

# Refined long time asymptotics for the Fisher-KPP fronts

James Nolen\*

Jean-Michel Roquejoffre†

Lenya Ryzhik‡

July 29, 2016

## Abstract

We study the one-dimensional Fisher-KPP equation, with an initial condition  $u_0(x)$  that coincides with the step function except on a compact set. A well-known result of M. Bramson in [3, 4] states that, as  $t \rightarrow +\infty$ , the solution converges to a traveling wave located at the position  $X(t) = 2t - (3/2)\log t + x_0 + o(1)$ , with the shift  $x_0$  that depends on  $u_0$ . U. Ebert and W. Van Saarloos have formally derived in [7, 17] a correction to the Bramson shift, arguing that  $X(t) = 2t - (3/2)\log t + x_0 - 3\sqrt{\pi}/\sqrt{t} + O(1/t)$ . Here, we prove that this result does hold, with an error term of the size  $O(1/t^{1-\gamma})$ , for any  $\gamma > 0$ . The interesting aspect of this asymptotics is that the coefficient in front of the  $1/\sqrt{t}$ -term does not depend on  $u_0$ .

## 1 Introduction

The goal of this paper is to provide a sharp large time asymptotics of the solutions the Fisher-KPP equation

$$u_t - u_{xx} = u - u^2, \quad t > 0, \quad x \in \mathbb{R}. \quad (1.1)$$

The initial condition  $u_{in}(x) = u(0, x)$  is a compactly supported perturbation of the step function: there exists  $L > 0$  so that  $u_{in}(x) \equiv 1$  for  $x < -L$  and  $u_{in}(x) \equiv 0$  for  $x \geq L$ . In addition, we assume that  $0 \leq u_{in}(x) \leq 1$  for all  $x \in \mathbb{R}$ , so that  $0 < u(t, x) < 1$  for all  $t > 0$  and  $x \in \mathbb{R}$ . The assumptions on the initial condition, especially as  $x \rightarrow -\infty$  can be significantly weakened, without any change in the result. The more stringent conditions are adopted purely for convenience, but we stress that the decay of  $u_0(x)$  as  $x \rightarrow +\infty$  does have to be faster than  $\exp(-x)$  for the results to hold. For a detailed study of this issue we refer to [1] where a related linear problem with similar properties has been studied.

This issue has a long history. The first contribution is that of Fisher [9], who identified the spreading velocity  $c_* = 2$  of the solutions via numerical computations and other arguments. In the same year, the pioneering KPP paper [13] proved that the solution of (1.1), starting from a step function, converges to a traveling wave profile in the following sense: there is a function

$$\sigma_\infty(t) = 2t + o(t), \quad \text{as } t \rightarrow +\infty,$$

such that

$$\lim_{t \rightarrow +\infty} u(t, x + \sigma_\infty(t)) = \phi(x). \quad (1.2)$$

---

\*Department of Mathematics, Duke University, Durham, NC 27708, USA; nolen@math.duke.edu

†Institut de Mathématiques (UMR CNRS 5219), Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse cedex, France; jean-michel.roquejoffre@math.univ-toulouse.fr

‡Department of Mathematics, Stanford University, Stanford CA, 94305, USA; ryzhik@math.stanford.edu

Here,  $\phi(x)$  is the profile of a traveling wave that connects the stable equilibrium  $u \equiv 1$  to the unstable equilibrium  $u \equiv 0$  and moves with the minimal speed  $c_* = 2$ :

$$\begin{aligned} -\phi'' - 2\phi' &= \phi - \phi^2, \\ \phi(-\infty) &= 1, \quad \phi(+\infty) = 0. \end{aligned} \tag{1.3}$$

Each solution  $\phi(\xi)$  of (1.3) is a shift of a fixed profile  $\phi_*(\xi)$ :  $\phi(\xi) = \phi_*(\xi + s)$ , with some fixed  $s \in \mathbb{R}$ . The function  $\phi_*(\xi)$  has the asymptotics

$$\phi_*(\xi) = (\xi + k)e^{-\xi} + O(e^{-(1+\omega_0)\xi}), \tag{1.4}$$

with two universal constants  $\omega_0 > 0$ ,  $k \in \mathbb{R}$ . The question whether the function  $\sigma_\infty(t)$  tends to a constant, or is a nontrivial sublinear function of time, was solved by Bramson [3], [4].

**Theorem 1.1** [3, 4] *There is a constant  $x_\infty$ , depending on the initial condition  $u_0(x)$ , such that*

$$u(t, x) = \phi_*(x - 2t + \frac{3}{2} \log t - x_\infty) + o(1), \text{ as } t \rightarrow +\infty, \tag{1.5}$$

*in the sense of uniform convergence on  $\mathbb{R}$ .*

Both papers by Bramson use probabilistic tools, and elaborate explicit computations. The reason why the probabilistic arguments are natural here is that (1.1) is related to the branching Brownian motion [14]. This connection brought a lot of recent activity on the Fisher-KPP equation in the probability and physics communities – see, for instance, [5, 6]. The results of [3, 4] were also proved by Lau [15], using the decrease of the number of intersection points between any two solutions of the parabolic Cauchy problem (1.1).

A short and simple proof of Theorem 1.1, solely relying on the PDE arguments, was given recently in [10, 16]: first, the estimate

$$\sigma_\infty(t) = 2t - \frac{3}{2} \log t + O(1)$$

was proved in [10], and then the full estimate

$$\sigma_\infty = 2t - \frac{3}{2} \log t + x_\infty, \tag{1.6}$$

with  $x_\infty$  depending on the initial datum, was proved in [16]. The ideas of [10] were developed in a more complex paper [11] to compute a logarithmic shift in a version of (1.1) with spatially periodic coefficients, a situation that had not been treated previously by the probabilistic methods.

The  $\log t$  correction in (1.6) is unusual: for reaction-diffusion equations of the type

$$u_t - u_{xx} = f(u), \quad t > 0, \quad x \in \mathbb{R}$$

one sees, most of the time, exponential in time convergence to a constant shift of a traveling wave, see for instance the classical Fife-McLeod paper [8]. This raises the question of the convergence rate in (1.5). That is, the issue is to estimate the error between

$$\sigma(t) = \sup\{x : u(t, x) = 1/2\}, \text{ and } \bar{\sigma}_\infty(t) := 2t - \frac{3}{2} \log t + x_\infty. \tag{1.7}$$

A very interesting paper of Ebert and Van Saarloos [7], completed in [17], performs a formal analysis of the convergence and states that

$$\sigma(t) = \bar{\sigma}_\infty(t) - \frac{3\sqrt{\pi}}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right). \tag{1.8}$$

A striking feature is that the predicted constant  $3\sqrt{\pi}$  in (1.8) does not depend on the initial condition, unlike the zero order term  $x_\infty$ .

Here, we prove a rigorous version of (1.8). We do this by constructing an approximate solution of (1.1), which is approached by the solutions of (1.1) at a rate almost equal to  $O(t^{-1})$ . Examination of the shift of the approximate solution provides the asymptotics of  $\sigma(t)$ .

## Main results

One of the main ingredients in this paper is the construction of an approximate solution which solves the equation up to a sufficiently small correction. Here is the precise result.

**Theorem 1.2** *For all  $\gamma \in (0, 1/8)$ , there is a one-parameter family  $(u_{app}(t, x + \lambda))_{\lambda \in \mathbb{R}}$  of the form*

$$u_{app}(t, x) = \phi_*(x - \tilde{\sigma}(t)) + u_0(t, x - \tilde{\sigma}(t)) + \frac{u_1(t, x - \tilde{\sigma}(t))}{\sqrt{t}},$$

with

$$\tilde{\sigma}(t) = 2t - \frac{3}{2} \log t - \frac{3\sqrt{\pi}}{\sqrt{t}} + O\left(\frac{1}{t^{1-\gamma}}\right). \quad (1.9)$$

The functions  $u_0(t, x)$  and  $u_1(t, x)$  are bounded and continuous, and supported in  $\{x > t^\gamma\}$ . In addition,  $u_0$  is of the class  $C^1$ , and  $u_1$  is  $C^1$  everywhere except at  $x = t^\gamma$ , where it has a jump of the  $x$ -derivative. The functions  $u_{app}(t, x)$  are approximate solutions to (1.1) in the sense that

$$\begin{aligned} \left| (\partial_t u_{app} - \partial_{xx}^2 u_{app} - u_{app} + u_{app}^2)(t, x + \tilde{\sigma}(t)) \right| &\leq C_\gamma t^{-1+2\gamma} (e^{-x} \mathbf{1}_{0 < x < t^\gamma} + \mathbf{1}_{x < 0}) \\ &+ C_\gamma t^{-3/2} e^{-x-x^2/((4+\gamma)t)} \mathbf{1}_{x > t^\gamma} + C_\gamma t^{-1+2\gamma} \delta(x - t^\gamma). \end{aligned} \quad (1.10)$$

