Fluctuations of solutions to Wigner equation with an Ornstein-Uhlenbeck potential

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Abstract

We consider energy fluctuations for solutions of the Schrödinger equation with an Ornstein-Uhlenbeck random potential when the initial data is spatially localized. The limit of the fluctuations of the Wigner transform satisfies a kinetic equation with random initial data. This result generalizes that of [13] where the random potential was assumed to be white noise in time.

1 Introduction

Solutions of the Schrödinger equation with a weakly random potential

$$i\frac{\partial\phi}{\partial t}+\frac{1}{2}\Delta\phi-\sqrt{\varepsilon}V(t,x)\phi=0,$$

and a small parameter $\varepsilon \ll 1$ behave non-trivially on the time scale $t \sim O(\varepsilon^{-1})$. The corresponding rescaled problem is

$$i\varepsilon\frac{\partial\phi_\varepsilon}{\partial t}+\frac{\varepsilon^2}{2}\Delta\phi_\varepsilon-\sqrt{\varepsilon}V(\frac{t}{\varepsilon},\frac{x}{\varepsilon})\phi_\varepsilon=0,$$

A convenient tool to study the energy distribution in this long time limit is via the Wigner transform [10, 15] of the solution defined as

$$W_\varepsilon(t,x,k)=\int e^{ik\cdot y}\phi_\varepsilon(t,x-\frac{\varepsilon y}{2})\bar{\phi}_\varepsilon(t,x+\frac{\varepsilon y}{2})\frac{dy}{(2\pi)^d}.$$

The weak limit $W(t,x,k)$ exists and is called the Wigner measure of the family ϕ_ε. As the weak limit of the energy density $|\phi_\varepsilon(t,x)|^2$ is, under very mild conditions, $\int W(t,x,k)dk$, the behavior of the Wigner measure is important.

The Wigner transform $W_\varepsilon(t,x,k)$ itself is a solution to an equation

$$\frac{\partial W_\varepsilon(t,x,k)}{\partial t}+k\cdot\nabla_x W_\varepsilon(t,x,k)=\frac{i}{\sqrt{\varepsilon}}\sum_{\sigma=\pm1}\sigma\int\frac{\hat{V}\left(t/\varepsilon,dp\right)}{(2\pi)^d}e^{ip\cdot x/\varepsilon}W_\varepsilon\left(t,x,k+\frac{\sigma p}{2}\right). \quad (1.1)$$

Here $\hat{V}(t,dp)$ is the (spatial) spectral measure corresponding to the random field $V(t,x)$. It has been shown under various assumptions on the random potential see [1, 8, 9, 16], that when the initial data

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W_0 for (1.1) is in $L^2(\mathbb{R}^{2d})$ the solutions converge in probability, as $\varepsilon \downarrow 0$, to $\bar{W}(t,x,k)$ the solution of a linear Boltzmann equation

$$\begin{aligned} &\partial_t \bar{W}(t,x,k) + k \cdot \nabla_x \bar{W}(t,x,k) = \mathcal{L}\bar{W}(t,x,k), \\ &\bar{W}(0,x,k) = W_0(x,k), \end{aligned} \tag{1.2}$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L}W(x,k) = \int \hat{R}(\frac{p^2-k^2}{2}, p-k)(W(x,p) - W(x,k))\frac{dp}{(2\pi)^{2d}}.$$

It is important, in particular, for inverse problems, to understand the fluctuations of W_ε around this self-averaging limit, as wave energy fluctuations are often large in practice [2, 3, 4]. As it was shown in [5, 6], the size of the fluctuations depends on the regularity of the initial W_0 – both spatially and wave vector localized singularities in W_0 produce stronger fluctuations than smooth initial energy distributions. Here, we study the fluctuations of the Wigner transform

$$Z_\varepsilon(t,x,k) = \varepsilon^{-1/2}[W_\varepsilon(t,x,k) - \bar{W}(t,x,k)]$$

when $W_0(x,k) = \delta(x)f(k)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, that is, the initial wave energy distribution is spatially localized but smoothly distributed in various directions. The fact that the fluctuations have the size $O(\sqrt{\varepsilon})$ comes from the singularity of the initial data – their size would be smaller were $W_0(x,k)$ more regular.

This problem was previously studied when the random potential $V(t,x)$ is white noise in time in [13] and the limit of Z_ε has been identified. In this paper we consider random potentials of Ornstein-Uhlenbeck type that have finite correlation time, and show that the gist of the result is similar to that in [13] – the limit $\bar{Z}$ is identified as a solution of a deterministic kinetic equation with a random initial data. This is because the main contribution to the fluctuations of Z_ε comes from the initial boundary time layer when the wave energy is very singular, and the fluctuations that are created later are of a smaller size since the wave field becomes spatially distributed. The analysis of the present paper is quite more involved than in [13] as it requires a completely different technique – it is impossible to get away with relying on a sophisticated version of the Ito formula, and one has to resort to a summation over all products of covariances that arise while computing the moments of multi-point statistics of a Gaussian potential, which is much more complicated technically. We refer the reader to [5, 6, 13] for a more detailed discussion of the motivation and related results.

The paper is organized as follows. Section 2 describes the detailed probabilistic setting of the problem, and the main result of the paper, Theorem 2.8. The rest of the paper contains the proof of Theorem 2.8 that is performed via a series of intermediate steps, outlined after the statement of this theorem.

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2 Preliminaries and the formulation of the main result

This section contains the background material that is necessary to make sense of the Wigner equation with a random Ornstein-Uhlenbeck potential. First, in Sections 2.1 and 2.2 we define the Ornstein-Uhlenbeck potential as a process taking values in an appropriate function Hilbert space. This material is somewhat standard but we were unable to find it in the literature. Section 2.3 contains the definition and basic properties of the scattering operator. These notions allow us to define in Section 2.4 the notion of a mild solution to the Wigner equation. We further recall the basic properties of the linear kinetic equation, and introduce the needed notation in Sections 2.5 and 2.6. Finally, the main result is formulated in Section 2.7.

