Convergence to a single wave in the Fisher-KPP equation

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Dedicated to H. Brezis, with admiration and respect

Abstract

We study the large time asymptotics of a solution of the Fisher-KPP reaction-diffusion equation, with an initial condition that is a compact perturbation of a step function. A well-known result of Bramson states that, in the reference frame moving as $2t-(3/2)\log t+x_\infty$, the solution of the equation converges as $t\to+\infty$ to a translate of the traveling wave corresponding to the minimal speed $c_*=2$. The constant x_∞ depends on the initial condition $u(0,x)$. The proof is elaborate, and based on probabilistic arguments. The purpose of this paper is to provide a simple proof based on PDE arguments.

1 Introduction

We consider the Fisher-KPP equation:

$$u_t-u_{xx}=u-u^2,\quad t>0,\ x\in\mathbb{R},\tag{1.1}$$

with an initial condition $u(0,x)=u_0(x)$ which is a compact perturbation of a step function, in the sense that there exist x_1 and x_2 so that $u_0(x)=1$ for all $x\le x_1$, and $u_0(x)=0$ for all $x\ge x_2$.

This equation has a traveling wave solution $u(t,x)=\phi(x-2t)$, moving with the minimal speed $c_*=2$, connecting the stable equilibrium $u\equiv 1$ to the unstable equilibrium $u\equiv 0$:

$$\begin{aligned}&-\phi''-2\phi'=\phi-\phi^2,\\&\phi(-\infty)=1,\quad \phi(+\infty)=0.\end{aligned}\tag{1.2}$$

Each solution $\phi(\xi)$ of (1.2) is a shift of a fixed profile $\phi_*(\xi)$: $\phi(\xi)=\phi_*(\xi+s)$, with some fixed $s\in\mathbb{R}$. The profile $\phi_*(\xi)$ satisfies the asymptotics

$$\phi_*(\xi)=(\xi+k)e^{-\xi}+O(e^{-(1+\omega_0)\xi}),\quad \xi\to+\infty,\tag{1.3}$$

with two universal constants $\omega_0>0$, $k\in\mathbb{R}$.

The large time behaviour of the solutions of this problem has a long history, starting with a striking paper of Fisher [10], which identifies the spreading velocity $c_*=2$ via numerical computations and other arguments. In the same year, the pioneering KPP paper [15] proved that the solution of (1.1), starting from a step function: $u_0(x)=1$ for $x\le 0$, $u_0(x)=0$ for $x\ge 0$, converges to ϕ_* in the following sense: there is a function

$$\sigma_\infty(t)=2t+o(t),\tag{1.4}$$

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such that

$$\lim_{t\to+\infty} u(t, x+\sigma_\infty(t)) = \phi_*(x).$$

Fisher has already made an informal argument that the $o(t)$ in (1.4) is of the order $O(\log t)$. An important series of papers by Bramson proves the following

Theorem 1.1 ([5], [6]) *There is a constant x_∞, depending on u_0, such that*

$$\sigma_\infty(t) = 2t - \frac{3}{2}\log t - x_\infty + o(1), \ \text{ as } t \to +\infty.$$

Theorem 1.1 was proved through elaborate probabilistic arguments. A generalization is provided by Lau [17], using the decrease of the number of intersection points for any pair of solutions of the parabolic Cauchy problem.

A natural question is to prove Theorem 1.1 with purely PDE arguments. In that spirit, a weaker version, precise up to the $O(1)$ term, (but valid also for a much more difficult case of the periodic in space coefficients), is the main result of [11, 12]:

$$\sigma(t) = 2t - \frac{3}{2}\log t + O(1) \text{ as } t \to +\infty. \tag{1.5}$$

Here, we will give a simple and robust proof of Theorem 1.1. These ideas are further developed to study the refined asymptotics of the solutions in [21].

The paper is organized as follows. In Section 2, we shortly describe some connections between the Fisher-KPP equation (1.1) and the branching Brownian motion. In Section 3, we explain, in an informal way, the strategy of the proof of the theorem: in a nutshell, the solution is slaved to the dynamics at $x = O(\sqrt{t})$. In Sections 4 and 5, we make the arguments of Section 3 rigorous.

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2 Probabilistic links and some related models

The time delay in models of the Fisher-KPP type has been the subject of various recent investigations, both from the PDE and probabilistic points of view. The Fisher-KPP equation appears in the theory of the branching Brownian motion (BBM) [19] as follows. Consider a BBM starting at $x = 0$ at time $t = 0$, with binary branching at rate 1. Let $X_1(t), \dots, X_{N_t}(t)$ be the descendants of the original particle at time t, arranged in the increasing order: $X_1(t) \le X_2(t) \le \dots \le X_{N_t}(t)$. Then, the probability distribution function of the maximum:

$$v(t, x) = \mathbb{P}(X_{N_t}(t) > x),$$

satisfies the Fisher-KPP equation

$$v_t = \frac{1}{2}v_{xx} + v - v^2,$$

with the initial data $v_0(x) = \mathbb{1}_{x\le 0}$. Therefore, Theorem 1.1 is about the median location of the maximal particle X_{N_t}. Building on the work of Lalley and Sellke [16], recent probabilistic analyses [1, 2, 3, 8, 7] of this particle system have identified a decorated Poisson-type point process which is the limit of the particle distribution "seen from the tip": there is a random variable $Z > 0$ such that the point process defined by the shifted particles $\{X_1(t) - c(t), \ldots, X_{N_t}(t) - c(t)\}$, with

$$c(t) = 2t - \frac{3}{2}\log t + \log Z,$$

has a well-defined limit process as $t \to \infty$. Furthermore, Z is the limit of the martingale

$$Z_t = \sum_k (2t - X_k(t))e^{X_k(t)-2t},$$

and

$$\phi_*(x) = 1 - \mathbb{E}\big[e^{-Ze^{-x}}\big] \text{ for all } x \in \mathbb{R}.$$

As we have mentioned, the logarithmic term in Theorem 1.1 arises also in inhomogeneous variants of this model. For example, consider the Fisher-KPP equation in a periodic medium:

$$u_t - u_{xx} = \mu(x)u - u^2 \tag{2.1}$$

where $\mu(x)$ is continuous and 1-periodic in $\mathbb{R}$, such that the principal periodic eigenvalue of the operator $-\partial_{xx} - \mu(x)$ is negative. Then there is a minimal speed $c_* > 0$ such that for each $c \geq c_*$, there is a unique pulsating front $U_c(t, x)$, up to a time shift [4, 13]. It was shown in [12] that there is $s_0 > 0$ such that, if $u(t, x)$ solves (2.1) with a nonnegative, nonzero, compactly supported initial condition $u_0(x)$, and $0 < s \leq s_0$, the s-level set of $u(t, x)$ satisfies

$$\sigma_s(t) = c_*t - \frac{3}{2\lambda_*}\log t + O(1).$$

This implies the convergence of $u(t, x - \sigma_s(t))$ to a closed subset of the family of minimal fronts. It is an open problem to determine whether convergence to a single front holds, not to mention the rate of this convergence. When $\mu(x) > 0$ everywhere, the solution u of the related model

$$u_t - u_{xx} = \mu(x)(u - u^2)$$

may be interpreted in terms of the extremal particle in a BBM with a spatially-varying branching rate [12].