The estimate in the right side includes the spatial behavior of the error – this is needed in the region where the solution is small. The different error sizes in the regions  $x < t^\gamma$  and  $x > t^\gamma$  in (1.10) come about because we need less precision in approximating the solution to the left of  $x = t^\gamma$ , where  $u$  is either  $O(1)$  or not too small, than to the right of  $x = t^\gamma$ , where  $u$  is “very small”. The delta function in the last term in the right side is not an issue, and can be, in principle, eliminated by a modification of the approximate solution. With this result in hand, the next task is to prove that the solutions of (1.1) converge to a shift of  $u_{app}$  at a certain rate. Our second main result is:

**Theorem 1.3** *For all  $\gamma > 0$ , there is  $C_\gamma > 0$  such that, for all  $t \geq 0$  and all  $x \in \mathbb{R}$ , we have, with  $\tilde{\sigma}(t)$  as in (1.9), and some  $x_\infty \in \mathbb{R}$ , depending on the initial condition  $u_{in}$ :*

$$|u(t, x + \tilde{\sigma}(t)) - u_{app}(t, x + \tilde{\sigma}(t) + x_\infty)| \leq \frac{C_\gamma(1 + |x|)e^{-x}}{t^{1-\gamma}}. \quad (1.11)$$

The corollary of this result is the following

**Corollary 1.4** *If we fix  $s \in (0, 1)$  and define the front position as  $\sigma_s(t) = \max\{x : u(t, x) = s\}$ , then  $\sigma_s(t)$  has an asymptotics of the form*

$$\sigma_s(t) = 2t - \frac{3}{2} \log t + x_\infty + \phi_*^{-1}(s) - \frac{3\sqrt{\pi}}{\sqrt{t}} + O\left(\frac{1}{t^{1-\gamma}}\right).$$

This confirms the Ebert-Van Saarloos prediction.

## Related works

The  $3\sqrt{\pi}$  prediction has already been verified by C. Henderson in [12], for a linearized moving boundary problem:

$$\begin{aligned} U_t - U_{xx} &= U, \quad t > 0, x > \sigma(t), \\ U(t, \sigma(t)) &= 0, \end{aligned} \tag{1.12}$$

and a compactly supported initial condition. The Dirichlet boundary condition serves the same purpose as the term  $(-u^2)$  in the KPP equation – when the moving boundary is chosen “correctly”, the solution of (1.12) does not grow or decay in time. Both solutions of (1.1) and (1.12) are governed by the “far ahead” tails where they are small – these are so called pulled fronts. The difference between (1.12) and the full KPP problem on the whole line is that (1.1) has an “inner” layer where the solution transitions from  $O(1)$  to very small values. The moving boundary in [12] is taken of the form

$$\sigma(t) = 2t - \frac{3}{2} \log t - \frac{c}{\sqrt{t}},$$

for  $t \geq 1$ . Then, if  $c = 3\sqrt{\pi}$ , there is  $\alpha_0 > 0$  such that

$$\left| \int_{\sigma(t)}^{+\infty} U(t, x) dx - \alpha_0 \right| \leq \frac{C \log t}{t}. \tag{1.13}$$

On the other hand, if  $c \neq 3\sqrt{\pi}$ , the convergence rate in (1.13) is of the order  $1/\sqrt{t}$ . We refer to a recent preprint [1] for a very detailed study of the same problem, according to the behavior of the initial condition at infinity.

As we have mentioned, an interesting feature of the problem is that the  $t^{-1/2}$  correction to the Bramson shift is universal, in the sense that it is independent of the initial datum. In addition, the analysis can be easily adapted to show that an identical result holds for more general equations of the form

$$u_t = u_{xx} + f(u),$$

with a KPP type nonlinearity:  $f \in C^1[0, 1]$ ,  $f(0) = f(1) = 0$ , and  $f(u) \leq f'(0)u$  for all  $u \in (0, 1)$ . In that case, the “ $3\sqrt{\pi}/\sqrt{t}$ ” term in the shift depends on the nonlinearity  $f(u)$  only through  $f'(0)$ , and the shape of the solution approaches the traveling wave profile at a rate almost  $O(t^{-1})$ . The preprint [2] explains why this last feature holds: if the  $t^{-1/2}$  correction were to depend of the value of the solution, this would entail wild oscillations to the front, that are not confirmed by the numerics. This result was a strong incentive for us to verify the actual value of the coefficient in front of  $1/\sqrt{t}$ .

**Organization of the paper.** In Section 2, we explain, in an informal way, why the results are likely to hold. We then prove Theorem 1.2 in Section 3, where we construct the approximate solution. In Section 4, we use the approximate solution to prove Theorem 1.3 and its corollaries.

**Acknowledgment.** JN was supported by NSF grant DMS-1351653, and LR by NSF grant DMS-1311903. JMR was supported by the European Union’s Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 321186 - ReaDi - “Reaction-Diffusion Equations, Propagation and Modelling”, as well as the ANR project NONLOCAL ANR-14-CE25-0013. LR and JMR thank the Labex CIMI for a PDE-probability quarter in Toulouse, in Winter 2014, out of which the idea of this paper grew and which provided a stimulating scientific environment for this project. The authors thank H. Berestycki for raising the issue of a mathematically rigorous proof of [7]. Finally, we thank E. Brunet and J. Berestycki for educating us about the problem, pointing out a computational mistake in an earlier version of this paper, and for making earlier versions of their preprint [2] available to us.

## 2 Strategy of the proofs

Consider the Cauchy problem (1.1) starting at  $t = 1$  for convenience of the notation:

$$\begin{aligned} u_t - u_{xx} &= u - u^2, \quad x \in \mathbb{R}, \quad t > 1, \\ u(1, x) &= u_{in}(x) = 1 - H(x) + v_0(x), \quad v_0 \text{ compactly supported,} \end{aligned} \quad (2.1)$$

and proceed with the standard sequence of changes of variables

$$x \mapsto x - 2t + (3/2) \log t, \quad u(t, x) = e^{-x} v(t, x) \quad (2.2)$$

so that  $v$  solves

$$v_t - v_{xx} - \frac{3}{2t}(v - v_x) + e^{-x} v^2 = 0, \quad x \in \mathbb{R}, \quad t > 1. \quad (2.3)$$

We stress that the removal of the exponential factor in (2.2) is critical for understanding the dynamics of  $u(t, x)$  as “basically diffusive”.

Observe that for any  $x_\infty \in \mathbb{R}$ , the function

$$V(x) = e^x \phi(x - x_\infty)$$

satisfies

$$V_t - V_{xx} + e^{-x} V^2 = 0, \quad (2.4)$$

and (2.3) is a perturbation of (2.4) for  $t \gg 1$ , and both of them are close to the diffusion equation for  $x \gg 1$ . Hence, everything relevant to the solutions of (2.3) should happen at the spatial scale  $x \sim \sqrt{t}$ . It is convenient to pass to the self-similar variables

$$\tau = \log t, \quad \eta = \frac{x}{\sqrt{t}}. \quad (2.5)$$

This transforms (2.3) into

$$w_\tau - \frac{\eta}{2} w_\eta - w_{\eta\eta} - \frac{3}{2} w + \frac{3}{2} e^{-\tau/2} w_\eta + e^{\tau - \eta \exp(\tau/2)} w^2 = 0, \quad \eta \in \mathbb{R}, \quad \tau > 0. \quad (2.6)$$

It is easy to see now why the linearized problem with the Dirichlet boundary condition at  $\eta = 0$  is a good approximation to (2.6). Indeed, for  $\eta < 0$ , the last term in the left side of (2.6) becomes very large, which forces  $w$  to be very small in this region. On the other hand, for  $\eta > 0$ , this term is very small, so it should not play any role in the dynamics of  $w$  for  $\eta > 0$ . The main step in the argument of [16] (see Lemma 5.1 therein) is a convergence result of the form

$$w(\tau, \eta) \sim \alpha_\infty \eta e^{\tau/2 - \eta^2/4}, \quad \eta > 0. \quad (2.7)$$

Specifically, as  $\tau \rightarrow \infty$ ,  $e^{-\tau/2} w(\tau, \eta)$  converges in  $L^2(0, \infty)$  to  $\alpha_\infty \eta e^{-\eta^2/4}$ . Therefore, we have (reverting to the variables of (2.3))

$$u(t, x) = e^{-x} v(t, x) \sim \alpha_\infty x e^{-x} e^{-x^2/(4t)}, \quad (2.8)$$

at least for  $x$  of the order  $O(\sqrt{t})$ . This, in view of the asymptotics (1.4) of the wave  $\phi_*$  at infinity determines the unique translation:

$$x_\infty = \log \alpha_\infty. \quad (2.9)$$

This argument gives the right insight for the construction of the approximate solution. The idea is to view  $1/\sqrt{t}$  as a small parameter, in terms of which one may expand the solution.

It is natural to identify two zones: the region near the front, that is,  $x \sim O(1)$  – it corresponds to  $\eta \sim e^{-\tau/2}$ , a very small region indeed, and the diffusive region, where  $x \sim \sqrt{t}$  and  $\eta \sim O(1)$ . The transition region is  $x \sim t^\gamma$ , with  $\gamma > 0$  small. We perform a classical asymptotic expansion of an inner solution in the region  $x \sim O(1)$ , approximating  $u$  near the front, and of an outer solution, approximating  $u$  at distances  $O(\sqrt{t})$  from the front. Matching the inner and outer expansions is done in the intermediate region  $x \sim t^\gamma$ .

