^+]$, respectively, while the singular term $\delta(x - t^\gamma)$ comes from the jump (3.51) in the derivative at the matching point $x = t^\gamma$. This estimate of $NL[V_{app}](t, x)$ will be used later in (5.11)

4 Convergence to a single wave

We now turn to the proof of Bramson's result, Theorem 1.4. The main step will be to establish the following.

Lemma 4.1 *There exists a constant α_∞ with the following property. For any $\gamma > 0$ and all $\varepsilon > 0$ we can find T_ε so that for all $t > T_\varepsilon$ we have*

$$|u(t, x_\gamma) - \alpha_\infty x_\gamma e^{-x_\gamma} e^{-x_\gamma^2/(4t)}| \leq \varepsilon x_\gamma e^{-x_\gamma} e^{-x_\gamma^2/(4t)}, \tag{4.1}$$

with $x_\gamma = t^\gamma$.

We postpone the proof of this lemma for the moment, and show how it is used.

We will consider the two "inner" problems

$$\begin{aligned} &u_t - u_{xx} - (2 - \frac{3}{2t})u_x - u + u^2 = 0 \quad x \leq x_\gamma(t) \\ &u(t, x_\gamma) = (\alpha_\infty \pm \varepsilon) x_\gamma e^{-x_\gamma - x_\gamma^2/4t}, \end{aligned} \tag{4.2}$$

and will show that their solutions converge to a pair steady solutions, separated only by an ε-translation. The problem is to understand, for a given $\alpha > 0$ and $t \geq 1$:

$$\begin{aligned} &u_t - u_{xx} - (2 - \frac{3}{2t})u_x - u + u^2 = 0, \quad x \leq x_\gamma(t) \\ &u(t, t^\gamma) = \alpha t^\gamma e^{-t^\gamma - t^{2\gamma-1}/4}. \end{aligned} \tag{4.3}$$

The function $v(t, x) = e^x u(t, x)$ solves

$$\begin{aligned} &v_t - v_{xx} + \frac{3}{2t}(v_x - v) + e^{-x} v^2 = 0, \quad x \leq t^\gamma \\ &v(t, t^\gamma) = \alpha t^\gamma e^{-t^{2\gamma-1}/4}. \end{aligned} \tag{4.4}$$

Since we anticipate that the tail is going to dictate the behavior of u, we choose the translate of the wave that matches exactly the behavior of u at the boundary $x = t^\gamma$: set

$$\psi(t, x) = e^x \phi_*(x + \zeta(t)). \tag{4.5}$$

Recall that $\phi_*(x)$ is the traveling wave profile. We look for a function $\zeta(t)$ such that

$$\psi(t, t^\gamma) = v(t, t^\gamma). \tag{4.6}$$

In view of the expansion (1.4) we should have, with some $\omega_0 > 0$:

$$e^{-\zeta(t)}(t^\gamma + \zeta(t) + k) + O(e^{-\omega_0 t^\gamma}) = \alpha t^\gamma e^{-1/4t^{1-2\gamma}},$$

which implies

$$\zeta(t) = -\log\alpha - (\log\alpha - k)t^{-\gamma} + O(t^{-2\gamma}),$$

and thus (for $\gamma \in (0, 1/3)$):

$$|\dot{\zeta}(t)| \leq \frac{C}{t^{1+\gamma}}.$$

The equation for the function ψ is

$$\psi_t - \psi_{xx} + \frac{3}{2t}(\psi_x - \psi) + e^{-x}\psi^2 = -\dot{\gamma}\psi + \dot{\gamma}\psi_x + \frac{3}{2t}(\psi_x - \psi) = O(\frac{x}{t}) = O(t^{-1+\gamma}), \quad x < t^\gamma.$$

Hence, the difference

$$s(t,x) = v(t,x) - \psi(t,x)$$

satisfies

$$\begin{aligned} &s_t - s_{xx} + \frac{3}{2t}(s_x - s) + e^{-x}(v+\psi)s = O(t^{-1+\gamma}), \quad |x| \le t^\gamma \\ &s(t,-t^\gamma) = O(e^{-t^\gamma}), \quad s(t,t^\gamma) = 0. \end{aligned} \tag{4.7}$$

Proposition 4.2 *We have* $\lim_{t\to+\infty} \sup_{|x|\le t^\gamma} |s(t,x)| = 0$.

Proof. The first Dirichlet eigenvalue for the Laplacian in $(-t^\gamma, t^\gamma)$ is $\pi^2/t^{2\gamma}$, hence trying to investigate (4.7) really amounts, heuristically, to solving the ODE

$$f'(t) + (1-2\gamma)t^{-2\gamma} f = \frac{1}{t^{1-\gamma}},$$

with the solution

$$f(t) = f(1)e^{(-t^{-2\gamma+1}+1)} + \int_1^t s^{\gamma-1} e^{(-t^{-2\gamma+1}+s^{-2\gamma+1})} ds.$$

Note that f tends to 0 a little faster than $t^{3\gamma-1}$ as soon as $\gamma < 1/3$. With this idea in mind, we are going to look for a super-solution to (4.7), in the form

$$\overline{s}(t,x) = t^{-\lambda} \sin(\frac{\pi}{2} + \frac{x}{t^{\gamma+\varepsilon}}), \tag{4.8}$$

where λ, γ and ε will be chosen small enough. We have, for $|x| \le t^\gamma$:

$$\begin{aligned} &\overline{s}(t,x) \sim t^{-\lambda}, \ -\overline{s}_{xx} = t^{-(2\gamma+2\varepsilon)}\overline{s}(t,x), \\ &\overline{s}_t = -\frac{\lambda}{t}\overline{s} + g(t,x), \quad |g(t,x)| \le \frac{C|x|}{t^{\lambda+\gamma+\varepsilon+1}} \le \frac{C}{t^{1+\varepsilon}}\overline{s}(t,x), \end{aligned} \tag{4.9}$$

and

$$\frac{3}{2t}(\overline{s}_x - \overline{s})(t,x) \le Ct^{-1}\overline{s}(t,x). \tag{4.10}$$

Gathering (4.9) and (4.10) we infer the existence of $q > 0$ such that, for t large enough:

$$\left(\partial_t - \partial_{xx} - \frac{3}{2t}(\partial_x - 1)\right)\overline{s}(t,x) \ge qt^{-(2\gamma+2\varepsilon)}\overline{s}(t,x) \ge \frac{q}{2}t^{-(2\gamma+2\varepsilon+\lambda)} \ge O(\frac{1}{t^{1-2\gamma}}),$$

as soon as γ, ε and λ are small enough. Because the right side of (4.7) does not depend on $\overline{s}$, the inequality extends to all $t \ge 1$ by replacing $\overline{s}$ by $A\overline{s}$, A large enough. □

Proof of Theorem 1.1 We are now ready to prove the theorem. Let $u_\alpha(t,x)$ be the solution of (4.3) and let $u(t,x)$ the solution of the original problem (2.2). It follows from Lemma 4.1 that for any $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that, for $t \ge t_\varepsilon$, we have

$$u_{\alpha_\infty - \varepsilon}(t,x) \le u(t,x) \le u_{\alpha_\infty + \varepsilon}(t,x),$$

for all $x \le C_\gamma t^\gamma$. From Proposition 4.2 we have

$$u_{\alpha_\infty \pm \varepsilon}(t,x) = \phi_*(x + \gamma_\pm(t)) + o_{t\to+\infty}(1),$$

uniformly in $x \in (-\infty, t^\gamma)$, with

$$\gamma_\pm(t) = -(1-t^{-\gamma})\log(\alpha_\infty \pm \varepsilon) + O(t^{-2\gamma}).$$

Because $\varepsilon > 0$ is arbitrary, we have in the end

$$\lim_{t\to+\infty} \big(u(t,x) - \phi_*(x + x_\infty)\big) = 0,$$

with $x_\infty = -\log\alpha_\infty$, uniformly on $\mathbb{R}$. This concludes the proof of Theorem 1.2. □

The rest of this section contains the proof of Lemma 4.1.