Models with temporal variation in the branching process have also been considered. In [9], Fang and Zeitouni studied the extremal particle of such a spatially homogeneous BBM where the branching particles satisfy

$$dX(t) = \sqrt{2}\kappa(t/T)\, dB(t)$$

between branching events, rather than following a standard Brownian motion. In terms of PDE, their study corresponds to the model

$$u_t = \kappa^2(t/T)u_{xx} + f(u), \quad 0 < t < T,\ x \in \mathbb{R}. \tag{2.2}$$

They proved that if κ is increasing, and and $f$is of the Fisher-KPP type, the shift is algebraic and not logarithmic in time: there exists $C > 0$ such that

$$\frac{T^{1/3}}{C} \leq X(T) - c_{eff}T \leq CT^{1/3}, \quad c_{eff} = 2\int_0^1 \kappa(s)ds.$$

In [20], we proved the asymptotics

$$X(T) = c_{eff}T - \bar{\nu}T^{1/3} + O(\log T), \text{ with } \bar{\nu} = \beta\int_0^1 \sigma(\tau)^{1/3}\dot{\sigma}(\tau)^{2/3}d\tau. \tag{2.3}$$

Here, $\beta < 0$ is the first zero of the Airy function. Maillard and Zeitouni [18] refined the asymptotics further, proving a logarithmic correction to (2.3), and convergence of $u(T)$ to a traveling wave.

3 Strategy of the proof of Theorem 1.1

Why converge to a traveling wave?

We first provide an informal argument for the convergence of the solution of the initial value problem to a traveling wave. Consider the Cauchy problem (1.1), starting at $t = 1$ for the convenience of the notation:

$$u_t - u_{xx} = u - u^2, \quad x \in \mathbb{R}, \quad t > 1, \tag{3.1}$$

and proceed with a standard sequence of changes of variables. We first go into the moving frame:

$$x \mapsto x - 2t + (3/2)\log t,$$

leading to

$$u_t - u_{xx} - (2 - \frac{3}{2t})u_x = u - u^2. \tag{3.2}$$

Next, we take out the exponential factor: set

$$u(t,x) = e^{-x}v(t,x)$$

so that v satisfies

$$v_t - v_{xx} - \frac{3}{2t}(v - v_x) + e^{-x}v^2 = 0, \qquad x \in \mathbb{R}, \quad t > 1. \tag{3.3}$$

Observe that for any shift $x_\infty \in \mathbb{R}$, the function $V(x) = e^x\phi(x - x_\infty)$ is a steady solution of

$$V_t - V_{xx} + e^{-x}V^2 = 0.$$

We regard (3.3) as a perturbation of this equation, and expect that $v(t,x) \to e^x\phi(x - x_\infty)$ as $t \to \infty$, for some $x_\infty \in \mathbb{R}$.

The self-similar variables

We note that for $x \to +\infty$, the term $e^{-x}v^2$ in (3.3) is negligible, while for $x \to -\infty$ the same term will create a large absorption and force the solution to be close to zero. For this reason, the linear Dirichlet problem

$$\begin{aligned} &z_t - z_{xx} - \frac{3}{2t}(z - z_x) = 0, \qquad x > 0 \\ &z(t,0) = 0 \end{aligned} \tag{3.4}$$

is a reasonable proxy for (3.3) for $x \gg 1$, and, as shown in [11, 12], it provides good sub- and super-solutions for $v(t,x)$. The main lesson of [11, 12] is that everything relevant to the solutions of (3.4) happens at the spatial scale $x \sim \sqrt{t}$, and their asymptotics may be unraveled by a self-similar change of variables. Here, we will accept the full nonlinear equation (3.3) and perform directly the self-similar change of variables

$$\tau = \log t, \quad \eta = \frac{x}{\sqrt{t}} \tag{3.5}$$

followed by a change of the unknown

$$v(\tau,\eta) = e^{\tau/2}w(\tau,\eta).$$

This transforms (3.3) into

$$w_\tau - \frac{\eta}{2} w_\eta - w_{\eta\eta} - w + \frac{3}{2} e^{-\tau/2} w_\eta + e^{3\tau/2 - \eta \exp(\tau/2)} w^2 = 0, \quad \eta \in \mathbb{R}, \quad \tau > 0. \tag{3.6}$$

This transformation strengthens the reason why the Dirichlet problem (3.4) appears naturally: for

$$\eta \ll -\tau e^{-\tau/2},$$

the last term in the left side of (3.6) becomes exponentially large, which forces w to be almost 0 in this region. On the other hand, for

$$\eta \gg \tau e^{-\tau/2},$$

this term is very small, so it should not play any role in the dynamics of w in that region. The transition region has width of the order $\tau e^{-\tau/2}$.

The choice of the shift

Also, through this change of variables, we can see how a particular translation of the wave will be chosen. Considering (3.4) in the self-similar variables, one can show – see [11, 14] – that, as $\tau \to +\infty$, we have

$$e^{-\tau/2} z(\tau, \eta) \sim \alpha_\infty \eta e^{-\eta^2/4}, \quad \eta > 0, \tag{3.7}$$

with some $\alpha_\infty > 0$. Therefore, taking (3.4) as an approximation to (3.3), we should expect that

$$u(t,x) = e^{-x} v(t,x) \sim e^{-x} z(t,x) \sim e^{-x} e^{\tau/2} \alpha_\infty \eta e^{-\eta^2/4} = \alpha_\infty x e^{-x} e^{-x^2/(4t)}, \tag{3.8}$$

at least for x of the order $O(\sqrt{t})$. This determines the unique translation: if we accept that u converges to a translate x_∞ of ϕ_*, then for large x (in the moving frame) we have

$$u(t,x) \sim \phi_*(x - x_\infty) \sim x e^{-x + x_\infty}. \tag{3.9}$$

Comparing this with (3.8), we infer that

$$x_\infty = \log \alpha_\infty.$$

The difficulty with this argument, apart from the justification of the approximation

$$u(t,x) \sim e^{-x} z(t,x),$$

is that each of the asymptotics (3.8) and (3.9) uses different ranges of x: (3.8) comes from the self-similar variables in the region $x \sim O(\sqrt{t})$, while (3.9) assumes x to be large but finite. However, the self-similar analysis does not tell us at this stage what happens on the scale $x \sim O(1)$. Indeed, it is clear from (3.6) that the error in the approximation (3.7) is at least of the order $O(e^{-\tau/2})$ – note that the right side in (3.7) is a solution of (3.6) without the last two terms in the left side. On the other hand, the scale $x \sim O(1)$ corresponds to $\eta \sim e^{-\tau/2}$. Thus, the leading order term and the error in (3.7) are of the same size for $x \sim O(1)$, which means that we can not extract information directly from (3.7) on that scale.