2.1 Basic notation

We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of rapidly decreasing functions of the (complex valued) Schwartz class and by $\mathcal{S}'(\mathbb{R}^d)$ the corresponding space of tempered distributions. Let

$$\mathcal{F}\psi(p) = \widehat{\psi}(p) := \int_{\mathbb{R}^d} \mathrm{e}^{-ip\cdot x}\psi(x)dx$$

be the Fourier transform of a function $\psi(x)$. We will also use the notation

$$\mathcal{F}_1(f)(q,k) := \int_{\mathbb{R}^d} \mathrm{e}^{-iq\cdot x} f(x,k)dx, \quad \mathcal{F}_2(f)(x,y) := \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot k} f(x,k)dk$$

for the partial Fourier transform of a function $f(x,k)$ in just one of the variables. The inverse Fourier transform is

$$\tilde{\mathcal{F}} f(x,k) = \int e^{iq\cdot x+iy\cdot k} f(q,y)\frac{dqdy}{(2\pi)^{2d}},$$

and the inverse Fourier transform in just one of the variables is defined similarly.

Given $s,u,\rho_1,\rho_2 \in \mathbb{R}$ we denote by $H^{s,u}_{\rho_1,\rho_2}$ the mixed Sobolev space with the norm

$$\|f\|^2_{H^{s,u}_{\rho_1,\rho_2}} := \int_{\mathbb{R}^{2d}} \theta_s(q)\theta_u(y)|\mathcal{F}(f\theta_{\rho_1/2}\otimes\theta_{\rho_2/2})|^2(q,y)dqdy, \quad f\in\mathcal{S}(\mathbb{R}^{2d}),$$

were $\theta_\rho(x) := (1+|x|^2)^{\rho/2}$. We will simply write $H^{s,u}$ when $\rho_1=\rho_2=0$. The corresponding Sobolev space for functions $f:\mathbb{R}^d\to\mathcal{C}$ depending only on one of the variables shall be denoted by H^s_ρ and H^s when $\rho=0$.

2.2 The Ornstein-Uhlenbeck potential

The Cameron-Martin reproducing kernel Hilbert space

Let $\hat{R}(p)\in L^1(\mathbb{R}^d)$ be a non-negative even function, and $\gamma(p)\in L^\infty(\mathbb{R}^d)$ be a uniformly positive even function:

$$0<\gamma_*\le\gamma(p)\le\Gamma_*, \quad \gamma(p)=\gamma(-p), \quad \forall\, p\in\mathbb{R}^d. \tag{2.1}$$

We assume that

$$\hat{R}(p)\le\frac{C}{(1+|p|^2)^{d/2+\delta}}, \quad \text{for all } p\in\mathbb{R}^d, \tag{2.2}$$

with some $C>0$ and $\delta>0$. Consider a stationary Gaussian random field $V(x)$ whose covariance function equals

$$R(x) := \int e^{ip\cdot x}\frac{d\mu(p)}{(2\pi)^d}, \tag{2.3}$$

where $d\mu(p)=\hat{R}(p)dp$ is a non-negative measure of finite mass. In order to describe the functional space that supports the law of the process consider the real Hilbert space $L^2_{(s)}(\mu)$ consisting of all functions $\psi\in L^2(\mu)$ that are complex even, that is, $\psi(-p)=\psi^*(p)$. Note that

$$\langle\psi_1,\psi_2\rangle_\mu := \int_{\mathbb{R}^d}\psi_1(p)\psi_2^*(p)\mu(dp),$$

is a real valued scalar product on $L^2_{(s)}(\mu)$. The following proposition holds, see Corollary 1 of [13].

Proposition 2.1 *Suppose that $\{\xi_n,\, n \ge 0\}$ is a sequence of i.i.d. standard normal random variables, and define the measure $\nu(dp, dq) = \delta(p+q)\mu(dp)dq$. Let also v_n be an orthonormal basis of $L^2_{(s)}(\mu)$. Then, for any function $\Psi \in L^2_{\mathbb{C}}(\mu) \cap L^1_{\mathbb{C}}(\nu)$ we have*

$$\mathbb{E}\left[\sum_{m,n\ge 0} \xi_n \xi_m \langle \Psi, v_n \otimes v_m\rangle_{\mu\otimes\mu}\right] = \int \Psi(p, -p)\mu(dp). \tag{2.4}$$

Let $\mathcal{H}_\mu$ be *the Cameron-Martin reproducing kernel Hilbert space* that corresponds to the Gaussian random field $V(x)$, that is, the subspace of $\mathcal{S}'(\mathbb{R}^d)$ given by

$$\mathcal{H}_\mu := \left[\tilde{\mathcal{F}}(\psi\mu) : \psi \in L^2_{(s)}(\mu)\right],$$

where, as we recall,

$$\tilde{\mathcal{F}}(\psi\mu)(x) := \int e^{ip\cdot x}\psi(p)\frac{\mu(dp)}{(2\pi)^d}.$$

It is a real Hilbert space, when considered with the scalar product induced from $L^2_{(s)}(\mu)$, that is, for all $\psi_1, \psi_2 \in L^2_{(s)}(\mu)$ we have

$$\langle \tilde{\mathcal{F}}(\psi_1\mu), \tilde{\mathcal{F}}(\psi_2\mu)\rangle_{\mathcal{H}_\mu} := \langle \psi_1, \psi_2\rangle_\mu.$$