The proof of Lemma 4.1

We first consider what happens on the diffusive scale $x \sim O(\sqrt{t})$. Our analysis centers on (2.6), which we write as

$$w_\tau + Lw + \frac{3}{2}e^{-\tau/2}w_\eta + e^{3\tau/2-\eta\exp(\tau/2)}w^2 = 0, \quad \eta \in \mathbb{R}, \quad \tau > 0, \tag{4.11}$$

where the operator L is defined as

$$Lv = -v_{\eta\eta} - \frac{\eta}{2}v_\eta - v, \tag{4.12}$$

The principal eigenfunction of L on the half-line $\eta > 0$ with the Dirichlet boundary condition at $\eta = 0$ is

$$\phi_0(\eta) = \frac{\eta}{2}e^{-\eta^2/4},$$

as $L\phi_0 = 0$. The operator L has a discrete spectrum in $L^2(\mathbb{R}_+)$, weighted by $e^{-\eta^2/8}$, its non-zero eigenvalues are $\lambda_k = k \geq 1$, and the corresponding eigenfunctions are $\phi_{k+1} = \phi_k''$. The principal eigenfunction of the adjoint operator is $\psi_0(\eta) = \eta$. The solution of the unperturbed version of (4.11) on a half-line

$$p_\tau + Lp = 0, \quad \eta > 0, \ p(\tau, 0) = 0, \tag{4.13}$$

thus satisfies

$$p(\tau,\eta) = \eta\,\frac{e^{-\eta^2/4}}{2\sqrt{\pi}}\int_0^{+\infty} \xi v_0(\xi)d\xi + O(e^{-\tau})e^{-\eta^2/7}, \text{ as } \tau \to +\infty, \tag{4.14}$$

and our task is to generalize this asymptotics to the full problem (4.11) on the whole line. We will prove the following:

Lemma 4.3 *Let $w(\tau,\eta)$ be the solution of (2.6) on $\mathbb{R}$, with the initial condition $w(0,\eta) := w_0(\eta)$ such that $w_0(\eta) = 0$ for all $\eta > M$, with some $M > 0$, and $w_0(\eta) = O(e^\eta)$ for $\eta < 0$. There exists $\alpha_\infty > 0$ and a function $h(\tau)$ such that*

$$\lim_{\tau\to+\infty} h(\tau) = 0,$$

and such that we have, for any $\gamma' \in (0, 1/2)$:

$$w(\tau,\eta) = (\alpha_\infty + h(\tau))\eta_+ e^{-\eta^2/4} + R(\tau,\eta)e^{-\eta^2/6}, \qquad \eta \in \mathbb{R},$$

with $|R(\tau,\eta)| \leq C_{\gamma'}e^{-(1/2-\gamma')\tau}$, where $\eta_+ = \max(0,\eta)$.

Lemma 4.1 is an immediate consequence of this result, as $x = t^\gamma$ corresponds to $\eta = e^{-(1/2-\gamma)\tau}$, so all we need to do is take $\gamma' < \gamma$ and use Lemma 4.3.

The approximate Dirichlet boundary condition

Let us first explain why the solution of (4.11) must approximately satisfy the Dirichlet boundary condition at $\eta = 0$. Recall that w is related to the solution of the original KPP problem via

$$w(\tau, \eta) = u(e^\tau, \eta e^{\tau/2}) e^{-\tau/2 + \eta e^{\tau/2}}.$$

The trivial a priori bound $0 < u(t, x) < 1$ implies that we have

$$0 < w(\tau, \eta) < e^{-\tau/2 + \eta e^{\tau/2}}, \quad \eta < 0, \tag{4.15}$$

and, in particular, we have

$$0 < w(\tau, -e^{-(1/2-\gamma)\tau}) \leq e^{-e^{\gamma\tau}}. \tag{4.16}$$

We also have

$$w_\tau(\tau, \eta) = u_t(e^\tau, \eta e^{\tau/2}) e^{\tau/2 + \eta e^{\tau/2}} + \frac{\eta}{2} u_x(e^\tau, \eta e^{\tau/2}) e^{\eta e^{\tau/2}} + (\frac{\eta}{2} e^{\tau/2} - \frac{1}{2}) u(e^\tau, \eta e^{\tau/2}) e^{-\tau/2 + \eta e^{\tau/2}},$$

so that

$$\begin{aligned} w_\tau(\tau, -e^{-(1/2-\gamma)\tau}) &= u_t(e^\tau, -e^{-(1/2-\gamma)\tau}) e^{\tau/2 - e^{\gamma\tau}} - e^{-(1/2-\gamma)\tau} u_x(e^\tau, -e^{-(1/2-\gamma)\tau}) e^{-e^{\gamma\tau}} \\ &\quad - (e^{\gamma\tau} + \frac{1}{2}) u(e^\tau, -e^{\gamma\tau}) e^{-\tau/2 - e^{\gamma\tau}} = O(e^{-\gamma e^{\gamma\tau}}), \end{aligned} \tag{4.17}$$

for any $\gamma > 0$. Thus, the solution of (4.11) satisfies

$$0 < w(\tau, -e^{-(1/2-\gamma)\tau}) \leq e^{-e^{\gamma\tau}}, \quad |w_\tau(\tau, -e^{-(1/2-\gamma)\tau})| \leq C e^{-\gamma e^{\gamma\tau}}, \tag{4.18}$$

which we will use as an approximate Dirichlet boundary condition at $\eta = 0$.

An upper barrier

It follows from (4.18) that an upper barrier for $w(\tau, \eta)$ is the solution of

$$\begin{aligned} &\overline{w}_\tau + L\overline{w} + \frac{3}{2} e^{-\tau/2} \overline{w}_\eta = 0, \quad \tau > 0, \quad \eta > -e^{-(1/2-\gamma)\tau}, \\ &w(\tau, -e^{-(1/2-\gamma)\tau}) = e^{-e^{\gamma\tau}}, \end{aligned} \tag{4.19}$$

with a compactly supported initial condition $\bar{w}_0(\eta) = \bar{w}(0, \eta)$ chosen so that

$$\bar{w}_0(\eta) \geq u_1(\eta) e^\eta.$$

That is, we have

$$w(\tau, \eta) \leq \bar{w}(\tau, \eta), \quad \text{for all } \tau > 0 \text{ and } \eta > -e^{-(1/2-\gamma)\tau}.$$

It is convenient to make a change of variables

$$\bar{w}(\tau, \eta) = \bar{p}(\tau, \eta + e^{-(1/2-\gamma)\tau}) + e^{-e^{\gamma\tau}} \gamma(\eta + e^{-(1/2-\gamma)\tau}),$$

where $\gamma(\eta)$ is a smooth monotonic function such that $\gamma(\eta) = 1$ for $0 \leq \eta < 1$ and $\gamma(\eta) = 0$ for $\eta > 2$. The function $\bar{p}$ satisfies

$$\bar{p}_\tau + L\bar{p} + (\gamma e^{-(1/2-\gamma)\tau} + \frac{3}{2} e^{-\tau/2}) \bar{p}_\eta = G(\tau, \eta) e^{-e^{\gamma\tau}}, \quad \eta > 0, \quad \bar{p}(\tau, 0) = 0, \tag{4.20}$$

with a smooth function $G(\tau,\eta)$ supported in $0 \le \eta \le 2$, and the initial condition

$$\bar{p}_0(\eta) = \bar{w}_0(\eta - 1) - e^{-1}\gamma(\eta).$$

We will allow (4.20) to run for a large time T, after which time we can treat the right side and the last term in the left side of (4.20) as a small perturbation. We will need the following lemma which follows from [14]:

Lemma 4.4 *Consider $\omega \in (0, 1/2)$ and $g(\tau,\eta)$ smooth, bounded, and compactly supported in $\mathbb{R}_+$. Let $p(\tau,\eta)$ solve*

$$|p_\tau + Lp| \le \varepsilon e^{-\omega\tau}(|p_\eta| + |p| + g(\tau,\eta)), \quad \tau > 0,\ \eta > 0, \qquad p(\tau, 0) = 0. \tag{4.21}$$

There exists $\varepsilon_0 > 0$ and $C > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, we have

$$p(\tau,\eta) = \eta\Big(\frac{e^{-\eta^2/4}}{2\sqrt{\pi}}\Big(\int_0^{+\infty} \xi v_0(\xi)d\xi + \varepsilon R_1(\tau,\eta)\Big) + \varepsilon e^{-\omega\tau} R_2(\tau,\eta)e^{-\eta^2/6} + e^{-\tau}R_3(\tau,\eta)e^{-\eta^2/6}\Big), \tag{4.22}$$

where $\|R_{1,2,3}(\tau,\cdot)\|_{C^3} \le C$ for all $\tau > 0$.

We now apply Lemma 4.4 to the function $\bar{p}(\tau,\eta)$ as follows: take T sufficiently large, so that this lemma applies for $\tau > T$ and $\omega \in (0, 1/2 - \gamma)$ because the exponentially decaying in τ terms are sufficiently small. Then for $\tau > T$ we have

$$\bar{p}(\tau,\eta) = \eta\Big(\frac{e^{-\eta^2/4}}{2\sqrt{\pi}}\Big(\int_0^{+\infty} \xi \bar{p}(T,\xi)d\xi + \varepsilon R_1(\tau,\eta)\Big) + \varepsilon e^{-\omega(\tau-T)} R_2(\tau,\eta)e^{-\eta^2/6} + e^{-(\tau-T)}R_3(\tau,\eta)e^{-\eta^2/6}\Big). \tag{4.23}$$

We claim that with a suitable choice of $\bar{w}_0$, the integral term in (4.23) is bounded from below:

$$\int_0^\infty \eta\bar{p}(\tau,\eta)d\eta \ge 1, \text{ for all } \tau > 0. \tag{4.24}$$

Multiplying (4.20) by η and integrating gives

$$\frac{d}{d\tau}\int_0^\infty \eta\bar{p}(\tau,\eta)d\eta = (\gamma e^{-(1/2-\gamma)\tau} + \frac{3}{2}e^{-\tau/2})\int_0^\infty \bar{p}(\tau,\eta)d\eta + e^{-e^{\gamma\tau}}\int G(\tau,\eta)\eta d\eta. \tag{4.25}$$

The function $G(\tau,\eta)$ need not have a sign, hence a priori we do not know that $\bar{p}(\tau,\eta)$ is positive everywhere. However, it follows from (4.20) that

$$\int_0^\infty \bar{p}(\tau,\eta)d\eta \ge -C_0,$$

for all $\tau > 0$, with the constant C_0 which does not depend on $\bar{w}_0(\eta)$ on the interval $[2,\infty)$. Thus, we deduce from (4.25) that for all $\tau > 0$ we have

$$\int_0^\infty \eta\bar{p}(\tau,\eta)d\eta \ge \int_0^\infty \eta\bar{w}_0(\eta)d\eta - C_0', \tag{4.26}$$

with, once again, C_0' independent of $\bar{w}_0$. Therefore, after possibly increasing $\bar{w}_0$ we may ensure that (4.24) holds.

It follows from (4.24) and (4.23) that there exists a sequence $\tau_n \to +\infty$, $C > 0$ and a function $\overline{W}_\infty(\eta)$ such that

$$C^{-1}\eta e^{-\eta^2/4} \le \overline{W}_\infty(\eta) \le C\eta e^{-\eta^2/4}, \tag{4.27}$$

and

$$\lim_{n\to+\infty} e^{\eta^2/8}|\bar{p}(\tau_n,\eta) - \overline{W}_\infty(\eta)| = 0, \tag{4.28}$$

uniformly in η on the half-line $\eta \geq 0$. The same bound for the function $\bar{w}(\tau,\eta)$ itself follows:

$$\lim_{n\to+\infty} e^{\eta^2/8}|\bar{w}(\tau_n,\eta) - \overline{W}_\infty(\eta)| = 0, \tag{4.29}$$

also uniformly in η on the half-line $\eta \geq 0$.

A lower barrier

A lower barrier for $w(\tau,\eta)$ is devised as follows. First, note that

$$e^{3\tau/2-\eta\exp(\tau/2)}w(\tau,\eta) \leq C_\gamma e^{-\exp(\gamma\tau/2)}$$

as soon as $\eta \geq e^{-(1/2-\gamma)\tau}$, with $\gamma \in (0,1/2)$, and $C_\gamma > 0$ is chosen sufficiently large. Thus, a lower barrier $\underline{w}(\tau,\eta)$ can be defined as the solution of

$$\underline{w}_\tau + L\underline{w} + \frac{3}{2}e^{-\tau/2}\underline{w}_\eta + C_\gamma e^{-\exp(\gamma\tau/2)}\underline{w} = 0, \quad \underline{w}(\tau, e^{-(1/2-\gamma)\tau}) = 0, \quad \eta > e^{-(1/2-\gamma)\tau}, \tag{4.30}$$

and with an initial condition $\underline{w}_0(\eta) \leq w_0(\eta)$. The same argument as for the upper barriers yields the uniform convergence of (possibly a subsequence of) $\underline{w}(\tau_n,\cdot)$ on the half-line $\eta \geq e^{-(1/2-\gamma)\tau}$ to a function $\underline{W}_\infty(\eta)$ which satisfies

$$C^{-1}\eta e^{-\eta^2/4} \leq \underline{W}_\infty(\eta) \leq C\eta e^{-\eta^2/4}, \tag{4.31}$$

so that

$$\lim_{n\to+\infty} e^{\eta^2/8}|\underline{w}(\tau_n,\eta) - \underline{W}_\infty(\eta)| = 0, \qquad \eta > 0. \tag{4.32}$$

Convergence of $w(\tau,\eta)$, proof of Lemma 4.3

Let X be the space of bounded uniformly continuous functions $u(\eta)$ such that $e^{\eta^2/8}u(\eta)$ is bounded and uniformly continuous on $\mathbb{R}_+$. We deduce from the convergence of the upper and lower barriers for (τ,η) that there exists a sequence $\tau_n \to +\infty$ such that $w(\tau_n,\cdot)$ converges to a limit $W_\infty \in X$, such that $W_\infty \equiv 0$ on $\mathbb{R}_-$, and $W_\infty(\eta) > 0$ for all $\eta > 0$. Our next step is bootstrap convergence along a sub-sequence, and show that the limit of $w(\tau,\eta)$ as $t \to +\infty$ exists in the space X. First, observe that the above convergence implies that the shifted functions

$$w_n(\tau,\eta) = w(\tau+\tau_n,\eta)$$

converge in X, uniformly on compact time intervals, as $n \to +\infty$ to the solution $w_\infty(\tau,\eta)$ of the linear problem

$$\begin{aligned} &(\partial_\tau + L)w_\infty = 0, \quad \eta > 0, \\ &w_\infty(\tau,0) = 0, \\ &w_\infty(0,\eta) = W_\infty(\eta). \end{aligned} \tag{4.33}$$