To overcome this issue, we proceed in two steps: first we use the self-similar variables to prove stabilization (that is, (3.8) holds) at the spatial scales $x \sim O(t^\gamma)$ with a small $\gamma > 0$, and not just at the diffusive scale $O(\sqrt{t})$. This boils down to showing that

$$w(\tau, \eta) \sim \alpha_\infty \eta e^{-\eta^2/4}$$

for the solution to (3.6), even for $\eta \sim e^{-(1/2-\gamma)\tau}$. Next, we show that this stabilization is sufficient to ensure the stabilization on the scale $x \sim O(1)$ and convergence to a unique wave. This is the core of the argument: everything happening at $x \sim O(1)$ should be governed by the tail of the solution – the fronts are pulled.

4 Convergence to a single wave as a consequence of the diffusive scale convergence

The proof of Theorem 1.1 relies on the following two lemmas. The first is a consequence of [11].

Lemma 4.1 *The solution of (3.2) with $u(1,x) = u_0(x)$ satisfies*

$$\lim_{x\to-\infty} u(t,x) = 1, \quad \lim_{x\to+\infty} u(t,x) = 0, \tag{4.1}$$

both uniformly in $t > 1$.

The main new step is to establish the following.

Lemma 4.2 *There exists a constant $\alpha_\infty > 0$ with the following property. For any $\gamma > 0$ and all $\varepsilon > 0$ we can find T_ε so that for all $t > T_\varepsilon$ we have*

$$|u(t,x_\gamma) - \alpha_\infty x_\gamma e^{-x_\gamma} e^{-x_\gamma^2/(4t)}| \le \varepsilon x_\gamma e^{-x_\gamma} e^{-x_\gamma^2/(6t)}, \tag{4.2}$$

with $x_\gamma = t^\gamma$.

We postpone the proof of this lemma for the moment, and show how it is used. A consequence of Lemma 4.2 is that the problem for the moment is to understand, for a given $\alpha > 0$, the behavior of the solutions of

$$\begin{aligned} &\frac{\partial u_\alpha}{\partial t} - \frac{\partial^2 u_\alpha}{\partial x^2} - (2 - \frac{3}{2t})\frac{\partial u_\alpha}{\partial x} - u_\alpha + u_\alpha^2 = 0, \quad x \le x_\gamma(t) \\ &u_\alpha(t,t^\gamma) = \alpha t^\gamma e^{-t^\gamma - t^{2\gamma-1}/4}, \end{aligned} \tag{4.3}$$

for $t > T_\varepsilon$, with the initial condition $u_\alpha(T_\varepsilon, x) = u(T_\varepsilon, x)$. In particular, we will show that $u_{\alpha_\infty \pm \varepsilon}(t,x)$ converge, as $t \to +\infty$, to a pair of steady solutions, separated only by an order $O(\varepsilon)$-translation. Note that the function $v(t,x) = e^x u_\alpha(t,x)$ solves

$$\begin{aligned} &v_t - v_{xx} + \frac{3}{2t}(v_x - v) + e^{-x}v^2 = 0, \quad x \le t^\gamma \\ &v(t,t^\gamma) = \alpha t^\gamma e^{-t^{2\gamma-1}/4}. \end{aligned} \tag{4.4}$$

Since we anticipate that the tail is going to dictate the behavior of u_α, we choose the translate of the wave that matches exactly the behavior of $u_\alpha(t,x)$ at the boundary $x = t^\gamma$: set

$$\psi(t,x) = e^x \phi_*(x + \zeta(t)). \tag{4.5}$$

Recall that $\phi_*(x)$ is the traveling wave profile. We look for a function $\zeta(t)$ in (4.5) such that

$$\psi(t,t^\gamma) = v(t,t^\gamma). \tag{4.6}$$

In view of the expansion (1.3), we should have, with some $\omega_0 > 0$:

$$e^{-\zeta(t)}(t^\gamma + \zeta(t) + k) + O(e^{-\omega_0 t^\gamma}) = \alpha t^\gamma e^{-1/(4t^{1-2\gamma})},$$

which implies

$$\zeta(t) = -\log\alpha - (\log\alpha - k)t^{-\gamma} + O(t^{-2\gamma}),$$

and thus (for $\gamma \in (0, 1/3)$), we have

$$|\dot{\zeta}(t)| \leq \frac{C}{t^{1+\gamma}}.$$

The equation for the function ψ is

$$\psi_t - \psi_{xx} + \frac{3}{2t}(\psi_x - \psi) + e^{-x}\psi^2 = -\dot{\zeta}\psi + \dot{\zeta}\psi_x + \frac{3}{2t}(\psi_x - \psi) = O(\frac{x}{t}) = O(t^{-1+\gamma}), \quad |x| < t^\gamma.$$

In addition, the left side above is exponentially small for $x < -t^\gamma$ because of the exponential factor in (4.5). Hence, the difference $s(t, x) = v(t, x) - \psi(t, x)$ satisfies

$$\begin{aligned} & s_t - s_{xx} + \frac{3}{2t}(s_x - s) + e^{-x}(v + \psi)s = O(t^{-1+\gamma}), \quad |x| \leq t^\gamma \\ & s(t, -t^\gamma) = O(e^{-t^\gamma}), \quad s(t, t^\gamma) = 0. \end{aligned} \tag{4.7}$$

Proposition 4.3 *We have*

$$\lim_{t \to +\infty} \sup_{|x| \leq t^\gamma} |s(t, x)| = 0. \tag{4.8}$$

Proof. The issue is whether the Dirichlet boundary conditions would be stronger than the force in the right side of (4.7). Since the principal Dirichlet eigenvalue for the Laplacian in $(-t^\gamma, t^\gamma)$ is $\pi^2/t^{2\gamma}$, investigating (4.7) is, heuristically, equivalent to solving the ODE

$$f'(t) + (1 - 2\gamma)t^{-2\gamma} f = \frac{1}{t^{1-\gamma}}. \tag{4.9}$$

The coefficient $(1 - 2\gamma)$ is chosen simply for convenience and can be replaced by another constant. The solution of (4.9) is

$$f(t) = f(1)e^{(-t^{-2\gamma+1}+1)} + \int_1^t s^{\gamma-1} e^{(-t^{-2\gamma+1}+s^{-2\gamma+1})} ds.$$

Note that $f(t)$ tends to 0 as $t \to +\infty$ a little faster than $t^{3\gamma-1}$ as soon as $\gamma < 1/3$, so the analog of (4.8) holds for the solutions of (4.9). With this idea in mind, we are going to look for a super-solution to (4.7), in the form

$$\overline{s}(t, x) = t^{-\lambda} \cos\left(\frac{x}{t^{\gamma+\varepsilon}}\right), \tag{4.10}$$

where λ, γ and ε will be chosen to be small enough. We now set $T_\varepsilon = 1$ for convenience. We have, for $|x| \leq t^\gamma$:

$$\begin{aligned} & \overline{s}(t, x) \sim t^{-\lambda}, \; -\overline{s}_{xx} = t^{-(2\gamma+2\varepsilon)}\overline{s}(t, x), \\ & \overline{s}_t = -\frac{\lambda}{t}\overline{s} + g(t, x), \quad |g(t, x)| \leq \frac{C|x|}{t^{\lambda+\gamma+\varepsilon+1}} \leq \frac{C}{t^{1+\varepsilon}}\overline{s}(t, x), \end{aligned} \tag{4.11}$$

and

$$\frac{3}{2t}(\overline{s}_x - \overline{s})(t, x) \leq Ct^{-1}\overline{s}(t, x). \tag{4.12}$$