Note that all elements of $\mathcal{H}_\mu$ are continuous functions, as $L^2_{(s)}(\mu) \subset L^1(\mu)$:

$$\int_{\mathbb{R}^d} |\psi(p)| d\mu \le \left(\mu(\mathbb{R}^d)\int_{\mathbb{R}^d} |\psi|^2 d\mu\right)^{1/2} < +\infty,$$

for any $\psi \in L^2_{(s)}(\mu)$. Suppose that $\mathcal{E}$ is a Hilbert space continuously embedded in $C(\mathbb{R}^n)$ such that $\mathcal{H}_\mu$ is its dense subset and the natural embedding $J : \mathcal{H}_\mu \to \mathcal{E}$ given by $Jf = f$, $f \in \mathcal{H}_\mu$ is a Hilbert-Schmidt operator. More explicitly one can take, for instance, $\mathcal{E} := H^m_{-\rho}$, where $\rho, m > d$, so all elements of $\mathcal{E}$ have continuous realizations. In that case, the embedding $J : \mathcal{H}_\mu \to \mathcal{E}$ is Hilbert-Schmidt, provided that

$$\int \theta_m(p)\hat{R}(p) < +\infty. \tag{2.5}$$

Indeed, suppose that $f_n(x) = \int e^{ip\cdot x} v_n(p)\mu(dp)$ be an orthonormal system in $\mathcal{H}_\mu$. We have

$$\begin{aligned}\sum_{n\ge 0} \|f_n\|^2_{H^m_{-\rho}} &= \sum_{n\ge 0}\int_{\mathbb{R}^d} \theta_m(q)|\mathcal{F}(f_n\theta_{-\rho/2})(q)|^2 dq \qquad (2.6)\\ &= \sum_{n\ge 0}\int_{\mathbb{R}^{5d}} \theta_m(q)e^{iq\cdot(x-x')}e^{-i(p\cdot x + p'\cdot x')}e_n(p)e_n(p')\theta_{-\rho/2}(x)\theta_{-\rho/2}(x')dqdxdx'\mu(dp)\mu(dp')\\ &= \int_{\mathbb{R}^{2d}} \theta_m(q)|\hat{\theta}_{-\rho/2}(-p+q)|^2\hat{R}(p)dqdp.\end{aligned}$$

Since $\theta_{-\rho/2} \in \cap_{k\ge 0} H^k$ for each $k \ge 0$ we can choose a constant $C_k > 0$ such that $|\hat{\theta}_{-\rho/2}(p)| \le C_k\theta_{-k}(p)$ for all $k \ge 0$. It is easy to observe that, for $k > m + d$ one choose a constant $C > 0$ such that

$$|\hat{\theta}_{-\rho/2}|^2 * \theta_m(p) \le C\theta_m(p)$$

for all p and the embedding is Hilbert-Schmidt, if (2.5) holds.

The covariance operator and the field $V(x)$

Denote $\zeta_f(U) := \langle U, f\rangle$ for given $f \in \mathcal{E}^*$, $U \in \mathcal{E}$ and define a bounded and symmetric linear operator $Q : \mathcal{E}^* \to \mathcal{E}^*$ by

$$\langle Qf, g\rangle_{\mathcal{E}^*} := \langle J^* f, J^* g\rangle_\mu, \quad \forall\, f, g \in \mathcal{E}^*. \tag{2.7}$$

Since J is Hilbert-Schmidt, so is J^* (see e.g. Appendix C of [7]), thus Q is of trace class. There exists therefore a unique Gaussian measure π on $\mathcal{E}$ corresponding to Q, see Section 2.3.2 of ibid, i.e. a Borel, probability measure such that for all $u_1, \ldots, u_n \in \mathcal{E}$ the joint law of random vector $(\zeta_{u_1}, \ldots, \zeta_{u_n})$ over $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \pi)$ is normal and for any $f, g \in \mathcal{E}^*$ we have

$$\int_{\mathcal{E}} \zeta_f(U)\zeta_g(U)\pi(dU) = \langle Qf, g\rangle_{\mathcal{E}^*}.$$

Suppose that $\{g_n,\, n \ge 0\}$ is an orthonormal basis in $\mathcal{E}^*$ consisting of eigenvectors of Q. Let $\lambda_n := \langle Qg_n, g_n\rangle_{\mathcal{E}^*}$ and $h_n := \lambda_n^{-1} J J^* g_n$, $n \ge 0$. Observe that

$$\langle g_n, h_m\rangle = \lambda_n^{-1}\langle J^* g_n, J^* g_m\rangle_{\mathcal{H}_\mu} = \delta_{mn}$$

thus $\{g_n,\, n \ge 0\}$, $\{h_n,\, n \ge 0\}$ form a bi-orthogonal system in $\mathcal{E}^*$ and $\mathcal{E}$, respectively. Note also that $J^* \mathfrak{U} J = Id_{\mathcal{H}_\mu}$, where $\mathfrak{U} : \mathcal{E} \to \mathcal{E}^*$ is the canonical unitary isomorphism coming from the Riesz representation theorem.