In addition, there exists $\alpha_\infty > 0$ such that $w_\infty(\tau,\eta)$ converges to

$$\bar{\psi}(\eta) = \alpha_\infty \eta e^{-\eta^2/4}, \quad \eta > 0,$$

in the topology of X. Thus, for any $\varepsilon > 0$ we may choose T_ε large enough so that

$$|w_\infty(\tau,\eta) - \alpha_\infty \eta e^{-\eta^2/4}| \le \varepsilon \eta e^{-\eta^2/8} \text{ for all } \tau > T_\varepsilon, \text{ and } \eta > 0. \tag{4.34}$$

Given T_ε we can find N_ε sufficiently large so that

$$|w(T_\varepsilon + \tau_n, \eta + e^{-(1/2-\gamma)T_\varepsilon}) - w_\infty(T_\varepsilon, \eta)| \le \varepsilon \eta e^{-\eta^2/8}, \text{ for all } n > N_\varepsilon. \tag{4.35}$$

In particular, we have

$$(\alpha_\infty - \varepsilon)\eta e^{-\eta^2/4} \le w(\tau_{N_\varepsilon} + T_\varepsilon, \eta + e^{-(1/2-\gamma)T_\varepsilon}) \le (\alpha_\infty + \varepsilon)\eta e^{-\eta^2/4}. \tag{4.36}$$

It follows, once again from Lemma 4.4 that any limit point ϕ_∞ of $w(\tau,\cdot)$ in X as $\tau \to +\infty$ satisfies

$$(\alpha_\infty - 2\varepsilon)\eta e^{-\eta^2/4} \le \phi_\infty(\eta) \le (\alpha_\infty + 2\varepsilon)\eta e^{-\eta^2/4}. \tag{4.37}$$

As $\varepsilon > 0$ is arbitrary, we conclude that $w(\tau,\eta)$ converges as $\tau \to +\infty$ to $\bar{\psi}(\eta)$. Taking into account Lemma 4.4 once again, we have proved Lemma 4.3, which implies Lemma 4.1.

5 The approximate solution is an approximation to the true solution

Let us sum up the situation. We have proved the existence of an asymptotic shift x_∞ such that we have, as $t \to +\infty$, uniformly on $\mathbb{R}$:

$$u(t,x) \to \phi_*(x - x_\infty).$$

Here, $u(t,x)$ is the solution of the KPP equation (2.2) (already in the logarithmically shifted frame), with an initial condition of the form (1.2). If $w(\tau,\eta)$ solves (2.6) (the full equation in the self-similar variables), we have shown (Lemma 4.3) that there is $\alpha_\infty > 0$ such that we have, as $\tau \to +\infty$, and uniformly in $\eta \in \mathbb{R}$:

$$w(\tau,\eta) \to \alpha_\infty \eta_+ e^{-\eta^2/4}.$$

Then the shift x_∞ is determined by

$$\alpha_\infty = e^{x_\infty}.$$

Without loss of generality, we will assume that the initial condition is such that

$$x_\infty = 0, \quad \alpha_\infty = 1.$$

We claim that Theorem 1.5 reduces to the following:

Theorem 5.1 *Given $\gamma > 0$ small, let $V^{app}(t,x)$ and $s(t)$ be, respectively, the approximate solution and the shift constructed in Section 3. There is $C_\gamma > 0$ such that, for all $(t,x) \in [1,\infty) \times \mathbb{R}$, we have*

$$|e^x u(t,x) - V^{app}(t,x)| \le \frac{C_\gamma(1+|x|)}{t^{1-\kappa\gamma}}, \tag{5.1}$$

where $\kappa > 0$ is independent of γ.

We will first show that the $t^{-1/2}$ term in the shift in Theorem 1.5 is a consequence of Theorem 5.1, and then prove Theorem 5.1.

5.1 The proof of Theorem 1.5

Let $u(t,x)$ be the solution of (2.2). From Theorem 5.1, we have, for x in an arbitrarily large compact set K and $t>0$ large:

$$\begin{aligned} u(t,x) &= e^{-x}V^-(t,x-s(t)) + O_K(t^{-1+\kappa\gamma}) \\ &= \phi_*(x-s(t)) + \frac{1}{\sqrt{t}}\Big[q_1^-\phi_*'(x) + v_1(t,x)e^{-x}\Big] + O_K(t^{-1+\kappa\gamma}) \\ &= \phi_*\Big(x+\frac{3\sqrt{\pi}}{I_1\sqrt{t}}\Big) + \frac{1}{\sqrt{t}}\Big(3\sqrt{\pi}\phi_*'(x) + v_1(t,x)e^{-x}\Big) + O(\frac{1}{t^{\kappa\gamma-1}}), \end{aligned} \tag{5.2}$$

for every $\gamma>0$. Recall that $v_1(t,x)$ is given by (3.15) and I_1 is given by (3.22). Note that for x in a fixed compact set, and t large we have

$$e^{-x}v_1(t,x) = \zeta(t)\Big[\frac{\phi_*'(x)}{\phi_1(t^\gamma)} + \frac{\phi_*'(x)}{I_1+O(e^{-\omega_0 t^\gamma})}\int_x^{t^\gamma}\frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy\Big]. \tag{5.3}$$

Observe that $\phi_1(y) = e^x\phi_*'(x) < 0$ and

$$\begin{aligned} &\int_{t^\gamma}^\infty \frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy = \Big(-I_1 + O(e^{-\omega_0 t^\gamma})\Big)\int_{t^\gamma}^\infty \frac{1}{\phi_1^2(y)}dy \\ &= \Big(I_1 + O(e^{-\omega_0 t^\gamma})\Big)\frac{1}{\phi_1(t^\gamma)}. \end{aligned}$$

Using this in (5.3) gives

$$e^{-x}v_1(t,x) = \frac{\zeta(t)\phi_*'(x)}{I_1+O(e^{-\omega_0 t^\gamma})}\int_x^\infty \frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}\phi_1(z)(V_0^-)^2(z)dz\Big)dy\Big]. \tag{5.4}$$

We conclude then from (5.2) that

$$u(t,x) = \phi_*\Big(x + \frac{3\sqrt{\pi}}{I_1\sqrt{t}} + \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{3\sqrt{\pi}}{I_1\sqrt{t}}r(x)\Big) + O(t^{-1+\gamma}). \tag{5.5}$$

The function $r(x)$ is given by

$$\begin{aligned} r(x) &= \int_x^\infty \frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{-z}|\phi_1(z)|(V_0^-)^2(z)dz\Big)dy \\ &= -\int_x^\infty \frac{1}{\phi_1^2(y)}\Big(\int_{-\infty}^y e^{2z}\phi_*'(z)\phi_*^2(z)dz\Big)dy \end{aligned} \tag{5.6}$$

and

$$I_1 = -\int_{\mathbb{R}} e^{2z}\phi_*'(z)\phi_*^2(z)dz > 0.$$

This finishes the proof of Theorem 1.5. □

The proof of Corollary 1.7

Let us fix $s\in(0,1)$ and let $\sigma_s(t)$ be defined by

$$\sigma(t) = \sup\{x:\ u(t,x)=s\},$$

and $\sigma_*^s = \phi_*^{-1}(s)$, so that $\phi_*(\sigma_*^s) = s$. From (5.5), we then have:

$$\sigma_* = \sigma_s(t) + \frac{3\sqrt{\pi}}{I_1\sqrt{t}} + \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{3\sqrt{\pi}r(\sigma_s(t))}{\sqrt{t}I_1} + O(t^{-1+\gamma}). \tag{5.7}$$

It follows that

$$\sigma(t) = \sigma_* - \frac{1}{\sqrt{t}}\left(\frac{3\sqrt{\pi}}{I_1} + 3\sqrt{\pi} + \frac{3\sqrt{\pi}}{I_1}\int_{\sigma_*}^{\infty}\frac{e^{-2y}}{(\phi_*')^2(y)}\left(\int_{-\infty}^{y}|\phi_*'(z)|\phi_*^2(z)e^{2z}dz\right)dy\right) + o(t^{-1/2}). \tag{5.8}$$