Gathering (4.11) and (4.12) we infer the existence of $q > 0$ such that, for t large enough:

$$\left(\partial_t - \partial_{xx} - \frac{3}{2t}(\partial_x - 1)\right)\overline{s}(t, x) \geq qt^{-(2\gamma+2\varepsilon)}\overline{s}(t, x) \geq \frac{q}{2}t^{-(2\gamma+2\varepsilon+\lambda)} \geq O(\frac{1}{t^{1-2\gamma}}),$$

as soon as γ, ε and λ are small enough. Because the right side of (4.7) does not depend on $\overline{s}$, the inequality extends to all $t \geq 1$ by replacing $\overline{s}$ by $A\overline{s}$, with A large enough, and (4.8) follows. □

Proof of Theorem 1.1

We are now ready to prove the theorem. Given $\varepsilon > 0$, take T_ε as in Lemma 4.2. Let $u_\alpha(t,x)$ be the solution of (4.3) for $t > T_\varepsilon$, and the initial condition $u_\alpha(T_\varepsilon, x) = u(T_\varepsilon, x)$. Here, $u(t,x)$ is the solution of the original problem (3.2). It follows from Lemma 4.2 that for any $t \geq T_\varepsilon$, we have

$$u_{\alpha_\infty - \varepsilon}(t,x) \leq u(t,x) \leq u_{\alpha_\infty + \varepsilon}(t,x),$$

for all $x \leq t^\gamma$. From Proposition 4.3, we have

$$e^x \big[u_{\alpha_\infty \pm \varepsilon}(t,x) - \phi_*(x + \zeta_\pm(t))\big] = o(1), \text{ as } t \to +\infty, \tag{4.13}$$

uniformly in $x \in (-t^\gamma, t^\gamma)$, with

$$\zeta_\pm(t) = -(1 - t^{-\gamma})\log(\alpha_\infty \pm \varepsilon) + O(t^{-2\gamma}).$$

Because $\varepsilon > 0$ is arbitrary, we have

$$\lim_{t \to +\infty} \big(u(t,x) - \phi_*(x + x_\infty)\big) = 0,$$

with $x_\infty = -\log\alpha_\infty$, uniformly on compact sets. Together with Lemma 4.1, this concludes the proof of Theorem 1.1. □

5 The diffusive scale $x \sim O(\sqrt{t})$ and the proof of Lemma 4.2

Our analysis starts with (3.6), which we write as

$$w_\tau + Lw + \frac{3}{2}e^{-\tau/2}w_\eta + e^{3\tau/2 - \eta\exp(\tau/2)}w^2 = 0, \quad \eta \in \mathbb{R}, \quad \tau > 0. \tag{5.1}$$

Here, the operator L is defined as

$$Lv = -v_{\eta\eta} - \frac{\eta}{2}v_\eta - v. \tag{5.2}$$

Its principal eigenfunction on the half-line $\eta > 0$ with the Dirichlet boundary condition at $\eta = 0$ is

$$\phi_0(\eta) = \frac{\eta}{2}e^{-\eta^2/4},$$

as $L\phi_0 = 0$. The operator L has a discrete spectrum in $L^2(\mathbb{R}_+)$, weighted by $e^{-\eta^2/8}$, its non-zero eigenvalues are $\lambda_k = k \geq 1$, and the corresponding eigenfunctions are related via

$$\phi_{k+1} = \phi_k''.$$

The principal eigenfunction of the adjoint operator

$$L^*\psi = -\psi_{\eta\eta} + \frac{1}{2}\partial_\eta(\eta\psi) - \psi$$

is $\psi_0(\eta) = \eta$. Thus, the solution of the unperturbed version of (5.1) on a half-line

$$p_\tau + Lp = 0, \quad \eta > 0, \; p(\tau, 0) = 0, \tag{5.3}$$

satisfies

$$p(\tau,\eta) = \eta\,\frac{e^{-\eta^2/4}}{2\sqrt{\pi}}\int_0^{+\infty}\xi v_0(\xi)d\xi + O(e^{-\tau})e^{-\eta^2/6}, \text{ as } \tau \to +\infty, \tag{5.4}$$

and our task is to generalize this asymptotics to the full problem (5.1) on the whole line. The weight $e^{-\eta^2/6}$ in (5.4) is, of course, by no means optimal. We will prove the following:

Lemma 5.1 *Let $w(\tau,\eta)$ be the solution of (3.6) on $\mathbb{R}$, with the initial condition $w(0,\eta)=w_0(\eta)$ such that $w_0(\eta)=0$ for all $\eta>M$, with some $M>0$, and $w_0(\eta)=O(e^{\eta})$ for $\eta<0$. There exists $\alpha_\infty>0$ and a function $h(\tau)$ such that $\lim_{\tau\to+\infty} h(\tau)=0$, and such that we have, for any $\gamma'\in(0,1/2)$:*

$$w(\tau,\eta)=(\alpha_\infty+h(\tau))\eta_+e^{-\eta^2/4}+R(\tau,\eta)e^{-\eta^2/6}, \qquad \eta\in\mathbb{R}, \tag{5.5}$$

with

$$|R(\tau,\eta)|\le C_{\gamma'}e^{-(1/2-\gamma')\tau},$$

and where $\eta_+=\max(0,\eta)$.

Once again, the weight $e^{-\eta^2/6}$ is not optimal. Lemma 4.2 is an immediate consequence of this result. Indeed,

$$u(t,x)=e^{-x}\sqrt{t}w(\log t,\frac{x}{\sqrt{t}}),$$

hence Lemma 5.1 implies, with $x_\gamma=t^\gamma$,

$$\begin{aligned} e^{x_\gamma}u(t,x_\gamma)-\alpha_\infty x_\gamma e^{-x_\gamma^2/(4t)} &= \sqrt{t}w\Big(\log t,\frac{x_\gamma}{\sqrt{t}}\Big)-\alpha_\infty x_\gamma e^{-x_\gamma^2/(4t)} \\ &= h(\log t)x_\gamma e^{-x_\gamma^2/(4t)}+\sqrt{t}R\Big(\log t,\frac{x_\gamma}{\sqrt{t}}\Big)e^{-x_\gamma^2/(6t)}. \end{aligned} \tag{5.6}$$

We now take T_ε so that $|h(\log t)|<\varepsilon/3$ for all $t>T_\varepsilon$. For the second term in the right side of (5.6) we write

$$\Big|R\Big(\log t,\frac{x_\gamma}{\sqrt{t}}\Big)\Big|\sqrt{t}e^{-x_\gamma^2/(6t)}\le Ct^{\gamma'}e^{-x_\gamma^2/(6t)}\le\varepsilon x_\gamma e^{-x_\gamma^2/(6t)} \tag{5.7}$$

for $t>T_\varepsilon$ sufficiently large, as soon as $\gamma'<\gamma$. This proves (4.2). Thus, the proof of Lemma 4.2 reduces to proving Lemma 5.1. We will prove the latter by a construction of an upper and lower barrier for w with the correct behaviors.