Once the translate  $x_\infty$  is selected, this also determines the translate of the approximate solution to which the solution is supposed to converge, at a rate faster than  $t^{-(1-\gamma)}$ , for all small  $\gamma$ . Everything reduces to proving that the difference between the true solution and the approximate solution will not exceed  $t^{\gamma-1}$ . The argument is long and technical, and is carried out in the self-similar variables (2.5). However, it relies on two simple ideas. The first is to transform the problem on the whole line into a Dirichlet problem on the half line, by a classical sequence of transformations and the final subtraction of the value of  $u$  at  $t^\gamma$ . The trouble is that the nonlinear term  $u^2$  in the original equation (1.1) provides, as usual, a term which may grow like  $e^{3\tau/2}$  in (2.6). The difficulty is overcome by noticing that its support shrinks as  $e^{-\tau/2}$ . A large part of the proof is devoted to estimating this term in the best way. For that, we first obtain weak estimates on the difference  $u - u_{app}$ , which still yield an improvement of the nonlinear term. This improvement entails a better estimate on  $u - u_{app}$ , and so on. As we have mentioned, the technical details are nontrivial.

### 3 The approximate solution

Instead of working directly with (2.3), we introduce the moving frame that incorporates a (still unknown) correction of the order  $t^{-1/2}$ , namely, instead of (2.2), we make a slightly different successive change of variables:

$$x \mapsto x - 2t + (3/2) \log t - \frac{\sigma}{\sqrt{t}}, \quad u(t, x) = e^{-x} v(t, x).$$

The function  $v$  satisfies

$$v_t - v_{xx} - \left(\frac{3}{2t} + \frac{\sigma}{2t^{3/2}}\right)(v - v_x) + e^{-x} v^2 = 0, \quad x \in \mathbb{R}, \quad t > 1. \quad (3.1)$$

Let us denote this nonlinear operator as

$$NL[v] = v_t - v_{xx} - \left(\frac{3}{2t} + \frac{\sigma}{2t^{3/2}}\right)(v - v_x) + e^{-x} v^2. \quad (3.2)$$

We will construct an approximate solution to (3.1), called  $V_{app}(t, x)$ . As we have mentioned, it is natural to consider an intermediate scale  $x \sim O(t^\gamma)$ , with some  $\gamma > 0$ , and seek an approximate solution to (3.1) in two different forms: one valid for  $x \leq t^\gamma$ , the other valid for  $x \geq t^\gamma$ :

$$V_{app}(t, x) = V^-(t, x) \text{ for } x < t^\gamma, \quad V_{app}(t, x) = V^+(t, x) \text{ for } x > t^\gamma.$$

The functions  $V^-$  and  $V^+$  will be matched at  $x = t^\gamma$ .

#### 3.1 The inner approximate solution $V^-$

Note that (3.1) contains terms that are either of order  $O(1)$ , or of the order  $O(t^{-1})$  and smaller. So, a natural first guess is to choose  $V^-(t, x) = V^-(x)$  and to discard the  $O(t^{-1})$  terms. In other words, we impose

$$-(V^-)'' + e^{-x}(V^-)^2 = 0.$$

A first choice is

$$V_0^-(x) = e^x \phi_*(x). \quad (3.3)$$

This function has the asymptotics:

$$V_0^-(x) \sim e^x \text{ as } x \rightarrow -\infty, \text{ and } V_0(x) \sim x \text{ as } x \rightarrow +\infty. \quad (3.4)$$

We will have to correct it slightly at  $x \sim t^\gamma$  in order to ensure the matching with  $V^+(t, x)$ . Hence, we choose  $V^-$  as

$$V^-(x) = V_0^-(x + \zeta(t)) = e^{x+\zeta(t)} \phi_*(x + \zeta(t)). \quad (3.5)$$

Here, the correction  $\zeta(t)$ , which will come from the matching procedure, will be of the order

$$\zeta(t) \sim O(t^{-1+2\gamma}), \quad \dot{\zeta}(t) \sim O(t^{-2+2\gamma}). \quad (3.6)$$

Let us now estimate  $NL[V^-]$ :

$$\begin{aligned} NL[V^-] &= \dot{\zeta} V_0^-(x + \zeta(t)) - (V_0^-)''(x + \zeta(t)) - \left(\frac{3}{2t} + \frac{\sigma}{2t^{3/2}}\right)(V_0^-(x + \zeta(t)) - (V_0^-)'(x + \zeta(t))) \\ &+ e^{-x}(V_0^-)^2(x + \zeta(t)) = \dot{\zeta} V_0^-(x + \zeta(t)) - \left(\frac{3}{2t} + \frac{\sigma}{2t^{3/2}}\right)(V_0^-(x + \zeta(t)) - (V_0^-)'(x + \zeta(t))) \\ &+ \left[e^{-x} - e^{-x-\zeta(t)}\right](V_0^-)^2(x + \zeta(t)). \end{aligned} \quad (3.7)$$

Note that all terms in (3.7), decay as  $e^x$  for  $x < 0$  because of (3.4). Taking also into account (3.6) gives

$$NL[V^-](t, x) = n_1(t, x)(\mathbf{1}_{0 < x < 2t^\gamma}(x) + \mathbf{1}_{\mathbb{R}_-}(x)e^x), \quad x \leq 2t^\gamma, \quad (3.8)$$

with

$$|n_1(t, x)| \leq Ct^{-1+2\gamma}. \quad (3.9)$$

### 3.2 The outer approximate solution $V^+$

In the outer region  $x > t^\gamma$ , we pass to the self-similar variables

$$\tau = \log t, \quad \eta = \frac{x + x_0}{\sqrt{t}}, \quad (3.10)$$

the shift  $x_0$  kept free for the moment. Our starting point is, again, (3.1), in the self-similar variables. The equation for  $V^+$  is

$$v_\tau - v_{\eta\eta} - \frac{\eta}{2}v_\eta + \left(\frac{3}{2} + \frac{\sigma}{2}e^{-\tau/2}\right)(e^{-\tau/2}v_\eta - v) + e^{\tau-\eta e^{\tau/2}+x_0}v^2 = 0. \quad (3.11)$$

We will set

$$Lv = -v_{\eta\eta} - \frac{\eta}{2}v_\eta - v. \quad (3.12)$$

As in the construction of  $V_{app}^-$ , we are not going to solve (3.11) exactly, but find an approximate solution. Strictly speaking, we only need  $V^+$  defined for  $x > t^\gamma$ , that is, for  $\eta > e^{-(1/2-\gamma)\tau}$  but we will define it for  $\eta \geq 0$ . We impose the boundary condition

$$V^+(\tau, 0) = 0, \quad (3.13)$$

which is consistent with the presence of the absorption term  $e^{\tau-\eta e^{\tau/2}}v^2$  in the left side of (3.11), which is huge as soon as  $\eta$  is just a little negative. As  $V^-(t, x)$  is of the order  $O(t^\gamma)$  at  $x = t^\gamma$ , to have a hope of a good matching we need

$$V^+(\tau, e^{-(1/2-\gamma)\tau}) \sim e^{\gamma\tau}.$$

On the other hand, the boundary condition (3.13) means that

$$V^+(\tau, e^{-(1/2-\gamma)\tau}) \sim \frac{\partial V^+(\tau, 0)}{\partial \eta} e^{-(1/2-\gamma)\tau},$$

thus we need

$$\frac{\partial V^+(\tau, 0)}{\partial \eta} \sim e^{\tau/2}.$$

Hence, it is natural to look for  $V^+$  in the form

$$V^+(\tau, \eta) = e^{\tau/2}V_0^+(\eta) + V_1^+(\eta).$$

Inserting this ansatz into (3.11) and collecting the leading order terms gives

$$LV_0^+ = 0, \tag{3.14}$$

and

$$(L - \frac{1}{2})V_1^+ + \frac{3}{2}(V_0^+)_\eta - \frac{\sigma}{2}V_0^+ = 0, \tag{3.15}$$

with the boundary conditions

$$V_i^+(0) = V_i^+(\infty) = 0, \quad i = 0, 1. \tag{3.16}$$

Setting

$$e_0(\eta) = \eta e^{-\eta^2/4} \text{ for } \eta > 0,$$

we have

$$V_0^+(\eta) = q_0^+ e_0(\eta), \tag{3.17}$$

the constant  $q_0^+$  being for the moment free. Once  $V_0^+$  is fixed, there is a unique solution  $V_1^+$  to (3.15), with  $e^{\eta^2/(4+\gamma)}V_1 \in L^2(\mathbb{R}_+)$ , because the spectrum of  $L$  is  $\{0, 1, 2, \dots\}$ .