In particular, the above implies that $\xi_n := \lambda_n^{-1/2}\zeta_{g_n}$, $n \ge 0$ is a sequence of independent, standard normal, random variables. Define an orthonormal base of $\mathcal{H}_\mu$ by $f_n := \lambda_n^{-1/2} J^* g_n$, $n \ge 0$ and let $\{e_n,\, n \ge 0\}$ be the corresponding orthonormal base on $L^2_{(s)}(\mu)$, given by $f_n = \tilde{\mathcal{F}}(e_n\mu)$. We have of course

$$f = \sum_{n\ge 0} \xi_n(Jf) f_n, \quad \forall\, f \in \mathcal{H}_\mu. \tag{2.8}$$

Let us define $V \in L^2(\pi; \mathcal{E})$

$$V := \sum_{n=0}^{\infty} \xi_n J f_n.$$

The series converges both a.s. and in the L^2-sense in $\mathcal{E}$. Moreover, the real valued random field

$$V(x) := \langle V, \delta_x\rangle = \sum_{n=0}^{\infty} \xi_n J f_n(x).$$

is stationary, with the covariance function given by (2.3). To abbreviate we shall also denote

$$\hat{V}(dp) := \sum_{n\ge 0} \xi_n e_n(p)\mu(dp). \tag{2.9}$$

Then

$$V(x) = \int e^{ip\cdot x}\frac{\hat{V}(dp)}{(2\pi)^d}, \quad \forall\, x \in \mathbb{R}^d,$$

using (2.8) we conclude that

$$\zeta_v = \int_{\mathbb{R}^d} \hat{v}(p)\hat{V}(dp) \tag{2.10}$$

when $J^* v = \tilde{\mathcal{F}}(\hat{v}\mu)$.

The definition of the Ornstein-Uhlenbeck process

Suppose that $\{V_t^{(n)},\, t\ge 0\}$, $n\ge 0$ are real valued jointly Gaussian processes such that

$$\mathbb{E}[V_t^{(n)}V_s^{(m)}]=(2\pi)^d\int e^{-\gamma(p)|t-s|}e_n(p)e_m(-p)\mu(dp) \tag{2.11}$$

for all $n,m\ge 0$ and $t,s\in\mathbb{R}$. Note that for each t fixed $\{V_t^{(n)},\, n\ge 0\}$ are idependent, standard normal random variables. Let V_t be an $\mathcal{E}$ valued process given by $V_t:=\sum_{n\ge 0}V_t^{(n)}Jf_n$. The convergence again takes place in the a.s. and L^2 sense.

The covariance function of the field $V(t,x):=\langle V_t,\delta_x\rangle=V_t(x)$ equals

$$\begin{aligned}\mathbb{E}\left[V_t(x)V_s(y)\right]&=\sum_{n,m\ge 0}\mathbb{E}[V_t^{(n)}V_s^{(m)}]Jf_n(x)Jf_m(y) \qquad (2.12)\\ &=\sum_{n,m\ge 0}(2\pi)^d\int e^{-\gamma(p)|t-s|}e_n(p)e_m^*(p)\mu(dp)\int e^{ip'\cdot x}e_n(p')\frac{\mu(dp')}{(2\pi)^d}\int e^{-iq\cdot y}e_m^*(q)\frac{\mu(dq)}{(2\pi)^d}.\end{aligned}$$

Now,

$$\sum_m\int e^{-\gamma(p)|t-s|}e_n(p)e_m^*(p)\frac{\mu(dp)}{(2\pi)^d}\int e^{-iq\cdot y}e_m^*(q)\frac{\mu(dq)}{(2\pi)^d}=\int e^{-\gamma(p)|t-s|}e_n(p)e^{ip\cdot y}\frac{\mu(dp)}{(2\pi)^{2d}},$$

hence

$$\begin{aligned}\mathbb{E}\left[V_t(x)V_s(y)\right]&=\sum_n\int e^{-\gamma(p)|t-s|}e_n(p)e^{-ip\cdot y}d\mu(p)\int e^{-ip'\cdot x}e_n(p')\frac{d\mu(p')}{(2\pi)^d}\\ &=\sum_n\int e^{-\gamma(p)|t-s|}e_n^*(p)e^{ip\cdot y}d\mu(p)\int e^{ip'\cdot x}e_n^*(p')d\mu(p')=\int e^{-\gamma(p)|t-s|+ip\cdot(y-x)}\frac{\mu(dp)}{(2\pi)^d}\\ &=R(t-s,x-y),\end{aligned}$$

where

$$R(t,x):=\int e^{ip\cdot x}e^{-\gamma(p)|t|}\hat{R}(p)\frac{dp}{(2\pi)^d},\quad (t,x)\in\mathbb{R}^{d+1}. \tag{2.13}$$

In the same way we can also prove that

$$\mathbb{E}\left[\langle V_t,\psi_1\rangle\langle V_s,\psi_2\rangle\right]=\int e^{-\gamma(p)|t-s|}\hat{\psi}_1(p)\hat{\psi}_2^*(p)\frac{\mu(dp)}{(2\pi)^d} \tag{2.14}$$

for any $\psi_1,\psi_2\in\mathcal{E}^*$ such that $J^*\psi_i=\tilde{\mathcal{F}}(\hat{\psi}_i\mu)$, where $\hat{\psi}_i\in L^2_{(s)}(\mu)$, $i=1,2$.

Homogeneous Wiener process

Recall that an $\mathcal{S}'(\mathbb{R}^d)$ -valued, Gaussian process $\{B_t,\, t\ge 0\}$ is called a *spatially homogeneous Wiener process* on $\mathbb{R}^d$ with the spectral measure m, see e.g. [7], if:

(M) for any $\psi\in\mathcal{S}(\mathbb{R}^d)$, $\{\langle B_t,\psi\rangle,\, t\ge 0\}$ is a real-valued and $\mathbb{E}\langle B_t,\psi\rangle=0$ for all $t\ge 0$,

(C) its covariance is of the form

$$\mathbb{E}\left[\,\langle B_t,\psi_1\rangle\langle B_s,\psi_2\rangle\right]=(2\pi)^{-d}\langle\hat{\psi}_1,\hat{\psi}_2\rangle_m(t\wedge s)\,,\quad \psi_1,\psi_2\in\mathcal{S}(\mathbb{R}^d),\ t,\ s\ge 0. \tag{2.15}$$

Suppose that m is such that the space $\mathcal{H}_m$ is Hilbert-Schmidt embedded in $\mathcal{E}$. One can show the following, see e.g. Proposition 4.1, p. 87 of [7].