The approximate Dirichlet boundary condition

Let us come back to why the solution of (5.1) must approximately satisfy the Dirichlet boundary condition at $\eta=0$. Recall that w is related to the solution of the original KPP problem via

$$w(\tau,\eta)=u(e^\tau,\eta e^{\tau/2})e^{-\tau/2+\eta e^{\tau/2}}.$$

The trivial a priori bound $0<u(t,x)<1$ implies that we have

$$0<w(\tau,\eta)<e^{-\tau/2+\eta e^{\tau/2}},\quad \eta<0, \tag{5.8}$$

and, in particular, we have

$$0<w(\tau,-e^{-(1/2-\gamma)\tau})\le e^{-e^{\gamma\tau}}. \tag{5.9}$$

We also have

$$w_\tau(\tau,\eta)=u_t(e^\tau,\eta e^{\tau/2})e^{\tau/2+\eta e^{\tau/2}}+\frac{\eta}{2}u_x(e^\tau,\eta e^{\tau/2})e^{\eta e^{\tau/2}}+(\frac{\eta}{2}e^{\tau/2}-\frac{1}{2})u(e^\tau,\eta e^{\tau/2})e^{-\tau/2+\eta e^{\tau/2}},$$

so that

$$
\begin{aligned}
w_\tau(\tau,-e^{-(1/2-\gamma)\tau}) &= u_t(e^\tau,-e^{-(1/2-\gamma)\tau})e^{\tau/2-e^{\gamma\tau}} - \frac{1}{2}e^{-(1/2-\gamma)\tau}u_x(e^\tau,-e^{-(1/2-\gamma)\tau})e^{-e^{\gamma\tau}}\\
&\quad - \frac{1}{2}(e^{\gamma\tau}+1)u(e^\tau,-e^{\gamma\tau})e^{-\tau/2-e^{\gamma\tau}}\\
&= O(e^{-\gamma e^{\gamma\tau}}),
\end{aligned}
\tag{5.10}
$$

for $\gamma>0$ sufficiently small. Thus, the solution of (5.1) satisfies

$$
\begin{aligned}
0 < w(\tau,-e^{-(1/2-\gamma)\tau}) \le&\ \ e^{-e^{\gamma\tau}},\\
|w_\tau(\tau,-e^{-(1/2-\gamma)\tau})| \le&\ \ Ce^{-\gamma e^{\gamma\tau}},
\end{aligned}
\tag{5.11}
$$

which we will use as an approximate Dirichlet boundary condition at $\eta=0$.

An upper barrier

Consider the solution of

$$
\begin{aligned}
&\overline{w}_\tau + L\overline{w} + \frac{3}{2}e^{-\tau/2}\overline{w}_\eta = 0,\quad \tau>0,\quad \eta > -e^{-(1/2-\gamma)\tau},\\
&\overline{w}(\tau,-e^{-(1/2-\gamma)\tau}) = e^{-e^{\gamma\tau}},
\end{aligned}
\tag{5.12}
$$

with a compactly supported initial condition $\bar{w}_0(\eta)=\bar{w}(0,\eta)$ chosen so that $\bar{w}_0(\eta)\ge u_1(\eta)e^\eta$. Here, $\gamma\in(0,1/2)$ should be thought of as a small parameter.

It follows from (5.11) that $\overline{w}(\tau,\eta)$ is an upper barrier for $w(\tau,\eta)$. That is, we have

$$
w(\tau,\eta)\le\bar{w}(\tau,\eta),\ \text{for all } \tau>0 \text{ and } \eta>-e^{-(1/2-\gamma)\tau}.
$$

It is convenient to make a change of variables

$$
\bar{w}(\tau,\eta)=\bar{p}(\tau,\eta+e^{-(1/2-\gamma)\tau})+e^{-e^{\gamma\tau}}g(\eta+e^{-(1/2-\gamma)\tau}),
\tag{5.13}
$$

where $g(\eta)$ is a smooth monotonic function such that $g(\eta)=1$ for $0\le\eta<1$ and $g(\eta)=0$ for $\eta>2$. The function $\bar{p}$ satisfies

$$
\bar{p}_\tau + L\bar{p} + (\gamma e^{-(1/2-\gamma)\tau}+\frac{3}{2}e^{-\tau/2})\bar{p}_\eta = G(\tau,\eta)e^{-e^{\gamma\tau}},\quad \eta>0,\quad \bar{p}(\tau,0)=0,
\tag{5.14}
$$

for $\tau>0$, with a smooth function $G(\tau,\eta)$ supported in $0\le\eta\le2$, and the initial condition

$$
\bar{p}_0(\eta)=\bar{w}_0(\eta-1)-e^{-1}g(\eta),
$$

which also is compactly supported.

We will allow (5.14) to run for a large time T, after which time we can treat the right side and the last term in the left side of (5.14) as a small perturbation. A variant of Lemma 2.2 from [11] implies that $\bar{p}(T,\eta)e^{\eta^2/6}\in L^2(\mathbb{R}_+)$ for all $T>0$, as well as the following estimate:

Lemma 5.2 *Consider $\omega\in(0,1/2)$ and $G(\tau,\eta)$ smooth, bounded, and compactly supported in $\mathbb{R}_+$. Let $p(\tau,\eta)$ solve*

$$
|p_\tau + Lp| \le \varepsilon e^{-\omega\tau}(|p_\eta|+|p|+G(\tau,\eta)),\quad \tau>0,\ \eta>0,\qquad p(\tau,0)=0.
\tag{5.15}
$$

with the initial condition $p_0(\eta)$ such that $p_0(\eta)e^{\eta^2/6}\in L^2(\mathbb{R}_+)$. There exists $\varepsilon_0>0$ and $C>0$ (depending on p_0) such that, for all $0<\varepsilon<\varepsilon_0$, we have

$$
p(\tau,\eta)=\eta\Big(\frac{e^{-\eta^2/4}}{2\sqrt{\pi}}\Big(\int_0^{+\infty}\xi p_0(\xi)d\xi+\varepsilon R_1(\tau,\eta)\Big)+\varepsilon e^{-\omega\tau}R_2(\tau,\eta)e^{-\eta^2/6}+e^{-\tau}R_3(\tau,\eta)e^{-\eta^2/6}\Big),
\tag{5.16}
$$

where $\|R_{1,2,3}(\tau,\cdot)\|_{C^3}\le C$ for all $\tau>0$.