We will need the derivative  $(V_1^+)_\eta(0)$  for the matching procedure. The (formal) adjoint of  $L$  satisfies

$$L^*(1 - \frac{\eta^2}{2}) = 0. \tag{3.18}$$

Multiplying (3.15) by  $1 - \eta^2/2$  and integrating by parts gives

$$(V_1^+)'(0) = \int_0^{+\infty} (1 - \frac{\eta^2}{2})(\frac{\sigma}{2}V_0^+ - \frac{3}{2}(V_0^+)')d\eta = -[\sigma + 3\sqrt{\pi}]q_0^+. \tag{3.19}$$



### Estimating the error

Let us denote by  $\mathcal{NL}[v]$  the nonlinear operator in the left side of (3.11). Then we have

$$|\mathcal{NL}[V^+]| \leq C e^{-\tau/2} \mathbf{1}_{\mathbb{R}_+}(\eta) e^{-\eta^2/(4+\gamma)}. \quad (3.20)$$

In the original variables, the function  $V^+$  has the form

$$V^+(t, x) = q_0^+(x + x_0) e^{-(x+x_0)^2/(4t)} + V_1^+ \left( \frac{x + x_0}{\sqrt{t}} \right), \quad (3.21)$$

and (3.20) implies that

$$|NL[V^+](t, x)| \leq C t^{-3/2} \mathbf{1}_{\{x+x_0>0\}} e^{-(x+x_0)^2/((4+\gamma)t)}, \quad \text{for } x \geq -x_0. \quad (3.22)$$

Here,  $NL[V^+]$  is as in (3.2).

### 3.3 Matching the inner and outer approximate solutions

Our next task is to choose the parameters so that the inner and outer approximate solutions match at  $x = t^\gamma$ . Ideally, we would like to match both  $V^-$  and  $V^+$  and their derivatives at this point. However,  $V^-$  and  $V^+$  are of the size  $O(t^\gamma)$  in this region – they are “large”, while their derivatives are  $O(1)$ . Thus, the key is to match  $V^-$  and  $V^+$  and the matching of the derivatives is less of an issue.

Recall that we have

$$V^-(t, t^\gamma) = t^\gamma + k + \zeta(t) + O(e^{-\omega_0 t^\gamma}) \quad (3.23)$$

while for  $V^+(t, t^\gamma)$ , using expression (3.21) we get

$$\begin{aligned} V^+(t, t^\gamma) &= t^{1/2} V_0^+ \left( \frac{t^\gamma + x_0}{\sqrt{t}} \right) + V_1^+ \left( \frac{t^\gamma + x_0}{\sqrt{t}} \right) \\ &= q_0^+ \left( (t^\gamma + x_0)(1 + O(t^{2\gamma-1})) - (\sigma + 3\sqrt{\pi}) t^{-1/2} (t^\gamma + x_0) \right) + O\left(\frac{1}{t^{1-2\gamma}}\right). \end{aligned} \quad (3.24)$$

Equating the terms of the order  $O(t^\gamma)$  and  $O(1)$  gives

$$q_0^+ = 1, \quad x_0 = k, \quad (3.25)$$

while those of the order  $O(t^{-1/2+\gamma})$  and  $O(t^{-1/2})$  give

$$\sigma = -3\sqrt{\pi}. \quad (3.26)$$

Finally, we choose  $\zeta(t)$  to eliminate the terms of the order higher than  $O(t^{-1/2})$ , which means that

$$\zeta(t) = O\left(\frac{1}{t^{1-2\gamma}}\right). \quad (3.27)$$

This implies, by inspection, that

$$\dot{\zeta}(t) = O\left(\frac{1}{t^{2-2\gamma}}\right).$$

Therefore, both conditions in (3.6) are satisfied.

Choosing the parameters in this way, we have matched the values of  $V^+$  and  $V^-$  at  $x = t^\gamma$ :

$$V^+(t, t^\gamma) = V^-(t, t^\gamma),$$

but we have no freedom left in terms of the parameters to match their derivatives at this point. This is a relatively minor inconvenience as  $NL[V_{app}]$  would then have a Dirac mass, of the size proportional to the jump in the derivatives. Taking into account (3.17) and (3.19), as well as (3.25)-(3.27), we see that these derivatives are given by:

$$V_x^+(t, t^\gamma) = e^{-(t^\gamma+k)^2/(4t)} - \frac{(t^\gamma+k)^2}{2t} e^{-(t^\gamma+k)^2/(4t)} + \frac{1}{\sqrt{t}} (V_1^+)_x \left( \frac{t^\gamma+k}{\sqrt{t}} \right) = 1 + O\left(\frac{1}{t^{1-2\gamma}}\right), \quad (3.28)$$

and,

$$V_x^-(t, t^\gamma) = (V_0^-)'(t^\gamma) + O(e^{-\omega_0 t^\gamma}) = 1 + O(e^{-\omega_0 t^\gamma}). \quad (3.29)$$

We conclude that with our choice of  $V^+$  and  $V^-$  the jump in the derivatives is very small:

$$V_x^+(t, t^\gamma) - V_x^-(t, t^\gamma) \sim O\left(\frac{1}{t^{1-2\gamma}}\right). \quad (3.30)$$

We could have avoided this jump by modifying slightly the approximate solution, at the expense of even longer formulas.

**Summary:** The full approximate solution  $V_{app}(t, x)$  for (3.1) is defined by

$$V_{app}(t, x) = V^-(t, x) \mathbf{1}_{x < t^\gamma} + V^+(t, x) \mathbf{1}_{x \geq t^\gamma}. \quad (3.31)$$

The inner and outer pieces have the form:

$$V^-(t, x) = e^{x+\zeta(t)} \phi_*(x + \zeta(t)), \quad \zeta(t) = O(t^{2\gamma-1}), \quad \dot{\zeta}(t) = O(t^{2\gamma-2}), \quad (3.32)$$

and

$$V^+(t, x) = (x+k)e^{-(x+k)^2/(4t)} + V_1^+ \left( \frac{x+k}{\sqrt{t}} \right), \quad (3.33)$$

The function  $V^+$  does not depend on the choice of  $\gamma$ , while  $V^-$  depends on  $\gamma$ , through the shift  $\zeta(t)$ .

Inserting the ansatz (3.31) into (3.1) yields, in view of (3.8)-(3.9) and (3.22), and taking into account that we use  $V^-$  for  $x < t^\gamma$  and  $V^+$  for  $x > t^\gamma$ :

$$|NL[V_{app}](t, x)| \leq Ct^{-1+2\gamma} (\mathbf{1}_{0 < x < t^\gamma} + e^x \mathbf{1}_{x < 0}) + Ct^{-3/2} e^{-x^2/((4+\gamma)t)} \mathbf{1}_{x > t^\gamma} + Ct^{-1+2\gamma} \delta(x - t^\gamma). \quad (3.34)$$

The first two terms come from  $NL[V^-]$  and  $NL[V^+]$ , respectively, while the singular term  $\delta(x - t^\gamma)$  comes from the jump (3.30) in the derivative at the matching point  $x = t^\gamma$ . This estimate is the main result of this section.

**Remark.** It is now clear why the  $t^{-1/2}$  term in the expansion of the front location does not depend on the initial datum, as it is determined by a matching procedure that is itself independent of  $u_0$ . It is another manifestation of the role played by the diffusive zone  $\{x \sim \sqrt{t}\}$ , which actually drives the dynamics of the solution. Let us recall that the shift  $x_\infty$  is also determined by the diffusive zone.

## 4 The approximate solution is an approximation to the true solution

From [16] (and from [3, 4]), we know that there is an asymptotic shift  $x_\infty$  such that, as  $t \rightarrow +\infty$ , we have  $u(t, x) \rightarrow \phi_*(x - x_\infty)$  uniformly on  $\mathbb{R}$ . Without loss of generality, we will assume that the initial condition is such that

$$x_\infty = 0.$$

As in Section 3, we will work in the frame moving as  $2t - (3/2) \log t - 3\sqrt{\pi}/t$ . If  $u(t, x)$  is the solution of the Fisher-KPP equation in this moving frame, then the function

$$v(t, x) = e^x u(t, x)$$

is a solution of

$$v_t - v_{xx} - \left( \frac{3}{2t} - \frac{3\sqrt{\pi}}{2t^{3/2}} \right) (v - v_x) + e^{-x} v^2 = 0, \quad x \in \mathbb{R}, \quad t > 1. \quad (4.1)$$

We have shown already that  $V_{app}$  defined by (3.31) is an approximate solution, and the convergence

$$u(t, x) \rightarrow \phi_\infty(x - x_\infty)$$

implies that

$$|v(t, x) - V_{app}(t, x)| \rightarrow 0.$$

Theorem 1.3 is an immediate consequence of the definition of  $V_{app}$  and the following bound on the error between  $v$  and  $V_{app}$ :

**Theorem 4.1** *Given  $\gamma > 0$  small, let  $V_{app}(t, x)$  be the approximate solution constructed in Section 3. There is  $C_\gamma > 0$  such that, for all  $(t, x) \in [1, \infty) \times \mathbb{R}$ , we have*

$$|e^x u(t, x) - V_{app}(t, x)| \leq \frac{C_\gamma(1 + |x|)}{t^{1-\gamma}}. \quad (4.2)$$

Corollary 1.4 also follows from Theorem 4.1. Let us fix  $s \in (0, 1)$ , let  $\sigma_s(t)$  be defined by

$$\sigma(t) = \sup\{x : u(t, x) = s\},$$

and set  $\sigma_*^s = \phi_*^{-1}(s)$ , so that  $\phi_*(\sigma_*^s) = s$ . From (4.2) and the definition of  $V^-$ , we then have:

$$\sigma_*^s = \sigma_s(t) + \frac{3\sqrt{\pi}}{\sqrt{t}} + O(t^{-1+\gamma}), \quad (4.3)$$

which is the claim of Corollary 1.4 in this moving frame.