We see that none of the quantities appearing in (5.8) depend on the initial datum, proving Theorem 1.5. □

5.2 The proof of Theorem 5.1

This is the most technical part of the paper, although the idea is really to apply a simple stability argument. We are going to work with the function

$$v(t,x) = e^x u(t,x),$$

and we will use the self-similar variables

$$\tau = \log t, \quad \eta = \frac{x}{\sqrt{t}} \tag{5.9}$$

most of the time. There, one may easily reduce the equation for v to an equation on a half-line $\eta > 0$, due to the very fast decay of v for $\eta < 0$. Then, we are left with an equation for $\eta > 0$ that is almost linear: it is perturbed by a nonlinear term whose support in η is essentially of the size $e^{-\tau/2}$. Moreover, we already know that $e^{-\tau/2}v(\tau,\eta)$ is equivalent, for large τ, to

$$\alpha_\infty \eta_+ e^{-\eta^2/4}.$$

However, the nonlinear term may be quite large in the small region $\eta \sim O(e^{-\tau/2})$. We use a bootstrap argument to show that it is in fact harmless, thus opening the way to a classical stability argument of the type [25].

Reduction to the Dirichlet problem

In view of (3.53), the difference

$$\widetilde{W}(t,x) = v(t,x) - V^{app}(t,x).$$

satisfies an equation

$$\widetilde{W}_t - \widetilde{W}_{xx} - \frac{3}{2t}(\widetilde{W} - \widetilde{W}_x) + e^{-x}(v + V^{app})\widetilde{W} = \widetilde{E}_1(t,x), \tag{5.10}$$

with a function $\widetilde{E}_1$ satisfying:

$$|\widetilde{E}_1(t,x)| \le Ct^{-1+2\gamma}(\mathbf{1}_{0<x<t^\gamma} + e^x\mathbf{1}_{x<0}) + Ct^{-3/2}e^{-x^2/((4+\gamma)t)}\mathbf{1}_{x>t^\gamma} + Ct^{-1+2\gamma}\delta(x - t^\gamma). \tag{5.11}$$

In order to reduce the equation for $\widetilde{W}$ to a Dirichlet problem in the self-similar variables, we proceed in several steps.

We first switch to

$$W_1(t,x) = \widetilde{W}(t,x) - \widetilde{W}(t,-t^\gamma)\psi(x+t^\gamma).$$

Here, $\psi(x)$ is a nonegative C^∞ function so that $\psi(x) = 1$ for $0 \le x \le 1$, and $\psi(x) = 0$ for $x \ge 2$, so that now $W_1(t,-t^\gamma) = 0$. This generates an additional term in the right side of (5.10) that we denote by $\widetilde{E}_2(t,x)$. Taking into account that

$$v(t,x) + V^{app}(t,x) = O(e^x) \text{ for } x < 0, \tag{5.12}$$

we obtain

$$|\widetilde{E}_2(t,x)| \le Ce^{-t^\gamma}\mathbf{1}_{[0,1]}(x+t^\gamma). \tag{5.13}$$

Next, we translate the origin to $x = -t^\gamma$: the function

$$W(t,x) = W_1(t,x+t^\gamma) = \widetilde{W}(t,x-t^\gamma) - \widetilde{W}(t,-t^\gamma)\psi(x) \tag{5.14}$$

satisfies

$$W_t - W_{xx} + \Big(\frac{\gamma}{t^{1-\gamma}} + \frac{3}{2t}\Big)W_x - \frac{3}{2t}W + e^{t^\gamma - x}(\widetilde{v} + \widetilde{V}^{app})W = G_1(t,x) + G_2(t,x) \tag{5.15}$$

for $x > 0$, with the Dirichlet condition $W(t,0) = 0$. Here, we have introduced

$$\widetilde{v}(t,x) = v(t,x-t^\gamma), \quad \widetilde{V}^{app}(t,x) = V^{app}(t,x-t^\gamma). \tag{5.16}$$

The functions $G_1(t.x)$ and $G_2(t,x)$ in (5.15) satisfy

$$\begin{aligned} |G_1(t,x)| = |\widetilde{E}_1(t,x-t^\gamma)| \quad \le \quad & Ct^{-1+2\gamma}(\mathbf{1}_{t^\gamma<x<2t^\gamma}(x) + e^{x-t^\gamma}\mathbf{1}_{x<t^\gamma}(x)) \\ & + Ct^{-3/2}e^{-(x-t^\gamma)^2/((4+\gamma)t)}\mathbf{1}_{x>2t^\gamma} + Ct^{-1+2\gamma}\delta(x-2t^\gamma), \end{aligned} \tag{5.17}$$

and

$$|G_2(t,x)| = |\widetilde{E}_2(t,x-t^\gamma)| \le Ce^{-t^\gamma}\mathbf{1}_{[0,1]}(x). \tag{5.18}$$

We now express (5.15) in the self-similar variables (5.9). This gives

$$W_\tau + \Big(L - \frac{1}{2}\Big)W + \Big(\gamma e^{\gamma\tau} + \frac{3}{2}\Big)e^{-\tau/2}W_\eta + e^{\tau + e^{\gamma\tau} - \eta e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})W = e^{-\eta^2/8}(E_1 + E_2), \tag{5.19}$$

with $E_1(\tau,\eta)$ satisfying

$$\begin{aligned} |E_1(\tau,\eta)| \le\ & Ce^{2\gamma\tau}e^{\eta^2/8}\mathbf{1}\Big(e^{-(\frac{1}{2}-\gamma)\tau} < \eta < 2e^{-(\frac{1}{2}-\gamma)\tau}\Big) \\ & + Ce^{2\gamma\tau}e^{\eta^2/8}e^{\eta e^{\tau/2} - e^{\gamma\tau}}\mathbf{1}\Big(0 < \eta < e^{-(\frac{1}{2}-\gamma)\tau}\Big) \\ & + Ce^{-\tau/2}e^{\eta^2/8}e^{-(\eta - e^{(-1/2+\gamma)\tau})^2/(4+\gamma)}\mathbf{1}\Big(\eta > 2e^{(-1/2+\gamma)\tau}\Big) \\ & + Ce^{2\gamma\tau}e^{\eta^2/8}\delta(\eta - 2e^{(-1/2+\gamma)\tau}) = E_{11} + E_{12} + E_{13} + E_{14}, \end{aligned} \tag{5.20}$$

and

$$|E_2(\tau,\eta)| \le e^{\eta^2/8}e^\tau e^{-e^{\tau\gamma}}\mathbf{1}\Big(0 < \eta < e^{-\tau/2}\Big). \tag{5.21}$$

Notice that the the support of E_{11}, E_{12}, E_{14} is very small, despite the larger prefactor, compared to E_{13} and E_2. Finally, we symmetrize the operator L by introducing the function

$$w(\tau,\eta) = e^{\eta^2/8}W(\tau,\eta), \tag{5.22}$$

which satisfies

$$w_\tau + \mathcal{M}w + e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})w = \sum_{i=1}^{2} E_i(\tau,\eta) + E_3(\tau,\eta), \qquad \eta > 0 \tag{5.23}$$

with the Dirichlet boundary condition $w(\tau, 0) = 0$. Here we have defined the operator

$$\mathcal{M}w = -w_{\eta\eta} + \left(\frac{\eta^2}{16} - \frac{5}{4}\right)w, \tag{5.24}$$

and set

$$\eta_\gamma(\tau) = e^{-(\frac{1}{2}-\gamma)\tau}, \qquad E_3(\tau,\eta) = -\left(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\right)(w_\eta - \frac{\eta}{4}w). \tag{5.25}$$

Strictly speaking, E_3 depends on w and w_η, but we omit this dependence for the notational purposes.