Proposition 2.2 *For any orthonormal basis $\{v_n\}$ of $L^2_{(s)}(m)$ there is a sequence of independent standard real-valued Wiener processes $\{B^{(n)}_t,\, t\ge 0\}$ such that*

$$B_t = \sum_n B^{(n)}_t \tilde{\mathcal{F}}\left(v_n m\right), \qquad t\ge 0, \tag{2.16}$$

where the series converges in the L^2 sense and $\mathbb{P}$-a.s in $\mathcal{E}$.

It is easy to calculate that

$$\mathbb{E}\left[\, B(t,x)B(s,y)\right] = [\tilde{\mathcal{F}}m](x-y)(t\wedge s)\,, \qquad x,y\in\mathbb{R}^d,\ t,s\ge 0.$$

Stochastic differential equation for the random potential

Let $f=\tilde{\mathcal{F}}(\varphi\mu)$, then we define $S^o(t):\mathcal{H}_\mu\to\mathcal{H}_\mu$ by

$$S^o(t)f := \int e^{ip\cdot x-\gamma(p)t}\varphi(p)\frac{\mu(dp)}{(2\pi)^d}.$$

The family of mappings $\{S^o(t),\, t\in\mathbb{R}\}$ forms a uniformly strongly continuous group on $\mathcal{H}_\mu$ with the generator $C^o:\mathcal{H}_\mu\to\mathcal{H}_\mu$ given by $C^of=-\tilde{\mathcal{F}}(\gamma\varphi\mu)$. Let $S(t):=JS^o(t)J^*\mathfrak{U}$, then $\{S(t),\, t\ge 0\}$ form a group that is continuous in the uniformly strong operator topology on $\mathcal{E}$ with the generator $C:=JC^oJ^*\mathfrak{U}$. Since the process $\{V_t,\, t\ge 0\}$ is Gaussian, equality (2.14) implies that

$$\mathbb{E}\left[\langle V_t,\psi\rangle\Big|\mathcal{F}_s\right] = \langle S(t-s)V_s,\psi\rangle \tag{2.17}$$

for any $\psi\in\mathcal{E}^*$ and $t\ge s$. Here $\{\mathcal{F}_t,\, t\ge 0\}$ is the natural filtration corresponding to the process.

One can directly verify the following.

Proposition 2.3 *The process $B_t := V_t - V_0 - \int_0^t CV_s ds,\ t\ge 0$ is homogeneous, Wiener on $\mathcal{E}$, non-anticipative w.r.t. the filtration $\{\mathcal{F}_t,\, t\ge 0\}$, with the spectral measure $\nu(dp):=2\gamma(p)\mu(dp)$.*

Let $\bar{e}_n(p):=(2\gamma(p))^{-1/2}e_n(p)$. It is an orthonormal base in $L^2_{(s)}(\nu)$. Thanks to Proposition 2.2 there exists a family of i.i.d. standard Brownian motions $\{B^{(n)}_t,\, t\ge 0\}$ such that

$$V_t - V_0 - \int_0^t CV_s ds = \sum_n B^{(n)}_t \mathcal{F}\left(\bar{e}_n\nu\right), \qquad t\ge 0. \tag{2.18}$$

The process $\{V_t,\, t\ge 0\}$ is Markovian, see e.g. [7] chapter 5, with an invariant measure π. The $L^2(\pi)$ extension of the transition semigroup is strongly continuous and we denote its generator by $\mathfrak{L}:D(\mathfrak{L})\to L^2(\pi)$. To abbreviate the notation we shall write

$$\hat{V}(s,dp) := \sum_{n\ge 0} V^{(n)}_s e_n(p)\mu(dp) \quad\text{and}\quad \hat{B}(ds,dp) := \sum_{n\ge 0} \bar{e}_n(p)dB^{(n)}_s\nu(dp). \tag{2.19}$$

Generator of the process

Denote by Π the class of polynomials in $L^2(\pi)$, defined as the span over the random variables of the form

$$\Phi := \prod_{i=1}^{n} \zeta_{v_i}, \tag{2.20}$$

where $v_1, \ldots, v_n \in \mathcal{E}^*$ and $n \geq 0$. In case $n = 0$ we adopt the convention $\Phi(\zeta) \equiv 1$. Elements of the form (2.20) are called *monomials* of degree n. It is well known, see e.g. Chapter 2 of [11], that Π is dense in $L^2(\pi)$ and forms a core of the generator $\mathfrak{Q}$ (see Theorem 13.15, p. 207 of ibid.), i.e. $\Pi \subset D(\mathfrak{Q})$ and $\{(\Phi, \mathfrak{Q}\Phi) : \Phi \in \Pi]$ is a dense subset of the graph of the generator in the epigraph norm.

Using (2.18) we can calculate easily, via an application of the Itô formula, the generator on Π. Namely for Φ of the form (2.20) we have

$$d\Phi(V_t) = \mathfrak{Q}\Phi(V_t)dt + \sum_{k=1}^{n} d\zeta_{v_k}(B_t) \prod_{i \neq k} \zeta_{v_i}(V_t) \tag{2.21}$$

and

$$\mathfrak{Q}\Phi = \sum_{k=1}^{n} \zeta_{C^* v_k} \prod_{i \neq k} \zeta_{v_i} + \frac{1}{2} \sum_{k \neq \ell} \langle R v_k, v_\ell \rangle_{\mathcal{E}^*} \prod_{i \neq k, \ell} \zeta_{v_i}. \tag{2.22}$$