### The proof of Theorem 4.1

This is the most technical part of the paper, although the idea is really to apply a simple stability argument. We will use the self-similar variables

$$\tau = \log t, \quad \eta = \frac{x}{\sqrt{t}} \quad (4.4)$$

most of the time. As we have noted, there, one may easily reduce the equation for  $v$  to an equation on a half-line  $\eta > 0$ , due to the very fast decay of  $v$  for  $\eta < 0$ . Then, we are left with an equation for  $\eta > 0$  that is almost linear: it is perturbed by a nonlinear term whose support in  $\eta$  is essentially of the size  $e^{-\tau/2}$ . Moreover, we already know that  $e^{-\tau/2} v(\tau, \eta)$  is equivalent, for large  $\tau$ , to

$$\alpha_\infty \eta_+ e^{-\eta^2/4}.$$

However, the nonlinear term may be quite large in the small region  $\eta \sim O(e^{-\tau/2})$ . We use a bootstrap argument to show that it is in fact harmless, thus opening the way to a classical Liapounov-Schmidt argument of the type [18].

## Reduction to the Dirichlet problem

In view of (3.34), the difference

$$\widetilde{W}(t, x) = v(t, x) - V_{app}(t, x).$$

satisfies an equation

$$\widetilde{W}_t - \widetilde{W}_{xx} - \left( \frac{3}{2t} - \frac{3\sqrt{\pi}}{2t^{3/2}} \right) (\widetilde{W} - \widetilde{W}_x) + e^{-x}(v + V_{app})\widetilde{W} = \widetilde{E}_1(t, x), \quad (4.5)$$

with a function  $\widetilde{E}_1$  satisfying:

$$|\widetilde{E}_1(t, x)| \leq Ct^{-1+2\gamma}(\mathbf{1}_{0 < x < t^\gamma} + e^x \mathbf{1}_{x < 0}) + Ct^{-3/2}e^{-x^2/((4+\gamma)t)}\mathbf{1}_{x > t^\gamma} + Ct^{-1+2\gamma}\delta(x - t^\gamma). \quad (4.6)$$

In order to reduce the equation for  $\widetilde{W}$  to a Dirichlet problem in the self-similar variables, we proceed in several steps.

We first switch to

$$W_1(t, x) = \widetilde{W}(t, x) - \widetilde{W}(t, -t^\gamma)\psi(x + t^\gamma).$$

Here,  $\psi(x)$  is a nonnegative  $C^\infty$  function so that  $\psi(x) = 1$  for  $0 \leq x \leq 1$ , and  $\psi(x) = 0$  for  $x \geq 2$ , so that now  $W_1(t, -t^\gamma) = 0$ . This generates an additional term in the right side of (4.5) that we denote by  $\widetilde{E}_2(t, x)$ . Taking into account that

$$v(t, x) + V_{app}(t, x) = O(e^x) \text{ for } x < 0, \quad (4.7)$$

we obtain

$$|\widetilde{E}_2(t, x)| \leq Ce^{-t^\gamma} \mathbf{1}_{[0,1]}(x + t^\gamma). \quad (4.8)$$

Next, we translate the origin to  $x = -t^\gamma$ : the function

$$W(t, x) = W_1(t, x + t^\gamma) = \widetilde{W}(t, x - t^\gamma) - \widetilde{W}(t, -t^\gamma)\psi(x) \quad (4.9)$$

satisfies

$$W_t - W_{xx} + \left( \frac{\gamma}{t^{1-\gamma}} + \frac{3}{2t} - \frac{3\sqrt{\pi}}{2t^{3/2}} \right) W_x - \left( \frac{3}{2t} - \frac{3\sqrt{\pi}}{2t^{3/2}} \right) W + e^{t^\gamma - x}(\widetilde{v} + \widetilde{V}_{app})W = G_1(t, x) + G_2(t, x) \quad (4.10)$$

for  $x > 0$ , with the Dirichlet condition  $W(t, 0) = 0$ . Here, we have introduced

$$\widetilde{v}(t, x) = v(t, x - t^\gamma), \quad \widetilde{V}_{app}(t, x) = V_{app}(t, x - t^\gamma). \quad (4.11)$$

The functions  $G_1(t, x)$  and  $G_2(t, x)$  in (4.10) satisfy

$$|G_1(t, x)| = |\widetilde{E}_1(t, x - t^\gamma)| \leq Ct^{-1+2\gamma}(\mathbf{1}_{t^\gamma < x < 2t^\gamma}(x) + e^{x-t^\gamma}\mathbf{1}_{x < t^\gamma}(x)) \\ + Ct^{-3/2}e^{-(x-t^\gamma)^2/((4+\gamma)t)}\mathbf{1}_{x > 2t^\gamma} + Ct^{-1+2\gamma}\delta(x - 2t^\gamma), \quad (4.12)$$

and

$$|G_2(t, x)| = |\widetilde{E}_2(t, x - t^\gamma)| \leq Ce^{-t^\gamma} \mathbf{1}_{[0,1]}(x). \quad (4.13)$$

We now express (4.10) in the self-similar variables (4.4). With  $L$  defined by (3.12), this gives

$$W_\tau + \left( L - \frac{1}{2} \right) W + e^{\tau + e^{\gamma\tau} - \eta e^{\tau/2}}(\widetilde{v} + \widetilde{V}_{app})W = - \left( \gamma e^{\gamma\tau} + \frac{3}{2} - \frac{3\sqrt{\pi}}{2} e^{-\tau} \right) e^{-\tau/2} W_\eta \\ - \frac{3\sqrt{\pi}}{2} e^{-\tau/2} W + e^{-\eta^2/8}(E_1 + E_2), \quad (4.14)$$

with  $E_1(\tau, \eta)$  satisfying

$$\begin{aligned}
|E_1(\tau, \eta)| &\leq Ce^{2\gamma\tau} e^{\eta^2/8} \mathbf{1}\left(e^{-(\frac{1}{2}-\gamma)\tau} < \eta < 2e^{-(\frac{1}{2}-\gamma)\tau}\right) \\
&\quad + Ce^{2\gamma\tau} e^{\eta^2/8} e^{\eta e^{\tau/2} - e^{\gamma\tau}} \mathbf{1}\left(0 < \eta < e^{-(\frac{1}{2}-\gamma)\tau}\right) \\
&\quad + Ce^{-\tau/2} e^{\eta^2/8} e^{-(\eta - e^{(-1/2+\gamma)\tau})^2/(4+\gamma)} \mathbf{1}\left(\eta > 2e^{(-1/2+\gamma)\tau}\right) \\
&\quad + Ce^{2\gamma\tau} e^{\eta^2/8} \delta(\eta - 2e^{(-1/2+\gamma)\tau}) = E_{11} + E_{12} + E_{13} + E_{14},
\end{aligned} \tag{4.15}$$

and

$$|E_2(\tau, \eta)| \leq e^{\eta^2/8} e^{\tau} e^{-e^{\tau\gamma}} \mathbf{1}\left(0 < \eta < e^{-\tau/2}\right). \tag{4.16}$$

Notice that the support of  $E_{11}$ ,  $E_{12}$ ,  $E_{14}$  is very small, despite the larger prefactor, compared to  $E_{13}$  and  $E_2$ . Finally, we symmetrize the operator  $L$  by introducing the function

$$w(\tau, \eta) = e^{\eta^2/8} W(\tau, \eta), \tag{4.17}$$

which satisfies

$$w_\tau + \mathcal{M}w + e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}} (\tilde{v} + \tilde{V}_{app})w = \sum_{i=1}^2 E_i(\tau, \eta) + E_3(\tau, \eta), \quad \eta > 0 \tag{4.18}$$

with the Dirichlet boundary condition  $w(\tau, 0) = 0$ . Here we have defined the operator

$$\mathcal{M}w = -w_{\eta\eta} + \left(\frac{\eta^2}{16} - \frac{5}{4}\right)w, \tag{4.19}$$

and set

$$\eta_\gamma(\tau) = e^{-(\frac{1}{2}-\gamma)\tau}, \quad E_3(\tau, \eta) = -\left(\gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2} - \frac{3\sqrt{\pi}}{2} e^{-3\tau/2}\right) \left(w_\eta - \frac{\eta}{4}w\right) - \frac{3\sqrt{\pi}}{2} e^{-\tau/2} w. \tag{4.20}$$

Strictly speaking,  $E_3$  depends on  $w$  and  $w_\eta$ , but we omit this dependence for the notational purposes.