Recall that, in these variables, V^{app} grows as $e^{\tau/2}$. An immediate consequence of Proposition 4.2, Lemma 4.3, and the definition of V^{app} is that

$$\lim_{\tau\to+\infty} e^{-\tau/2}\|w(\tau,.)\|_{L^2(\mathbb{R}_+)} = 0. \tag{5.26}$$

Our goal is to improve this $o(e^{\tau/2})$ bound on w to an exponentially decaying estimate for w. Eventually, we will obtain:

$$\|w(\tau,\cdot)\|_{L^2(\mathbb{R}_+)} + \|w(\tau,\cdot)\|_\infty \le C_\gamma e^{(-1/2+100\gamma)\tau}. \tag{5.27}$$

From $o(e^{\tau/2})$ to $O(e^{10\gamma\tau})$ asymptotics for w

The principal eigenfunction of the self-adjoint operator $\mathcal{M}$ with the Dirichlet boundary condition at $\eta = 0$ is

$$e_0(\eta) = c_0\eta e^{-\eta^2/8}, \qquad \mathcal{M}e_0 = -\frac{e_0}{2}, \tag{5.28}$$

with the constant c_0 chosen so that $\|e_0\|_{L^2(\mathbb{R}_+)} = 1$. We decompose the solution of (5.23) as

$$w(\tau) = \langle e_0, w(\tau)\rangle e_0 + w^\perp(\tau), \qquad \int_{\mathbb{R}_+} e_0(\eta)w^\perp(\tau,\eta)d\eta = 0. \tag{5.29}$$

Step 1. We will first obtain a bound for $\langle e_0, w\rangle$. We have, projecting (5.23) onto e_0 and using (5.29):

$$\frac{d\langle e_0, w\rangle}{d\tau} - \frac{\langle e_0, w\rangle}{2} + \langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})w\rangle = \sum_{i=1}^{3}\langle e_0, E_i(\tau)\rangle. \tag{5.30}$$

Let us bound the various perturbative terms in (5.30). The terms involving E_1 and E_2 in the right side are easily treated. In view of (5.20) we have

$$|\langle e_0, E_1(\tau)\rangle| \le Ce^{-(\frac{1}{2}-3\gamma)\tau}. \tag{5.31}$$

and (5.21) implies

$$|\langle e_0, E_2(\tau)\rangle| \le Ce^{-e^{\gamma\tau}} \le Ce^{-(\frac{1}{2}-3\gamma)\tau}, \tag{5.32}$$

as well. As for the term involving E_3, using (5.25) and integrating by parts, we get

$$|\langle e_0, E_3(\tau)\rangle| = \left(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\right)\left|\langle e_0', w\rangle + \langle e_0, \frac{\eta}{4} w\rangle\right|. \tag{5.33}$$

Because of (5.26), we obtain

$$|\langle e_0, E_3(\tau)\rangle| \leq Ce^{2\gamma\tau}. \tag{5.34}$$

It finally remains to estimate the last term in the left side of (5.30), and some care should be given to it: although the exponential term is small outside of the very small set $0 < \eta < \eta_\gamma$, it could be very large (of the order e^τ) there. This will be compensated by the smallness of the factor $v + V^{app}$. Let us recall (5.12) and (5.16) which imply that in the self-similar variables

$$|\widetilde{v}(\tau,\eta) + \widetilde{V}^{app}(\tau,\eta)|,\ |w(\tau,\eta)| \leq Ce^{\eta e^{\tau/2} - e^{\gamma\tau}} = Ce^{e^{\tau/2}(\eta - \eta_\gamma(\tau))} \text{ for } 0 \leq \eta \leq \eta_\gamma(\tau). \tag{5.35}$$

Let us decompose the inner product

$$Q(\tau) = \langle e_0, e^{\tau + (\eta_\gamma(\tau) - \eta)e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})w\rangle = \int_0^{\eta_\gamma(\tau)} + \int_{\eta_\gamma(\tau)}^{\infty} = I_1 + I_2, \tag{5.36}$$

For $\eta \leq \eta_\gamma(\tau)$ we use the bound $0 \leq e_0(\eta) \leq c_0\eta$. Using (5.35), we obtain

$$\begin{aligned} I_1 &\leq \int_0^{\eta_\gamma(\tau)} e_0(\eta) e^{\tau + (\eta_\gamma(\tau) - \eta)e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})|w| d\eta \leq C\int_0^{\eta_\gamma(\tau)} \eta e^{\tau + (\eta - \eta_\gamma(\tau))e^{\tau/2}} d\eta \\ &\leq C\eta_\gamma(\tau) e^\tau e^{-\tau/2} = Ce^{\gamma\tau}. \end{aligned} \tag{5.37}$$

As for I_2, let us apply Lemma 4.3 to infer that

$$|\widetilde{v}(\tau,\eta) + \widetilde{V}^{app}(\tau,\eta)|,\ |w(\tau,\eta)| \leq C(1 + \eta e^{\tau/2})$$

for all $\eta \in \mathbb{R}$. This implies

$$\begin{aligned} I_2 &\leq \int_{\eta_\gamma(\tau)}^{\infty} e_0(\eta) e^{\tau + (\eta_\gamma(\tau) - \eta)e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})|w| d\eta \leq C\int_{\eta_\gamma(\tau)}^{\infty} \eta e^{\tau + (\eta_\gamma(\tau) - \eta)e^{\tau/2}}(1 + \eta e^{\tau/2})^2 d\eta \\ &\leq Ce^{2\tau}\int_{\eta_\gamma(\tau)}^{\infty} \eta^3 e^{-(\eta - \eta_\gamma(\tau))e^{\tau/2}} d\eta \leq C(\eta_\gamma(\tau))^3 e^{3\tau/2} \leq Ce^{3\gamma\tau}, \end{aligned} \tag{5.38}$$

and therefore,

$$|Q(\tau)| \leq Ce^{3\gamma\tau}. \tag{5.39}$$

Putting everything together, we infer that

$$\frac{d\langle e_0, w\rangle}{d\tau} - \frac{\langle e_0, w\rangle}{2} = \varphi(\tau), \tag{5.40}$$

with

$$|\varphi(\tau)| \leq Ce^{3\gamma\tau}.$$

We see that

$$\frac{d}{dt}\Big(\langle e_0, w\rangle e^{-\tau/2}\Big) = \varphi(\tau)e^{-\tau/2}. \tag{5.41}$$

Taking into account (5.26), we can integrate (5.41) from τ to $+\infty$ leading to

$$\langle e_0, w(\tau)\rangle = -\int_\tau^{+\infty} e^{(\tau - \tau')/2}\varphi(\tau')d\tau', \tag{5.42}$$

hence

$$|\langle e_0, w(\tau)\rangle| \le C \int_\tau^{+\infty} e^{(\tau-\tau')/2} e^{3\gamma\tau'} d\tau' \le C_\gamma e^{3\gamma\tau}. \tag{5.43}$$

This bound will be improved in the next step.