Here $\prod_{i \neq k}$ (resp. $\prod_{i \neq k,\ell}$) denotes the product over all $i = 1, \ldots, n$-s excluding k (resp. k, ℓ), the summation $\sum_{k \neq \ell}$ extends over all distinct $1 \leq k, \ell \leq n$. In addition, $J^* C^* v_k = -\tilde{\mathcal{F}}(\gamma \hat{v}_k \mu)$, if $J^* v_k = \tilde{F}(\hat{v}_k \mu)$,

$$\langle R v_k, v_\ell \rangle_{\mathcal{E}^*} = 2 \int_{\mathbb{R}^d} \gamma(p) \hat{v}_k(p) \hat{v}_\ell(p) \frac{\mu(dp)}{(2\pi)^d}.$$

and $(\zeta_{v_1}(B_t), \ldots, \zeta_{v_n}(B_t))$ is an n dimensional Brownian motion with the covariance matrix $[\langle R v_k, v_\ell \rangle_{\mathcal{E}^*}]$, $k, \ell = 1, \ldots, n$. In particular for the first degree polynomial given by (2.10) we obtain

$$\mathfrak{Q}\zeta_v = -\int_{\mathbb{R}^d} \gamma(p) \hat{v}(p) \hat{V}(dp). \tag{2.23}$$

2.3 Definition of the scattering operator

We define an operator valued function

$$\mathcal{K} : C(\mathbb{R}^{3d}) \times \mathbb{R}^d \to \mathcal{L}(\mathcal{H}_\mu, C(\mathbb{R}^{2d}))$$

assigning to a function $\psi \in C(\mathbb{R}^{3d})$ and $z \in \mathbb{R}^d$ an operator $\mathcal{K}[\psi, z] \in \mathcal{L}(\mathcal{H}_\mu, C(\mathbb{R}^{2d}))$ setting

$$\mathcal{K}[\psi, z]u(x, k) := -i \sum_{\sigma = \pm 1} \sigma \int_{\mathbb{R}^d} e^{ip \cdot z} \psi\left(x, z, k + \frac{\sigma p}{2}\right) \hat{u}(p) \frac{\mu(dp)}{(2\pi)^d} \tag{2.24}$$

for $u := \tilde{\mathcal{F}}(\hat{u}\mu)$, where $\hat{u} \in L^2_{(s)}(\mu)$. We let

$$\mathcal{K}_\varepsilon[\psi]u(x, k) = \mathcal{K}[\psi, \frac{x}{\varepsilon}]u(x, k).$$

Proposition 2.4 *Suppose that for a given $\varepsilon > 0$ and $s \in \mathbb{R}$*

$$a_\varepsilon := 2 \sup_{q\in\mathbb{R}^d} \int_{\mathbb{R}^d} \theta_{-s}\left(q + \frac{p}{\varepsilon}\right)\theta_s(q)\mu(dp) < +\infty. \tag{2.25}$$

Then,

$$\sum_{n\geq 0} \|\mathcal{K}_\varepsilon[\psi]f_n\|^2_{H^{s,0}} \leq a_\varepsilon \|\psi_\varepsilon\|^2_{H^{s,0}}, \tag{2.26}$$

where $\psi_\varepsilon(x,k) := \psi(x, x/\varepsilon, k) \in H^{s,0}$ and $\psi \in C(\mathbb{R}^{3d})$. Moreover,

$$\sum_{n\geq 0} \|\mathcal{K}[\psi, z]f_n\|^2_{H^{s,0}} \leq 2\mu(\mathbb{R}^d)\|\psi(\cdot, z; \cdot)\|^2_{H^{s,0}} \tag{2.27}$$

for any $z \in \mathbb{R}^d$ and $\psi \in C(\mathbb{R}^{3d})$, such that $\psi(\cdot, z; \cdot) \in H^{s,0}$.

Proof. We only prove (2.27), the proof of (2.26) is given in [13]. Note that

$$\sum_{n\geq 0} \|\mathcal{K}[\psi, z]f_n\|^2_{H^{s,0}} = \sum_n \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^d} \Phi(z,p,q,k)e_n(p)\mu(dp)\right|^2 \theta_s(q)dqdk, \tag{2.28}$$

where

$$\Phi(z,p,q,k) := \sum_{\sigma=\pm 1} \int_{\mathbb{R}^d} \sigma e^{i(q\cdot x + p\cdot z)}\psi(x, z; k - \frac{\sigma p}{2})dx.$$

We have

$$\sum_n \left| \int_{\mathbb{R}^d} \Phi(z,p,q,k)e_n(p)\mu(dp)\right|^2 = \sum_n \int_{\mathbb{R}^d} \Phi(z,p,q,k)e_n(p)\mu(dp) \int_{\mathbb{R}^d} \Phi^*(z,p',q,k)e_n(-p')\mu(dp')$$
$$= \sum_n \int_{\mathbb{R}^{2d}} \Phi(z,p,q,k)\Phi^*(z,-p',q,k)e_n(p)e_n(p')\mu(dp)\mu(dp'). \tag{2.29}$$

Therefore, by Proposition 2.1, we obtain

$$\sum_n \left| \int_{\mathbb{R}^d} \Phi(z,p,q,k)e_n(p)\mu(dp)\right|^2 = \int_{\mathbb{R}^d} |\Phi(z,p,q,k)|^2 \mu(dp),$$

and, consequently, the utmost left hand side of (2.28) equals

$$\int_{\mathbb{R}^{2d}} |\Phi(z,p,q,k)|^2 \theta_s(q)\mu(dp)dq.$$

Now, write

$$\Phi_\pm(z,p,q,k) := \pm \int_{\mathbb{R}^d} e^{-i(q\cdot x + p\cdot z)}\psi(x, z, k \pm \frac{p}{2})dx,$$

so that $\Phi = \Phi_- + \Phi_+$, and, moreover,

$$\int_{\mathbb{R}^d} |\Phi_\pm(z,p,q,k)|^2 dk = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i(q\cdot x + p\cdot z)}\psi(x,z,k)dx\right|^2 dk.$$