Recall that, in the self-similar variables,  $V_{app}$  grows as  $e^{\tau/2}$ . From the convergence result of [16] (Lemma 5.1, in particular) and the definition of  $V_{app}$  it follows that

$$\lim_{\tau \rightarrow +\infty} e^{-\tau/2} \|w(\tau, \cdot)\|_{L^2(\mathbb{R}_+)} = 0. \tag{4.21}$$

Our goal is to improve this  $o(e^{\tau/2})$  bound on  $w$  to an exponentially decaying estimate for  $w$ . Eventually, we will obtain:

$$\|w(\tau, \cdot)\|_{L^2(\mathbb{R}_+)} + \|w(\tau, \cdot)\|_\infty \leq C_\gamma e^{(-1/2+100\gamma)\tau}. \tag{4.22}$$

### From $o(e^{\tau/2})$ to $O(e^{10\gamma\tau})$ asymptotics for $w$

The principal eigenfunction of the self-adjoint operator  $\mathcal{M}$  with the Dirichlet boundary condition at  $\eta = 0$  is

$$e_0(\eta) = c_0 \eta e^{-\eta^2/8}, \quad \mathcal{M}e_0 = -\frac{e_0}{2}, \tag{4.23}$$

with the constant  $c_0$  chosen so that  $\|e_0\|_{L^2(\mathbb{R}_+)} = 1$ . We decompose the solution of (4.18) as

$$w(\tau) = \langle e_0, w(\tau) \rangle e_0 + w^\perp(\tau), \quad \int_{\mathbb{R}_+} e_0(\eta) w^\perp(\tau, \eta) d\eta = 0. \tag{4.24}$$

**Step 1.** We will first obtain a bound for  $\langle e_0, w \rangle$ . We have, projecting (4.18) onto  $e_0$  and using (4.24):

$$\frac{d\langle e_0, w \rangle}{d\tau} - \frac{\langle e_0, w \rangle}{2} + \langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\tilde{v} + \tilde{V}_{app})w \rangle = \sum_{i=1}^3 \langle e_0, E_i(\tau) \rangle. \quad (4.25)$$

Let us bound the various perturbative terms in (4.25). The terms involving  $E_1$  and  $E_2$  in the right side are easily treated. In view of (4.15) we have

$$|\langle e_0, E_1(\tau) \rangle| \leq C e^{-(\frac{1}{2}-3\gamma)\tau}. \quad (4.26)$$

and (4.16) implies

$$|\langle e_0, E_2(\tau) \rangle| \leq C e^{-e^{\gamma\tau}} \leq C e^{-(\frac{1}{2}-3\gamma)\tau}, \quad (4.27)$$

as well. As for the term involving  $E_3$ , using (4.20) and integrating by parts, we get

$$|\langle e_0, E_3(\tau) \rangle| \leq \left( \gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{9e^{-\tau/2}}{2} \right) \left( |\langle e'_0, w \rangle| + |\langle e_0, \frac{\eta}{4}w \rangle| + |\langle e_0, w \rangle| \right). \quad (4.28)$$

Because of (4.21), we obtain

$$|\langle e_0, E_3(\tau) \rangle| \leq C e^{2\gamma\tau}. \quad (4.29)$$

It finally remains to estimate the last term in the left side of (4.25), and some care should be given to it: although the exponential term is small outside of the very small set  $0 < \eta < \eta_\gamma$ , it could be very large (of the order  $e^\tau$ ) there. This will be compensated by the smallness of the factor  $v + V_{app}$ . Let us recall (4.7) and (4.11) which imply that in the self-similar variables

$$|\tilde{v}(\tau, \eta) + \tilde{V}_{app}(\tau, \eta)|, |w(\tau, \eta)| \leq C e^{\eta e^{\tau/2} - e^{\gamma\tau}} = C e^{e^{\tau/2}(\eta - \eta_\gamma(\tau))} \text{ for } 0 \leq \eta \leq \eta_\gamma(\tau). \quad (4.30)$$

Let us decompose the inner product

$$Q(\tau) = \langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\tilde{v} + \tilde{V}_{app})w \rangle = \int_0^{\eta_\gamma(\tau)} + \int_{\eta_\gamma(\tau)}^\infty = I_1 + I_2, \quad (4.31)$$

For  $\eta \leq \eta_\gamma(\tau)$  we use the bound  $0 \leq e_0(\eta) \leq c_0\eta$ . Using (4.30), we obtain

$$\begin{aligned} I_1 &\leq \int_0^{\eta_\gamma(\tau)} e_0(\eta) e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\tilde{v} + \tilde{V}_{app})|w|d\eta \leq C \int_0^{\eta_\gamma(\tau)} \eta e^{\tau+(\eta-\eta_\gamma(\tau))e^{\tau/2}}d\eta \\ &\leq C \eta_\gamma(\tau) e^\tau e^{-\tau/2} = C e^{\gamma\tau}. \end{aligned} \quad (4.32)$$

As for  $I_2$ , we have that

$$|\tilde{v}(\tau, \eta) + \tilde{V}_{app}(\tau, \eta)|, |w(\tau, \eta)| \leq C(1 + \eta e^{\tau/2})$$

for all  $\eta \in \mathbb{R}$ . This implies

$$\begin{aligned} I_2 &\leq \int_{\eta_\gamma(\tau)}^\infty e_0(\eta) e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\tilde{v} + \tilde{V}_{app})|w|d\eta \leq C \int_{\eta_\gamma(\tau)}^\infty \eta e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(1 + \eta e^{\tau/2})^2d\eta \\ &\leq C e^{2\tau} \int_{\eta_\gamma(\tau)}^\infty \eta^3 e^{-(\eta-\eta_\gamma(\tau))e^{\tau/2}}d\eta \leq C(\eta_\gamma(\tau))^3 e^{3\tau/2} \leq C e^{3\gamma\tau}, \end{aligned} \quad (4.33)$$

and therefore,

$$|Q(\tau)| \leq Ce^{3\gamma\tau}. \quad (4.34)$$

Putting everything together, we infer that

$$\frac{d\langle e_0, w \rangle}{d\tau} - \frac{\langle e_0, w \rangle}{2} = \varphi(\tau), \quad (4.35)$$

with

$$|\varphi(\tau)| \leq Ce^{3\gamma\tau}.$$

We see that

$$\frac{d}{dt} \left( \langle e_0, w \rangle e^{-\tau/2} \right) = \varphi(\tau) e^{-\tau/2}. \quad (4.36)$$

Taking into account (4.21), we can integrate (4.36) from  $\tau$  to  $+\infty$  leading to

$$\langle e_0, w(\tau) \rangle = - \int_{\tau}^{+\infty} e^{(\tau-\tau')/2} \varphi(\tau') d\tau', \quad (4.37)$$

hence

$$|\langle e_0, w(\tau) \rangle| \leq C \int_{\tau}^{+\infty} e^{(\tau-\tau')/2} e^{3\gamma\tau'} d\tau' \leq C_{\gamma} e^{3\gamma\tau}. \quad (4.38)$$

This bound will be improved in the next step.

**Step 2.** Now, we bound  $w^{\perp}(\tau)$ . We multiply ((4.18)) by  $w^{\perp}$ , and integrate by parts:

$$\frac{1}{2} \frac{d\|w^{\perp}\|^2}{d\tau} + \langle \mathcal{M}w^{\perp}, w^{\perp} \rangle + \int_{\mathbb{R}_+} e^{\tau+(\eta_{\gamma}(\tau)-\eta)e^{\tau/2}} (\tilde{v} + \tilde{V}_{app}) w w^{\perp} d\eta = \sum_{i=1}^3 \int_{\mathbb{R}_+} E_i w^{\perp} d\eta. \quad (4.39)$$

We denoted here the  $L^2(\mathbb{R}_+)$  norm by  $\|\cdot\|$ . Once again, we need to bound the perturbative terms in (4.39). Let us start with the less standard term:

$$q(w) := \int_{\mathbb{R}_+} e^{\tau+(\eta_{\gamma}(\tau)-\eta)e^{\tau/2}} (\tilde{v} + \tilde{V}_{app}) w w^{\perp} d\eta = J_1(\tau) + J_2(\tau),$$

with the two terms coming from the decomposition (4.24) for  $w$ . We have

$$J_1(\tau) = \langle e_0, w(\tau) \rangle \int_{\mathbb{R}_+} e^{\tau+(\eta_{\gamma}(\tau)-\eta)e^{\tau/2}} (\tilde{v} + \tilde{V}_{app}) e_0 w^{\perp} d\eta. \quad (4.40)$$

We know from Step 1 that

$$\langle e_0, e^{\tau+(\eta_{\gamma}(\tau)-\eta)e^{\tau/2}} (\tilde{v} + \tilde{V}_{app}) |w\rangle = |Q(\tau)| \leq Ce^{3\gamma\tau}.$$

Together with (4.38) this gives

$$|J_1(\tau)| \leq Ce^{6\gamma\tau}. \quad (4.41)$$

Furthermore,  $J_2(\tau)$  is positive, so we do not need to estimate it.