Step 2. Now, we bound $w^\perp(\tau)$. We multiply ((5.23)) by $w^\perp$, and integrate by parts:

$$\frac{1}{2}\frac{d\|w^\perp\|^2}{d\tau} + \langle \mathcal{M} w^\perp, w^\perp\rangle + \int_{\mathbb{R}_+} e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}} (\widetilde{v} + \widetilde{V}^{app}) w w^\perp d\eta = \sum_{i=1}^{3} \int_{\mathbb{R}_+} E_i w^\perp d\eta. \tag{5.44}$$

We denoted here the $L^2(\mathbb{R}_+)$ norm by $\|\cdot\|$. Once again, we need to bound the perturbative terms in (5.44). Let us start with the less standard term:

$$q(w) := \int_{\mathbb{R}_+} e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}} (\widetilde{v} + \widetilde{V}^{app}) w w^\perp d\eta = J_1(\tau) + J_2(\tau),$$

with the two terms coming from the decomposition (5.29). We have

$$J_1(\tau) = \langle e_0, w(\tau)\rangle \int_{\mathbb{R}_+} e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}} (\widetilde{v} + \widetilde{V}^{app}) e_0 w^\perp d\eta. \tag{5.45}$$

We know from Step 1 that

$$\langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}} (\widetilde{v} + \widetilde{V}^{app}) w\rangle = |Q(\tau)| \le C e^{3\gamma\tau}.$$

Together with (5.43) this gives

$$|J_1(\tau)| \le C e^{6\gamma\tau}. \tag{5.46}$$

Furthermore, $J_2(\tau)$ is positive, so we do not need to estimate it.

As for the three terms in the right side of (5.44), in view of (5.20) we have, with some constant $C_\gamma > 0$: first,

$$|\langle w^\perp, E_{11}\rangle| + |\langle w^\perp, E_{12}\rangle| \le \gamma\|w^\perp\|^2 + \frac{1}{4\gamma}(\|E_{11}\|^2 + \|E_{12}\|^2) \le \gamma\|w^\perp\|^2 + C_\gamma e^{5\gamma\tau} e^{-\tau/2}, \tag{5.47}$$

while for E_{13} we have

$$|\langle w^\perp, E_{13}\rangle| \le \gamma\|w^\perp\|^2 + \frac{1}{4\gamma}\|E_{13}\|^2 \le \gamma\|w^\perp\|^2 + \frac{C}{\gamma} e^{-\tau}. \tag{5.48}$$

Finally, for E_{14} we have

$$\begin{aligned} |\langle w^\perp, E_{14}\rangle| &\le C e^{2\gamma\tau} |w^\perp(\tau, 2e^{-\tau/2+\gamma\tau})| \le C e^{2\gamma\tau} e^{-\frac{\tau}{4}+\frac{\gamma}{2}\tau} \|\partial_x w^\perp(\tau,\cdot)\|_{L^2} \\ &\le C e^{2\gamma\tau} e^{-\frac{\tau}{4}+\frac{\gamma}{2}\tau} (1 + \langle M w^\perp, w^\perp\rangle + \|w^\perp\|^2). \end{aligned} \tag{5.49}$$

For E_2 we may simply estimate

$$|\langle w^\perp, E_2\rangle| \le \gamma\|w^\perp\|^2 + \frac{1}{4\gamma}\|E_2\|^2 \le \gamma\|w^\perp\|^2 + C_\gamma e^{2\tau-2e^{\gamma\tau}} e^{-\tau/2} \le \gamma\|w^\perp\|^2 + C_\gamma e^{-\tau/2}. \tag{5.50}$$

As for E_3, we have

$$\Big|\int_{\mathbb{R}_+} (w_\eta - \frac{\eta}{4} w) w^\perp d\eta\Big| \le \int \eta (w^\perp)^2 d\eta + C_\gamma \langle e_0, w\rangle^2 + \gamma\|w^\perp\|^2 \le C\|w^\perp\|^2 + C\langle \mathcal{M} w^\perp, w^\perp\rangle + C_\gamma \langle e_0, w\rangle^2, \tag{5.51}$$

hence

$$|\langle w^\perp, E_3\rangle| \le C_\gamma e^{(-1/2+\gamma)\tau}\Big(\|w^\perp\|^2 + \langle \mathcal{M}w^\perp, w^\perp\rangle + \langle e_0, w\rangle^2\Big). \tag{5.52}$$

Putting everything together, remembering that

$$\langle \mathcal{M}w^\perp, w^\perp\rangle \ge \frac{\|w\|^2}{2},$$

yields

$$\frac{1}{2}\frac{d\|w^\perp\|^2}{d\tau} + \left(\frac{1}{2} - \gamma - C_\gamma e^{-(\frac{1}{4}-\frac{3\gamma}{2})\tau}\right)\|w^\perp\|^2 \le |J_1(\tau)| \le Ce^{6\gamma\tau}. \tag{5.53}$$

This implies

$$\|w^\perp\| \le C_\gamma e^{3\gamma\tau}. \tag{5.54}$$

Because of (5.43), this bound all holds for the full solution: $\|w\| \le C_\gamma e^{3\gamma\tau}$. By the parabolic regularity, together with our estimates on the perturbative terms, we also infer that

$$\|w^\perp\|_{L^\infty([0,A])} \le C_A e^{5\gamma\tau},$$

for A large. The L^∞ estimates on the perturbative terms in the equation (5.23) for w imply that for $\eta \ge A$ sufficiently large, $w(\tau,\eta)$ can not attain its maximum at a point $\eta > A$ where it is larger than than $Ce^{5\gamma\tau}$, thus we are finally ready to conclude that

$$\|w\|_{L^2(\mathbb{R}_+)} + \|w\|_\infty \le C_\gamma e^{10\gamma\tau}. \tag{5.55}$$

From the $O(e^{10\gamma\tau})$ growth to $O(e^{-(\frac{1}{2}-100\gamma)\tau})$ decay for w

The next step is to improve the "slow" $O(e^{10\gamma\tau})$ growth in (5.55) to actual decay in time. Let us come back to (5.30), the equation for $\langle e_0, w\rangle$:

$$\frac{d\langle e_0, w\rangle}{d\tau} - \frac{\langle e_0, w\rangle}{2} + \langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})w\rangle = \sum_{i=1}^{3}\langle e_0, E_i(\tau)\rangle. \tag{5.56}$$

The bounds (5.31) and (5.32) are already of the "good" size $O(e^{-(1/2-3\gamma)\tau})$, and the already obtained bound (5.55) allows us to improve (5.34) to

$$|\langle e_0, E_3(\tau)\rangle| = \left(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\right)\big|\langle e_0', w\rangle + \langle e_0, \frac{\eta}{4}w\rangle\big| \le Ce^{(-1/2+15\gamma)\tau}. \tag{5.57}$$

Thus, what really limits the decay improvement for $\langle e_0, w\rangle$ is the integral

$$Q(\tau) = \langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\widetilde{v} + \widetilde{V}^{app})w\rangle, \tag{5.58}$$

that we have so far only managed to bound by $Ce^{3\gamma\tau}$ (see (5.39)). We have already noted that the integrand could be very large only for η of the order

$$\eta_\gamma(\tau) = e^{(-1/2+\gamma)\tau}.$$

On the other hand, we have the boundary condition $w(\tau, 0) = 0$. It is therefore natural to ask whether w has a bounded linear growth in a neighborhood of $\eta = 0$. If this is so, this will bring a small factor of the order η in the integrand, which will, in turn, make the integral be of a smaller

order. This is what we are going to prove now. Indeed, by the Kato inequality, equation (5.23) for w yields, writing out explicitly the operator $\mathcal{M}$:

$$\partial_\tau|w| - |w|_{\eta\eta} + \Big(\frac{\eta^2}{16} - \frac{5}{4}\Big)|w| + \Big(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\Big)\Big(\partial_\eta|w| - \frac{\eta}{4}|w|\Big) \le Ce^{-(\frac{1}{2}-\gamma)\tau} \tag{5.59}$$
$$+ Ce^{2\gamma\tau}\mathbf{1}\Big(0 < \eta < 2\eta_\gamma(\tau)\Big) + Ce^{2\gamma\tau}\delta(\eta - 2\eta_\gamma(\tau)).$$