For any $\varepsilon > 0$, we may choose T sufficiently large, and $\omega \in (0, 1/2 - \gamma)$ so that

$$|\bar{p}_\tau + L\bar{p}| \le \varepsilon e^{-\omega(\tau - T)}(|\bar{p}_\eta| + |G(\tau,\eta)|), \quad \tau > T,\ \eta > 0, \qquad p(\tau, 0) = 0. \tag{5.17}$$

This follows from (5.14). Then, applying Lemma 5.2 for $\tau > T$, we have

$$\bar{p}(\tau,\eta) = \eta\Big(\frac{e^{-\eta^2/4}}{2\sqrt{\pi}}\Big(\int_0^{+\infty}\xi\bar{p}(T,\xi)d\xi + \varepsilon R_1(\tau,\eta)\Big) + \varepsilon e^{-\omega(\tau-T)}R_2(\tau,\eta)e^{-\eta^2/6} + e^{-(\tau-T)}R_3(\tau,\eta)e^{-\eta^2/6}\Big). \tag{5.18}$$

We claim that with a suitable choice of $\bar{w}_0$, the integral term in (5.18) is bounded from below:

$$\int_0^\infty \eta\bar{p}(\tau,\eta)d\eta \ge 1, \text{ for all } \tau > 0. \tag{5.19}$$

Indeed, multiplying (5.14) by η and integrating gives

$$\frac{d}{d\tau}\int_0^\infty \eta\bar{p}(\tau,\eta)d\eta = (\gamma e^{-(1/2-\gamma)\tau} + \frac{3}{2}e^{-\tau/2})\int_0^\infty \bar{p}(\tau,\eta)d\eta + e^{-e^{\gamma\tau}}\int G(\tau,\eta)\eta d\eta. \tag{5.20}$$

The function $G(\tau,\eta)$ need not have a sign, hence a priori we do not know that $\bar{p}(\tau,\eta)$ is positive everywhere. However, it follows from (5.14) that the negative part of $\bar{p}$ is bounded as

$$\int_0^\infty \bar{p}(\tau,\eta)d\eta \ge -C_0,$$

for all $\tau > 0$, with the constant C_0 which does not depend on $\bar{w}_0(\eta)$ on the interval $[2,\infty)$. Thus, we deduce from (5.20) that for all $\tau > 0$ we have

$$\int_0^\infty \eta\bar{p}(\tau,\eta)d\eta \ge \int_0^\infty \eta\bar{w}_0(\eta)d\eta - C_0', \tag{5.21}$$

with, once again, C_0' independent of $\bar{w}_0$. Therefore, after possibly increasing $\bar{w}_0$ we may ensure that (5.19) holds.

It follows from (5.19) and (5.18) that there exists a sequence $\tau_n \to +\infty$, $C > 0$ and a function $\overline{W}_\infty(\eta)$ such that

$$C^{-1}\eta e^{-\eta^2/4} \le \overline{W}_\infty(\eta) \le C\eta e^{-\eta^2/4}, \tag{5.22}$$

and

$$\lim_{n\to+\infty} e^{\eta^2/8}|\bar{p}(\tau_n,\eta) - \overline{W}_\infty(\eta)| = 0, \tag{5.23}$$

uniformly in η on the half-line $\eta \ge 0$. The same bound for the function $\bar{w}(\tau,\eta)$ itself follows:

$$\lim_{n\to+\infty} e^{\eta^2/8}|\bar{w}(\tau_n,\eta) - \overline{W}_\infty(\eta)| = 0, \tag{5.24}$$

also uniformly in η on the half-line $\eta \ge 0$.

A lower barrier

A lower barrier for $w(\tau,\eta)$ is devised as follows. First, note that the upper barrier for $w(\tau,\eta)$ we have constructed above implies that

$$e^{3\tau/2 - \eta\exp(\tau/2)}w(\tau,\eta) \le C_\gamma e^{-\exp(\gamma\tau/2)},$$

as soon as

$$\eta \geq e^{-(1/2-\gamma)\tau},$$

with $\gamma \in (0,1/2)$, and $C_\gamma > 0$ is chosen sufficiently large. Thus, a lower barrier $\underline{w}(\tau,\eta)$ can be defined as the solution of

$$\underline{w}_\tau + L\underline{w} + \frac{3}{2}e^{-\tau/2}\underline{w}_\eta + C_\gamma e^{-\exp(\gamma\tau/2)}\underline{w} = 0, \quad \underline{w}(\tau, e^{-(1/2-\gamma)\tau}) = 0, \quad \eta > e^{-(1/2-\gamma)\tau}, \tag{5.25}$$

and with an initial condition $\underline{w}_0(\eta) \leq w_0(\eta)$. This time it is convenient to make the change of variables

$$\underline{w}(\tau,\eta) = \underline{z}(\tau, \eta - e^{-(1/2-\gamma)\tau})$$

so that

$$\underline{z}_\tau + L\underline{z} + (-\gamma e^{-(1/2-\gamma)\tau} + \frac{3}{2}e^{-\tau/2})\underline{z}_\eta + C_\gamma e^{-\exp(\gamma\tau/2)}\underline{z} = 0, \quad \eta > 0, \quad \underline{z}(\tau,0) = 0, \tag{5.26}$$

We could now try to use an abstract stable manifold theorem to prove that

$$\underline{I}(\tau) := \int_0^\infty \eta\underline{z}(\tau,\eta)d\eta \geq c_0 > 0, \quad \text{for all } \tau > 0. \tag{5.27}$$

That is, $\underline{I}(\tau)$ remains uniformly bounded away from 0. However, to keep this paper self-contained, we give a direct proof of (5.27). We look for a sub-solution to (5.26) in the form

$$\underline{p}(\tau,\eta) = \left(\zeta(\tau)\phi_0(\eta) - q(\tau)\eta e^{-\eta^2/8}\right)e^{-F(\tau)}, \tag{5.28}$$

where

$$F(\tau) = \int_0^\tau C_\gamma e^{-\exp(\gamma s/2)}\, ds,$$

and with the functions $\zeta(\tau)$ and $q(\tau)$ satisfying

$$\zeta(\tau) \geq \zeta_0 > 0, \quad \dot{\zeta}(\tau) < 0, \quad q(\tau) > 0, \quad q(\tau) = O(e^{-\tau/4}). \tag{5.29}$$

In other words, we wish to devise $\underline{p}(\tau,\eta)$ as in (5.28)-(5.29) such that

$$\underline{p}(0,\eta) \leq \underline{z}(0,\eta) = w_0(\eta+1), \tag{5.30}$$

and

$$\mathcal{L}(\tau)\underline{p} \leq 0, \tag{5.31}$$

with

$$\mathcal{L}(\tau)\underline{p} = \underline{p}_\tau + L\underline{p} + (-\gamma e^{-(1/2-\gamma)\tau} + \frac{3}{2}e^{-\tau/2})\underline{p}_\eta.$$