As for the three terms in the right side of (4.39), in view of (4.15) we have, with some constant  $C_{\gamma} > 0$ : first,

$$|\langle w^{\perp}, E_{11} \rangle| + |\langle w^{\perp}, E_{12} \rangle| \leq \gamma \|w^{\perp}\|^2 + \frac{1}{4\gamma} (\|E_{11}\|^2 + \|E_{12}\|^2) \leq \gamma \|w^{\perp}\|^2 + C_{\gamma} e^{5\gamma\tau} e^{-\tau/2}, \quad (4.42)$$

while for  $E_{13}$  we have

$$|\langle w^\perp, E_{13} \rangle| \leq \gamma \|w^\perp\|^2 + \frac{1}{4\gamma} \|E_{13}\|^2 \leq \gamma \|w^\perp\|^2 + \frac{C}{\gamma} e^{-\tau}. \quad (4.43)$$

Finally, for  $E_{14}$  we have

$$\begin{aligned} |\langle w^\perp, E_{14} \rangle| &\leq C e^{2\gamma\tau} |w^\perp(\tau, 2e^{-\tau/2+\gamma\tau})| \leq C e^{2\gamma\tau} e^{-\frac{\tau}{4}+\frac{7}{2}\tau} \|\partial_x w^\perp(\tau, \cdot)\|_{L^2} \\ &\leq C e^{2\gamma\tau} e^{-\frac{\tau}{4}+\frac{7}{2}\tau} (1 + \langle \mathcal{M}w^\perp, w^\perp \rangle + \|w^\perp\|^2). \end{aligned} \quad (4.44)$$

For  $E_2$  we may simply estimate

$$|\langle w^\perp, E_2 \rangle| \leq \gamma \|w^\perp\|^2 + \frac{1}{4\gamma} \|E_2\|^2 \leq \gamma \|w^\perp\|^2 + C_\gamma e^{2\tau-2e\gamma\tau} e^{-\tau/2} \leq \gamma \|w^\perp\|^2 + C_\gamma e^{-\tau/2}. \quad (4.45)$$

As for  $E_3$ , we have

$$\left| \int_{\mathbb{R}_+} (w_\eta - \frac{\eta}{4} w) w^\perp d\eta \right| \leq \int \eta^2 (w^\perp)^2 d\eta + C_\gamma \langle e_0, w \rangle^2 + \gamma \|w^\perp\|^2 \leq C \|w^\perp\|^2 + C \langle \mathcal{M}w^\perp, w^\perp \rangle + C_\gamma \langle e_0, w \rangle^2, \quad (4.46)$$

hence

$$|\langle w^\perp, E_3 \rangle| \leq C_\gamma e^{(-1/2+\gamma)\tau} \left( \|w^\perp\|^2 + \langle \mathcal{M}w^\perp, w^\perp \rangle + \langle e_0, w \rangle^2 \right). \quad (4.47)$$

Putting everything together, remembering that

$$\langle \mathcal{M}w^\perp, w^\perp \rangle \geq \frac{\|w\|^2}{2},$$

yields

$$\frac{1}{2} \frac{d\|w^\perp\|^2}{d\tau} + \left( \frac{1}{2} - \gamma - C_\gamma e^{-(\frac{1}{4}-\frac{3\gamma}{2})\tau} \right) \|w^\perp\|^2 \leq |J_1(\tau)| \leq C e^{6\gamma\tau}. \quad (4.48)$$

This implies

$$\|w^\perp\| \leq C_\gamma e^{3\gamma\tau}. \quad (4.49)$$

Because of (4.38), this bound all holds for the full solution:  $\|w\| \leq C_\gamma e^{3\gamma\tau}$ . By the parabolic regularity, together with our estimates on the perturbative terms, we also infer that

$$\|w^\perp\|_{L^\infty([0,A])} \leq C_A e^{5\gamma\tau},$$

for  $A$  large. The  $L^\infty$  estimates on the perturbative terms in the equation (4.18) for  $w$  imply that for  $\eta \geq A$  sufficiently large,  $w(\tau, \eta)$  can not attain its maximum at a point  $\eta > A$  where it is larger than than  $C e^{5\gamma\tau}$ , thus we are finally ready to conclude that

$$\|w\|_{L^2(\mathbb{R}_+)} + \|w\|_\infty \leq C_\gamma e^{10\gamma\tau}. \quad (4.50)$$

**From the  $O(e^{10\gamma\tau})$  growth to  $O(e^{-(\frac{1}{2}-100\gamma)\tau})$  decay for  $w$**

The next step is to improve the ‘‘slow’’  $O(e^{10\gamma\tau})$  growth in (4.50) to actual decay in time. Let us come back to (4.25), the equation for  $\langle e_0, w \rangle$ :

$$\frac{d\langle e_0, w \rangle}{d\tau} - \frac{\langle e_0, w \rangle}{2} + \langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e\tau/2} (\tilde{v} + \tilde{V}_{app}) w \rangle = \sum_{i=1}^3 \langle e_0, E_i(\tau) \rangle. \quad (4.51)$$



The bounds (4.26) and (4.27) are already of the “good” size  $O(e^{-(1/2-3\gamma)\tau})$ , and the already obtained bound (4.50) allows us to improve (4.29) to

$$|\langle e_0, E_3(\tau) \rangle| \leq \left( \gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{9e^{-\tau/2}}{2} \right) \left( |\langle e'_0, w \rangle| + |\langle e_0, \frac{\eta}{4} w \rangle| + |\langle e_0, w \rangle| \right) \leq C e^{(-1/2+15\gamma)\tau}. \quad (4.52)$$

Thus, what really limits the decay improvement for  $\langle e_0, w \rangle$  is the integral

$$Q(\tau) = \langle e_0, e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}} (\tilde{v} + \tilde{V}_{app}) w \rangle, \quad (4.53)$$

that we have so far only managed to bound by  $Ce^{3\gamma\tau}$  (see (4.34)). We have already noted that the integrand could be very large only for  $\eta$  of the order

$$\eta_\gamma(\tau) = e^{(-1/2+\gamma)\tau}.$$

On the other hand, we have the boundary condition  $w(\tau, 0) = 0$ . It is therefore natural to ask whether  $w$  has a bounded linear growth in a neighborhood of  $\eta = 0$ . If this is so, this will bring a small factor of the order  $\eta$  in the integrand, which will, in turn, make the integral be of a smaller order. This is what we are going to prove now. Indeed, by the Kato inequality, equation (4.18) for  $w$  yields, writing out explicitly the operator  $\mathcal{M}$ :

$$\begin{aligned} \partial_\tau |w| - |w|_{\eta\eta} + \left( \frac{\eta^2}{16} - \frac{5}{4} \right) |w| + g(\tau) \left( \partial_\eta |w| - \frac{\eta}{4} |w| \right) \\ \leq C e^{-(\frac{1}{2}-\gamma)\tau} + C e^{2\gamma\tau} \mathbf{1}(0 < \eta < 2\eta_\gamma(\tau)) + C e^{2\gamma\tau} \delta(\eta - 2\eta_\gamma(\tau)), \end{aligned} \quad (4.54)$$

with

$$g(\tau) = \left( \gamma e^{-(\frac{1}{2}-\gamma)\tau} + \frac{3e^{-\tau/2}}{2} - \frac{3\sqrt{\pi}}{2} e^{-3\tau/2} \right).$$

Let  $a \in (0, 1)$  be small enough so that (4.54) implies

$$\partial_\tau |w| - |w|_{\eta\eta} - 10|w| + g(\tau) \partial_\eta |w| \leq C e^{2\gamma\tau} + C e^{2\gamma\tau} \delta(\eta - 2\eta_\gamma(\tau)), \quad (4.55)$$

for  $\eta \in (0, a)$  with the boundary conditions

$$|w|(\tau, 0) = 0, \quad |w|(\tau, a) \leq C e^{10\gamma\tau}, \quad (4.56)$$

which is achievable, due to (4.50). Let us write

$$|w|(\tau, \eta) \leq C e^{10\gamma\tau} \psi(\tau, \eta) + e^{2\gamma\tau} \phi(\tau, \eta),$$

with the function  $\psi(\tau, \eta) \geq 0$  such that

$$\begin{aligned} \partial_\tau \psi - \psi_{\eta\eta} - 11\psi + g(\tau) \partial_\eta \psi &= C e^{-8\gamma\tau}, \\ \psi(\tau, 0) &= 0, \quad \psi(\tau, a) = 1. \end{aligned} \quad (4.57)$$