Let $a \in (0,1)$ be small enough so that (5.59) implies

$$\partial_\tau|w| - |w|_{\eta\eta} - 10|w| + \Big(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\Big)\partial_\eta|w| \le Ce^{2\gamma\tau} + Ce^{2\gamma\tau}\delta(\eta - 2\eta_\gamma(\tau)), \tag{5.60}$$

for $\eta \in (0,a)$ with the boundary conditions

$$|w|(\tau,0) = 0, \quad |w|(\tau,a) \le Ce^{10\gamma\tau}, \tag{5.61}$$

which is achievable, due to (5.55). Let us write

$$|w|(\tau,\eta) \le Ce^{10\gamma\tau}\psi(\tau,\eta) + e^{2\gamma\tau}\phi(\tau,\eta),$$

with the function $\psi(\tau,\eta) \ge 0$ such that

$$\begin{aligned} &\partial_\tau\psi - \psi_{\eta\eta} - 11\psi + \Big(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\Big)\partial_\eta\psi = Ce^{-8\gamma\tau}, \\ &\psi(\tau,0) = 0, \quad \psi(\tau,a) = 1. \end{aligned} \tag{5.62}$$

Possibly decreasing a, we may ensure that the principal eigenvalue λ_a of the Dirichlet Laplacian on the interval $(0,2a)$ is sufficiently large, say, $\lambda_a > 100$. Then there exists a constant $C > 0$ so that

$$\psi(\tau,\eta) \le C\eta. \tag{5.63}$$

We choose the function $\phi \ge 0$ so that it satisfies

$$\partial_\tau\phi - \phi_{\eta\eta} - 11\phi + \Big(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\Big)\partial_\eta\phi = C\delta(\eta - 2e^{(-1/2+\gamma)\tau}), \tag{5.64}$$

with the boundary conditions

$$\phi(\tau,0) = 0, \quad \phi(\tau,a) = 0. \tag{5.65}$$

Let us prove that

$$\phi(\tau,\eta) \le C\eta. \tag{5.66}$$

We have $\phi(\tau,\eta) = \phi_0(\tau,\eta) + \phi_1(\tau,\eta)$ with

$$\begin{aligned} &-\partial_{\eta\eta}\phi_0 = C\delta(\eta - 2e^{(-1/2+\gamma)\tau}) \\ &\phi_0(\tau,0) = \phi_0(\tau,a) = 0, \end{aligned}$$

and

$$\begin{aligned} &\partial_\tau\phi_1 - \partial_{\eta\eta}\phi_1 - 11\phi_1 + \Big(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\Big)\partial_\eta\phi_1 \\ &= -\partial_\tau\phi_0 - \Big(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2}\Big)\partial_\eta\phi_0 + 11\phi_0, \\ &\phi_1(\tau,0) = 0, \quad \phi_1(\tau,a) = 0. \end{aligned}$$

The function ϕ_0 is easily computed:

$$\phi_0(\tau,\eta) = \begin{cases} C\dfrac{(a-\xi_\gamma(\tau))}{a}\eta, & \eta \le \xi_\gamma(\tau) \\ C\dfrac{(a-\eta)\xi_\gamma(\tau)}{a}, & \eta \ge \xi_\gamma(\tau) \end{cases} \tag{5.67}$$

with $\xi_\gamma(\tau) = 2e^{(-1/2+\gamma)\tau}$. So, all the quantities ϕ_0, $\partial_\eta\phi_0$ and $\partial_\tau\phi_0$ are uniformly bounded, hence (recall $\lambda_a \ge 100$) we have (5.66). It follows that

$$|w(\tau,\eta)| \le Ce^{10\gamma\tau}\eta \ \text{ for } \tau \ge 0 \text{ and } 0 \le \eta \le a. \tag{5.68}$$

Returning to $Q(\tau)$ given by (5.58), we deduce, using (5.35) and (5.68) the following improvement of (5.37):

$$\begin{aligned} I_1 &\le \int_0^{\eta_\gamma(\tau)} e_0(\eta)e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\widetilde{v}+\widetilde{V}^{app})|w|d\eta \le Ce^{10\gamma\tau}\int_0^{\eta_\gamma(\tau)}\eta^2e^\tau d\eta \\ &\le Ce^{10\gamma\tau}[\eta_\gamma(\tau)]^3e^\tau = Ce^{(-1/2+20\gamma)\tau}, \end{aligned} \tag{5.69}$$

while (5.38) can be improved to

$$\begin{aligned} I_2 &\le \int_{\eta_\gamma(\tau)}^a e_0(\eta)e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\widetilde{v}+\widetilde{V}^{app})|w|d\eta + Ce^{3\tau/2}e^{-a/2e^{\tau/2}} \\ &\le Ce^{10\gamma\tau}\int_{\eta_\gamma(\tau)}^a \eta e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(1+\eta e^{\tau/2})\eta d\eta \\ &\le Ce^{10\gamma\tau}e^\tau e^{\tau/2}\int_{\eta_\gamma(\tau)}^a \eta^3 e^{-(\eta-\eta_\gamma(\tau))e^{\tau/2}}d\eta \le Ce^{10\gamma\tau}(\eta_\gamma(\tau))^3e^\tau \le Ce^{(-1/2+20\gamma)\tau}. \end{aligned} \tag{5.70}$$

Equation (5.56) for $\langle e_0, w(\tau)\rangle$ now gives

$$|\langle e_0, w(\tau)\rangle| \le C\int_\tau^{+\infty} e^{-(\tau-\tau')/2}e^{(-\frac{1}{2}+20\gamma)\tau}d\tau' \le Ce^{-(\frac{1}{2}-20\gamma)\tau}. \tag{5.71}$$

Moreover, equation (5.53) for $w^\perp$ shows that the only "slightly large" term that potentially can make $w^\perp(\tau,\eta)$ grow in τ is $J_1(\tau)$ given by (5.45)

$$J_1(\tau) = \langle e_0, w(\tau)\rangle \int_{\mathbb{R}_+} e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}+x_0}(\widetilde{v}+\widetilde{V}^{app})e_0w^\perp d\eta. \tag{5.72}$$

However, we may now use (5.71) to bootstrap (5.46) to

$$|J_1(\tau)| \le Ce^{(-1/2+40\gamma)\tau}. \tag{5.73}$$

Using this in (5.53) gives us

$$\|w^\perp\| \le Ce^{-(\frac{1}{2}-50\gamma)\tau}. \tag{5.74}$$

This implies the same estimate for the full solution w. As in the passage from (5.54) to (5.55) we obtain

$$\|w\|_{L^2(\mathbb{R}_+)} + \|w\|_\infty \le C_\gamma e^{(-1/2+100\gamma)\tau}. \tag{5.75}$$

Concluding the proof of Theorem 5.1

The last step seems to yield a $t^{\gamma-1/2}$ decay for w. However, recall that we want a $t^{\gamma-1}$ estimate. To obtain it, it suffices to remember that $w(\tau,\eta)$ solves a Dirichlet problem, hence w has an extra η factor. To realize that, it suffices to argue just as in the proof of estimate (5.68), up to the fact that, this time, the slow $e^{10\gamma\tau}$ growth is replaced by the decay $e^{-(1/2-100\gamma)\tau}$. Repeating this argument, we end up with

$$|w(\tau,\eta)| \leq C_\gamma \eta e^{-(1/2-100\gamma)\tau}. \tag{5.76}$$

To obtain the full Theorem 5.1, it suffices to unzip (5.76) by reverting to the (t,x) variables. We obtain

$$|v(t,x) - V^{app}(t,x)| \leq \frac{C}{t^{\frac{1}{2}-100\gamma}} \frac{x+t^\gamma}{\sqrt{t}}, \qquad \text{for } \ x > -t^\gamma + 2, \ \ t \geq 1. \tag{5.77}$$

This implies (5.1). □

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