Notice that the choice of $F(\tau)$ in (5.28) has eliminated a low order term involving $C_\gamma e^{-\exp(\gamma\tau/2)}$. For convenience, let us define

$$h(\tau) = -\gamma e^{-(1/2-\gamma)\tau} + \frac{3}{2}e^{-\tau/2},$$

which appears in (5.26). Because $L\phi_0 = 0$ and because

$$L(\eta e^{-\eta^2/8}) = \eta L e^{-\eta^2/8} = (\frac{\eta^2}{16} - \frac{3}{4})\eta e^{-\eta^2/8},$$

we find that

$$\mathcal{L}(\tau)\underline{p} = \dot{\zeta}\phi_0 + \zeta h(\tau)\phi_0' - \left(\dot{q} + (\frac{\eta^2}{16} - \frac{3}{4})q\right)\eta e^{-\eta^2/8} + q\frac{\eta^2}{4}e^{-\eta^2/8}h(\tau) - qe^{-\eta^2/8}h(\tau).$$

Let us write this as

$$\eta^{-1}e^{\eta^2/8}\mathcal{L}(\tau)\underline{p} = \dot{\zeta}\eta^{-1}\phi_0 e^{\eta^2/8} + \eta^{-1}h(\tau)\left(\zeta e^{\eta^2/8}\phi_0' + q\left(\frac{\eta^2}{4} - 1\right)\right) - \left(\dot{q} + (\frac{\eta^2}{16} - \frac{3}{4})q\right). \quad (5.32)$$

Our goal is to choose $\zeta(\tau)$ and $q(\tau)$ such that (5.29) holds and the right side of (5.32) is non-positive after a certain time τ_0, possibly quite large. However, and this is an important point, this time τ_0 will not depend on the initial condition $w_0(\eta)$.

Let us restrict the small parameter γ to the interval $(0, 1/4)$. Observe that if $\tau_0 > 0$ is sufficiently large, then $h(\tau) < 0$ and $|h(\tau)| \le e^{-\tau/4}$ for all $\tau \ge \tau_0$. As $\phi_0(\eta) = \eta e^{-\eta^2/4}$, note that in (5.32) both $\phi_0'(\eta)e^{\eta^2/8}$ and $\phi_0(\eta)e^{\eta^2/8}$ are bounded functions. In particular, if τ_0 is large enough then

$$|\phi_0' e^{\eta^2/8}h(\tau)| \le e^{-\tau/4}$$

for all $\tau \ge \tau_0$, $\eta \ge 0$.

Note also that for all $\eta \ge \eta_1 = \sqrt{28}$ we have

$$\frac{\eta^2}{16} - \frac{3}{4} \ge 1 \qquad \text{and} \qquad \frac{\eta^2}{4} - 1 \ge 0. \quad (5.33)$$

Therefore, on the interval $\eta \in [\eta_1, \infty)$ and for $\tau \ge \tau_0$, (5.32) is bounded by

$$\eta^{-1}e^{\eta^2/8}\mathcal{L}(\tau)\underline{p} \;\le\; \eta^{-1}h(\tau)\zeta e^{\eta^2/8}\phi_0' - (\dot{q} + q) \le \zeta(\tau)e^{-\tau/4} - (\dot{q} + q)\,,$$

assuming $q(\tau) > 0$ and $\dot{\zeta} < 0$. Hence, if $q(\tau)$ and $\zeta(\tau)$ are chosen to satisfy the differential inequality

$$\dot{q} + q - e^{-\tau/4}\zeta \ge 0, \quad \tau \ge \tau_0, \quad (5.34)$$

then we will have

$$\mathcal{L}(\tau)\underline{p} \le 0 \text{ for } \tau \ge \tau_0 \text{ and } \eta \ge \eta_1, \quad (5.35)$$

provided that $\dot{\zeta} \le 0$, as presumed in (5.29). Still assuming $\dot{\zeta} \le 0$ on $(\tau_0, +\infty)$, a sufficient condition for (5.34) to be satisfied is:

$$\dot{q} + q \ge e^{-\tau/4}\zeta(\tau_0), \quad \tau \ge \tau_0.$$

Hence, we choose

$$q(\tau) = e^{-(\tau-\tau_0)} + \frac{4}{3}e^{-\tau/4}\zeta(\tau_0). \quad (5.36)$$

Note that $q(\tau)$ satisfies the assumptions on q in (5.29).

Let us now deal with the range $\eta \in [0, \eta_1]$. The function $\eta^{-1}\phi_0(\eta)$ is bounded on $\mathbb{R}$ and it is bounded away from 0 on $[0, \eta_1]$. Define

$$\varepsilon_1 = \min_{\eta \in [0, \eta_\gamma]} \eta^{-1}\phi_0(\eta)e^{\eta^2/8} > 0.$$

As $h(\tau) < 0$ for $\tau \ge \tau_0$, on the interval $[0, \eta_1]$, we can bound (5.32) by

$$\eta^{-1}e^{\eta^2/8}\mathcal{L}(\tau)\underline{p} \;\le\; \varepsilon_1\dot{\zeta}(\tau) + \eta^{-1}h(\tau)\left(\zeta e^{\eta^2/8}\phi_0' - q\right) - \left(\dot{q} - \frac{3}{4}q\right). \quad (5.37)$$

For $\eta \in [1, \eta_1]$, where $\eta^{-1} < 1$, we have

$$\eta^{-1} e^{\eta^2/8} \mathcal{L}(\tau) \underline{p} \quad \leq \quad \varepsilon_1 \dot{\zeta}(\tau) + e^{-\tau/4}(\zeta + q) - \left(\dot{q} - \frac{3}{4} q \right). \tag{5.38}$$

To make this non-positive, we choose ζ to satisfy

$$\varepsilon_1 \dot{\zeta}(\tau) \leq \dot{q} - \frac{3}{4} q - e^{-\tau/4}(\zeta + q) = e^{-\tau/4} \zeta(\tau_0) - \frac{7}{4} q(\tau) - e^{-\tau/4}(\zeta(\tau) + q(\tau)), \tag{5.39}$$

where the last equalilty comes from (5.36). Assuming $\dot{\zeta} < 0$, we have $\zeta(\tau) < \zeta(\tau_0)$, so a sufficient condition for (5.39) to hold when $\tau \geq \tau_0$ is simply

$$\varepsilon_1 \dot{\zeta}(\tau) \quad \leq \quad -3q(\tau). \tag{5.40}$$

For η near 0, the dominant term in (5.37) is $\eta^{-1} h(\tau) \left(\zeta e^{\eta^2/8} \phi_0' - q \right)$. Define

$$\varepsilon_2 = \min_{\eta \in [0,1]} \phi_0'(\eta) e^{\eta^2/8} > 0.$$

Therefore, if we can arrange that $\zeta(\tau) > q(\tau)/\varepsilon_2$, then for $\eta \in [0, 1]$, we have $\zeta e^{\eta^2/8} \phi_0' - q \geq 0$, so

$$\eta^{-1} h(\tau) \left(\zeta e^{\eta^2/8} \phi_0' - q \right) \leq 0.$$

In this case,

$$\eta^{-1} e^{\eta^2/8} \mathcal{L}(\tau) \underline{p} \quad \leq \quad \varepsilon_1 \dot{\zeta}(\tau) - \left(\dot{q} - \frac{3}{4} q \right). \tag{5.41}$$

which is non-positive for $\tau \geq \tau_0$, due to (5.39). In summary, we will have $\mathcal{L}(\tau)\underline{p} \leq 0$ in the interval $\eta \in [0, \eta_1]$ and $\tau \geq \tau_0$ if ζ satisfies (5.40) and $\zeta(\tau) > q(\tau)/\varepsilon_2$ for $\tau \geq \tau_0$. In view of this, we let $\zeta(\tau)$ have the form

$$\zeta(\tau) = a_2 + a_3 e^{-(\tau - \tau_0)/4}.$$

Thus, (5.40) holds if

$$-\frac{\varepsilon_1 a_3}{4} e^{-(\tau - \tau_0)/4} \leq -3q = -3e^{-(\tau - \tau_0)} - 4e^{-\tau/4}(a_2 + a_3), \quad \tau \geq \tau_0.$$