Possibly decreasing  $a$ , we may ensure that the principal eigenvalue  $\lambda_a$  of the Dirichlet Laplacian on the interval  $(0, 2a)$  is sufficiently large, say,  $\lambda_a > 100$ . Then there exists a constant  $C > 0$  so that

$$\psi(\tau, \eta) \leq C\eta. \quad (4.58)$$

We choose the function  $\phi \geq 0$  so that it satisfies

$$\partial_\tau \phi - \phi_{\eta\eta} - 11\phi + g(\tau) \partial_\eta \phi = C \delta(\eta - 2e^{(-1/2+\gamma)\tau}), \quad (4.59)$$

with the boundary conditions

$$\phi(\tau, 0) = 0, \quad \phi(\tau, a) = 0. \quad (4.60)$$

Let us prove that

$$\phi(\tau, \eta) \leq C\eta. \quad (4.61)$$

We have  $\phi(\tau, \eta) = \phi_0(\tau, \eta) + \phi_1(\tau, \eta)$  with

$$\begin{aligned} -\partial_{\eta\eta}\phi_0 &= C\delta(\eta - 2e^{(-1/2+\gamma)\tau}) \\ \phi_0(\tau, 0) &= \phi_0(\tau, a) = 0, \end{aligned}$$

and

$$\partial_\tau\phi_1 - \partial_{\eta\eta}\phi_1 - 11\phi_1 + g(\tau)\partial_\eta\phi_1 = -\partial_\tau\phi_0 - g(\tau)\partial_\eta\phi_0 + 11\phi_0,$$

with the boundary conditions

$$\phi_1(\tau, 0) = 0, \quad \phi_1(\tau, a) = 0.$$

The function  $\phi_0$  is easily computed:

$$\phi_0(\tau, \eta) = \begin{cases} C\frac{(a - \xi_\gamma(\tau))}{\eta}\eta, & \eta \leq \xi_\gamma(\tau) \\ C\frac{(a - \eta)\xi_\gamma(\tau)}{a}, & \eta \geq \xi_\gamma(\tau) \end{cases} \quad (4.62)$$

with  $\xi_\gamma(\tau) = 2e^{(-1/2+\gamma)\tau}$ . So, all the quantities  $\phi_0$ ,  $\partial_\eta\phi_0$  and  $\partial_\tau\phi_0$  are uniformly bounded, hence (recall  $\lambda_a \geq 100$ ) we have (4.61). It follows that

$$|w(\tau, \eta)| \leq Ce^{10\gamma\tau}\eta \quad \text{for } \tau \geq 0 \text{ and } 0 \leq \eta \leq a. \quad (4.63)$$

Returning to  $Q(\tau)$  given by (4.53), we deduce, using (4.30) and (4.63) the following improvement of (4.32):

$$\begin{aligned} I_1 &\leq \int_0^{\eta_\gamma(\tau)} e_0(\eta)e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\tilde{v} + \tilde{V}_{app})|w|d\eta \leq Ce^{10\gamma\tau} \int_0^{\eta_\gamma(\tau)} \eta^2 e^\tau d\eta \\ &\leq Ce^{10\gamma\tau}[\eta_\gamma(\tau)]^3 e^\tau = Ce^{(-1/2+20\gamma)\tau}, \end{aligned} \quad (4.64)$$

while (4.33) can be improved to

$$\begin{aligned} I_2 &\leq \int_{\eta_\gamma(\tau)}^a e_0(\eta)e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(\tilde{v} + \tilde{V}_{app})|w|d\eta + Ce^{3\tau/2}e^{-a/2e^{\tau/2}} \\ &\leq Ce^{10\gamma\tau} \int_{\eta_\gamma(\tau)}^a \eta e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}}(1 + \eta e^{\tau/2})\eta d\eta \\ &\leq Ce^{10\gamma\tau} e^\tau e^{\tau/2} \int_{\eta_\gamma(\tau)}^a \eta^3 e^{-(\eta-\eta_\gamma(\tau))e^{\tau/2}} d\eta \leq Ce^{10\gamma\tau}(\eta_\gamma(\tau))^3 e^\tau \leq Ce^{(-1/2+20\gamma)\tau}. \end{aligned} \quad (4.65)$$

Equation (4.51) for  $\langle e_0, w(\tau) \rangle$  now gives

$$|\langle e_0, w(\tau) \rangle| \leq C \int_\tau^{+\infty} e^{-(\tau-\tau')/2} e^{(-\frac{1}{2}+20\gamma)\tau} d\tau' \leq Ce^{-(\frac{1}{2}-20\gamma)\tau}. \quad (4.66)$$

Moreover, equation (4.48) for  $w^\perp$  shows that the only ‘‘slightly large’’ term that potentially can make  $w^\perp(\tau, \eta)$  grow in  $\tau$  is  $J_1(\tau)$  given by (4.40)

$$J_1(\tau) = \langle e_0, w(\tau) \rangle \int_{\mathbb{R}_+} e^{\tau+(\eta_\gamma(\tau)-\eta)e^{\tau/2}+x_0}(\tilde{v} + \tilde{V}_{app})e_0 w^\perp d\eta. \quad (4.67)$$

However, we may now use (4.66) to bootstrap (4.41) to

$$|J_1(\tau)| \leq C e^{(-1/2+40\gamma)\tau}. \quad (4.68)$$

Using this in (4.48) gives us

$$\|w^\perp\| \leq C e^{-(\frac{1}{2}-50\gamma)\tau}. \quad (4.69)$$

This implies the same estimate for the full solution  $w$ . As in the passage from (4.49) to (4.50) we obtain

$$\|w\|_{L^2(\mathbb{R}_+)} + \|w\|_\infty \leq C_\gamma e^{(-1/2+100\gamma)\tau}. \quad (4.70)$$

### Concluding the proof of Theorem 4.1

The last step seems to yield a  $t^{\gamma-1/2}$  decay for  $w$ . However, recall that we want a  $t^{\gamma-1}$  estimate. To this end, it suffices to remember that  $w(\tau, \eta)$  solves a Dirichlet problem, hence  $w$  should have an extra  $\eta$  factor. To show that, it suffices to argue just as in the proof of estimate (4.63), up to the fact that, this time, the slow  $e^{10\gamma\tau}$  growth is replaced by the decay  $e^{-(1/2-100\gamma)\tau}$ . Repeating this argument, we end up with

$$|w(\tau, \eta)| \leq C_\gamma \eta e^{-(1/2-100\gamma)\tau}. \quad (4.71)$$

To obtain the conclusion of Theorem 4.1, it suffices to unzip (4.71), reverting to the  $(t, x)$  variables. We obtain

$$|v(t, x) - V_{app}(t, x)| \leq \frac{C}{t^{\frac{1}{2}-100\gamma}} \frac{x+t^\gamma}{\sqrt{t}}, \quad \text{for } x > -t^\gamma + 2, \quad t \geq 1. \quad (4.72)$$

This implies Theorem 4.1.  $\square$

## References

- [1] J. Berestycki, E. Brunet, S.C. Harris, M.I. Roberts, Vanishing corrections for the position in a linear model of FKPP fronts, preprint, 2015. <http://arxiv.org/abs/1510.03329>
- [2] J. Berestycki, E. Brunet, A note of the convergence of the Fisher-KPP front centred around its  $\alpha$ -level, preprint, 2016. <http://http://arxiv.org/pdf/1603.06005>
- [3] M.D. Bramson, Maximal displacement of branching Brownian motion, *Comm. Pure Appl. Math.* **31**, 1978, 531–581.
- [4] M.D. Bramson, Convergence of solutions of the Kolmogorov equation to travelling waves, *Mem. Amer. Math. Soc.* **44**, 1983.
- [5] E. Brunet and B. Derrida, A branching random walk seen from the tip, *Jour. Stat. Phys.* **143** (2011), pp. 420–446.
- [6] E. Brunet and B. Derrida, Statistics at the tip of a branching random walk and the delay of traveling waves. *Eur. Phys. Lett.* **87**, 60010, 2009.
- [7] U. Ebert and W. Van Saarloos, Front propagation into unstable states: universal algebraic convergence towards uniformly translating pulled fronts, *Phys. D* **146**, 2000, 1–99.
- [8] P.C. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, *Arch. Rat. Mech. Anal.*, **65**, 1977, 335–361.
- [9] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* **7**, 1937, 353–369.

- [10] F. Hamel, J. Nolen, J.-M. Roquejoffre and L. Ryzhik, A short proof of the logarithmic Bramson correction in Fisher-KPP equations, *Netw. Het. Media* **8**, 2013, 275–289.
- [11] F. Hamel, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik, The logarithmic time delay of KPP fronts in a periodic medium, *J. European Math. Society* **18** (2016), pp. 465-505.
- [12] C. Henderson, Population stabilization in branching Brownian motion with absorption, to appear in *Comm. Math. Sci.*, 2015.
- [13] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. État Moscou, Sér. Inter. A* **1**, 1937, 1–26.
- [14] H.P. McKean, Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov, *Comm. Pure Appl. Math.* **28** 1975, 323–331.
- [15] K.-S. Lau, On the nonlinear diffusion equation of Kolmogorov, Petrovskii and Piskunov, *Jour. Diff. Eqs.* **59**, 1985, 44-70.
- [16] J. Nolen, J.-M. Roquejoffre and L. Ryzhik, Convergence to a single wave in the Fisher-KPP equation, Preprint, 2016. <http://arxiv.org/abs/1604.02994>
- [17] W. Van Saarloos, Front propagation into unstable states, *Phys. Reports* **386**, 2003, 29–222.
- [18] D.H. Sattinger, Weighted norms for the stability of traveling waves, *J. Diff. Eq.* **25**, 1977, 130–144.