Hence, the sum on the utmost left hand side of (2.28) is bounded by

$$2\int_{\mathbb{R}^{3d}} \left| \int_{\mathbb{R}^d} e^{i(q\cdot x + p\cdot z)}\psi(x,z,k)dx\right|^2 \theta_s(q)\mu(dp)dkdq = 2\mu(\mathbb{R}^d)\|\psi(\cdot, z, \cdot)\|^2_{H^{s,0}}, \tag{2.30}$$

hence (2.27) holds. $\Box$

The above result shows in particular that for each $\varepsilon > 0$ and such that $\psi_\varepsilon \in H^{s,0}$ operator $u \mapsto \mathcal{K}_\varepsilon[\psi_\varepsilon]u$ can be extended to a Hilbert-Schmidt operator from $\mathcal{H}_\mu$ to $H^{s,0}$.

Let us fix $\psi \in \mathcal{S}(\mathbb{R}^{3d})$ and define $H^{s,0}$-valued, square integrable random element on $\mathcal{E}$ as follows

$$\mathcal{K}^o_\varepsilon[\psi]U := \sum_{n\geq 0} \xi_n(U)\mathcal{K}_\varepsilon[\psi]f_n, \quad \text{for } \pi \text{ a.s. } U \in \mathcal{E}. \tag{2.31}$$

Thanks to Proposition 2.4 the right hand side of (2.31) is $L^2(\pi)$ convergent in $H^{s,0}$ and the limiting object is defined as an element of $L^2(\pi; H^{s,0})$ - the space of all square integrable, $H^{s,0}$-valued, random elements with the appropriate norm. Formally speaking $\mathcal{K}^o_\varepsilon[\psi]Ju = \mathcal{K}_\varepsilon[\psi]u$ for $u \in \mathcal{H}_\mu$ (the left hand side needs not really be defined on Ju). For that reason and to simplify matters whenever it will not lead to a confusion we drop the superscript in the notation of the random element appearing on the right hand side of (2.31).

Similarly, when $\psi(\cdot, z, \cdot) \in H^{s,0}$ for any $z \in \mathbb{R}^d$ we can define

$$\mathcal{K}[\psi, z]U := \sum_{n\geq 0} \xi_n(U)\mathcal{K}[\psi, z]f_n. \tag{2.32}$$

Note also that for any $0 < t_1 < t_2 < T$ and $z \in \mathbb{R}^d$

$$\mathcal{K}[\psi, z]V_{t_2} - \mathcal{K}[\psi, z]V_{t_1} = \sum_{n\geq 0}(V^{(n)}_{t_2} - V^{(n)}_{t_1})\mathcal{K}_\varepsilon[\psi]f_n.$$

Hence,

$$\mathbb{E}\|\mathcal{K}[\psi, z]V_{t_2} - \mathcal{K}[\psi, z]V_{t_1}\|^2_{H^{s,0}} = \sum_{n,m\geq 0} \mathbb{E}\left[(V^{(n)}_{t_2} - V^{(n)}_{t_1})(V^{(m)}_{t_2} - V^{(m)}_{t_1})\right]\langle \mathcal{K}_\varepsilon[\psi]f_n, \mathcal{K}_\varepsilon[\psi]f_m\rangle_{H^{s,0}}$$

$$= \sum_{n,m\geq 0} \mathbb{E}\left[(V^{(n)}_{t_2} - V^{(n)}_{t_1})(V^{(m)}_{t_2} - V^{(m)}_{t_1})\right] \int\limits_{\mathbb{R}^{4d}} \varPhi(z, p, q, k)\varPhi^*(z, p', q, k)e_n(p)e^*_m(p')\theta_s(q)\mu(dp)\mu(dp')dqdk$$

$$= (2\pi)^d \sum_{n,m\geq 0} \int [e^{-\gamma(p_1)|t_2-t_1|} - 1]e_n(p_1)e_m(-p_1)\mu(dp_1)$$

$$\times \int_{\mathbb{R}^{4d}} \varPhi(z, p, q, k)\varPhi^*(z, p', q, k)e_n(p)e^*_m(p')\theta_s(q)\mu(dp)\mu(dp')dqdk$$

$$= (2\pi)^d \int_{\mathbb{R}^{3d}} [e^{-\gamma(p_1)|t_2-t_1|} - 1]|\varPhi(z, p, q, k)|^2\theta_s(q)\mu(dp)\mu(dp')dqdk \leq C|t_2 - t_1|\|\psi\|^2_{H^{s,0}}.$$

This, according to Corollary 11. 8 of [Ledoux-Talagrand], and due to properties of Gaussian elements, see Lemma 3. 7 and Corollary 3.9 of ibid. suffices to find an $H^{s,0}$ valued, Hölder continuous modification of $\{\mathcal{K}[\psi, z]V_t,\, t \geq 0\}$.