Hence it suffices that

$$\frac{\varepsilon_1 a_3}{4} \geq 3 + 4e^{-\tau_0/4}(a_2 + a_3)$$

holds; this may be achieved with $a_2, a_3 > 0$ if τ_0 is large enough. Then we may take a_2 large enough so that $\zeta(\tau) > q(\tau)/\varepsilon_2$ also holds for $\tau \geq \tau_0$; this condition translates to:

$$a_2 + a_3 e^{-(\tau - \tau_0)/4} \geq \frac{1}{\varepsilon_2} \left(e^{-(\tau - \tau_0)} + \frac{4}{3} e^{-\tau/4}(a_2 + a_3) \right), \quad \tau \geq \tau_0.$$

This also is attainable with $a_2 > \frac{1}{\varepsilon_2}$ and $a_3 > 0$ if τ_0 is chosen large enough. This completes the construction of the subsolution $\underline{p}(\tau, \eta)$ in (5.28).

Let us come back to our subsolution $\underline{z}(\tau, \eta)$. From the strong maximum principle, we know that $\underline{z}(\tau_0, \eta) > 0$ and $\partial_\eta \underline{z}(\tau_0, 0) > 0$. Hence, there is $\lambda_0 > 0$ such that

$$w(\tau_0, \eta) \geq \lambda_0 \underline{p}(\tau_0, \eta),$$

where $\underline{p}$ is given by (5.28) with ζ and q defined above, and we have for $\tau \geq \tau_0$:

$$\underline{w}(\tau,\eta) \geq \lambda_0 p(\tau,\eta).$$

This, by (5.29), bounds the quantity $\underline{I}(\tau)$ uniformly from below, so that (5.29) holds with a constant $c_0 > 0$ that depends on the initial condition w_0.

Therefore, just as in the study of the upper barrier, we obtain the uniform convergence of (possibly a subsequence of) $\underline{w}(\tau_n,\cdot)$ on the half-line $\eta \geq e^{-(1/2-\gamma)\tau}$ to a function $\underline{W}_\infty(\eta)$ which satisfies

$$C^{-1}\eta e^{-\eta^2/4} \leq \underline{W}_\infty(\eta) \leq C\eta e^{-\eta^2/4}, \tag{5.42}$$

and such that

$$\lim_{n\to+\infty} e^{\eta^2/8}|\underline{w}(\tau_n,\eta) - \underline{W}_\infty(\eta)| = 0, \qquad \eta > 0. \tag{5.43}$$

Convergence of $w(\tau,\eta)$: proof of Lemma 5.1

Let X be the space of bounded uniformly continuous functions $u(\eta)$ such that $e^{\eta^2/8}u(\eta)$ is bounded and uniformly continuous on $\mathbb{R}_+$. We deduce from the convergence of the upper and lower barriers for $w(\tau,\eta)$ (and ensuing uniform bounds for w) that there exists a sequence $\tau_n \to +\infty$ such that $w(\tau_n,\cdot)$ itself converges to a limit $W_\infty \in X$, such that $W_\infty \equiv 0$ on $\mathbb{R}_-$, and $W_\infty(\eta) > 0$ for all $\eta > 0$. Our next step is to bootstrap the convergence along a sub-sequence, and show that the limit of $w(\tau,\eta)$ as $\tau \to +\infty$ exists in the space X. First, observe that the above convergence implies that the shifted functions $w_n(\tau,\eta) = w(\tau+\tau_n,\eta)$ converge in X, uniformly on compact time intervals, as $n \to +\infty$ to the solution $w_\infty(\tau,\eta)$ of the linear problem

$$\begin{aligned} &(\partial_\tau + L)w_\infty = 0, \quad \eta > 0, \\ &w_\infty(\tau,0) = 0, \\ &w_\infty(0,\eta) = W_\infty(\eta). \end{aligned} \tag{5.44}$$

In addition, there exists $\alpha_\infty > 0$ such that $w_\infty(\tau,\eta)$ converges to $\bar{\psi}(\eta) = \alpha_\infty \eta e^{-\eta^2/4}$, in the topology of X as $\tau \to +\infty$. Thus, for any $\varepsilon > 0$ we may choose T_ε large enough so that

$$|w_\infty(\tau,\eta) - \alpha_\infty \eta e^{-\eta^2/4}| \leq \varepsilon\eta e^{-\eta^2/8} \text{ for all } \tau > T_\varepsilon, \text{ and } \eta > 0. \tag{5.45}$$

Given T_ε we can find N_ε sufficiently large so that

$$|w(T_\varepsilon + \tau_n, \eta + e^{-(1/2-\gamma)T_\varepsilon}) - w_\infty(T_\varepsilon,\eta)| \leq \varepsilon\eta e^{-\eta^2/8}, \text{ for all } n > N_\varepsilon. \tag{5.46}$$

In particular, we have

$$\alpha_\infty \eta e^{-\eta^2/4} - 2\varepsilon\eta e^{-\eta^2/8} \leq w(\tau_{N_\varepsilon} + T_\varepsilon, \eta + e^{-(1/2-\gamma)T_\varepsilon}) \leq \alpha_\infty \eta e^{-\eta^2/4} + 2\varepsilon\eta e^{-\eta^2/8}. \tag{5.47}$$

We may now construct the upper and lower barriers for the function $w(\tau + \tau_{N_\varepsilon} + T_\varepsilon, \eta + e^{-(1/2-\gamma)T_\varepsilon})$, exactly as we have done before. It follows, once again from Lemma 5.2 applied to these barriers that any limit point ϕ_∞ of $w(\tau,\cdot)$ in X as $\tau \to +\infty$ satisfies

$$(\alpha_\infty - C\varepsilon)\eta e^{-\eta^2/4} \leq \phi_\infty(\eta) \leq (\alpha_\infty + C\varepsilon)\eta e^{-\eta^2/4}. \tag{5.48}$$

As $\varepsilon > 0$ is arbitrary, we conclude that $w(\tau,\eta)$ converges in X as $\tau \to +\infty$ to $\bar{\psi}(\eta) = \alpha_\infty \eta e^{-\eta^2/4}$. Taking into account Lemma 5.2 once again, applied to the upper and lower barriers for $w(\tau,\eta)$ constructed starting from any time $\tau > 0$, we have proved Lemma 5.1, which implies Lemma 4.2.

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