2.4 The solution of the Wigner equation with a random potential

Denote by $\{S_0(t),\, t \in \mathbb{R}\}$ a group of operators $S_0(t)f(x, k) := f(x - kt, k)$ that corresponds to generator A

$$A\psi(x, k) := -k \cdot \nabla_x\psi(x, k). \tag{2.33}$$

It can be shown, see [13], that

Proposition 2.5 *The group $\{S_0(t),\, t \in \mathbb{R}\}$ is strongly continuous on any space $H^{s,u}$ for $s, u \in \mathbb{R}$.*

Equation(1.1) can be recast in the following form (recall that $W_\varepsilon(t,x,k)$ is real valued)

$$\begin{aligned}&\partial_t W_\varepsilon(t) = AW_\varepsilon(t) + \varepsilon^{-1/2}\mathcal{K}_\varepsilon[W_\varepsilon(t), V_{t/\varepsilon}], \\ &W_\varepsilon(0,x,k) = W_0(x,k).\end{aligned} \tag{2.34}$$

This, in turn, leads to a mild formulation

$$W_\varepsilon(t) = S_0(t)W_0 + \int_0^t S_0(t-s)\mathcal{K}_\varepsilon[W_\varepsilon(s), V_{s/\varepsilon}]ds \tag{2.35}$$

for a.s. realization of $\{V_t,\, t\ge 0\}$, $n \ge 0$ satisfying (2.11). Performing the Fourier transform on both sides of (2.35) we get

$$\hat W_\varepsilon(t,q,k) = e^{-iq\cdot kt}\hat W_0(q,k) + \varepsilon^{-1/2} i \sum_{\sigma=\pm 1}\sigma \int_0^t\int \mathrm{e}^{-iq\cdot k(t-s)}\hat W_\varepsilon(s, q-p/\varepsilon, k+\sigma p/2)\frac{\hat V\left(s/\varepsilon, dp\right)ds}{(2\pi)^d}, \tag{2.36}$$

Iterating the right hand side of (2.36) we obtain that, at least formally, the solution should be given by the following series:

$$W_\varepsilon(t,x,k) = \sum_{n\ge 0}\mathcal{W}_{n,\varepsilon}(t,x,k), \tag{2.37}$$

where $\mathcal{W}_{0,\varepsilon}(t) := S_0(t)W_0$ and $\widehat{\mathcal{W}}_{n,\varepsilon}(t,q,k) = \mathcal{F}_1(\mathcal{W}_{n,\varepsilon}(t))(q,k)$ is given by

$$\begin{aligned}&\widehat{\mathcal{W}}_{0,\varepsilon}(t,q,k) = \mathrm{e}^{-ik\cdot qt}\hat W_0(q,t), \qquad (2.38)\\ &\widehat{\mathcal{W}}_{n,\varepsilon}(t,q,k) = \left(\frac{\varepsilon^{-1/2} i}{(2\pi)^d}\right)^n \sum_{\sigma_1,\ldots,\sigma_n=\pm 1}\sigma_1\ldots\sigma_n\int_{\Delta_n(t)}\int \exp\left\{-i\sum_{j=0}^n Q_j\cdot K_j(s_j - s_{j+1})\right\}\\ &\times\prod_{j=1}^n \hat V\left(\frac{s_j}{\varepsilon}, dp_j\right)\hat W_0(Q_n, K_n)d\mathbf{s}^{(n)},\end{aligned}$$

for $n\ge 1$. Here,

$$Q_j := q - \frac{1}{\varepsilon}\sum_{m=1}^j p_m \quad K_j := k + \frac{1}{2}\sum_{m=1}^j \sigma_m p_m, \tag{2.39}$$

with the conventions of writing $Q_0 := Q$, $K_0 := K$, $s_0 := t$, $s_{n+1} := 0$ and $d\mathbf{s}^{(n)} := ds_1\ldots ds_n$ and $d\mathbf{p}^{(n)} := dp_1\ldots dp_n$ and $\Delta_n(t,s) := [(s_1,\ldots,s_n) : t\ge s_1\ge\ldots\ge s_n\ge s]$ is an n-dimensional simplex. In case $s=0$ we shall simply write $\Delta_n(t)$. The Duhamel solution of (2.34) is defined as the sum of the series (2.37) in $H^{s,0}$.

Another notion of solution that can be introduced in the context of equation (2.34) is a weak solution. A stochastic process $\{W_\varepsilon(t),\, t\ge 0\}$ with trajectories belonging to $C([0,+\infty); H^{s,0})$ is called a weak solution if

$$\langle W_\varepsilon(t),\phi\rangle = \langle W_0,\phi\rangle - \int_0^t \left\langle W_\varepsilon(s), \left\{A\phi + \varepsilon^{-1/2}\mathcal{K}_\varepsilon[\phi]V_{s/\varepsilon}\right\}\right\rangle ds \tag{2.40}$$

for any $\phi\in H^{-s,0}$. Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-s,0}$ and $H^{s,0}$.

Theorem 2.6 *For a given $W_0\in H^{s,0}$ there exists a unique mild solution of* (2.34). *In addition, it is also a weak soltuion of the equation. In addition, the series appearing on the right hand side of* (2.37) *is convergent in $H^{s,0}$ in the $L^p(\mathbb{P})$ sense for any $p\in[1,+\infty)$. There exists a version of $\{W_\varepsilon(t),\, t\ge 0\}$ whose paths belong to $C([0,+\infty); H^{s,0})$. It is also a unique weak solution of* (2.34).

The proof is standard and can be done following the methods of [7].

2.5 The kinetic scattering operator

Given a function $\lambda \in H^{s,0}$ we define $\lambda_1(x,z,k;U)$ for π a.s. $U \in \mathcal{E}$ as a random element belonging to the domain of the generator $\mathfrak{Q}$ for each (x,z,k) that is differentiable in the $L^2(\pi)$ sense w.r.t. z variable such that

$$\mathcal{K}[\lambda, z]U(x,k) = [k \cdot \nabla_z + \mathfrak{Q}]\lambda_1(x,z,k;U) \tag{2.41}$$

and

$$\langle \lambda_1(x,k,z)\rangle_\pi = 0. \tag{2.42}$$

Suppose that $\lambda(x,k)$ is such that π-a.s. $\lambda_1(\cdot, z, \cdot) \in H^{s,0}$. Let

$$\mathcal{L}\lambda(x,k) := \langle \mathcal{K}[\lambda_1, z; \cdot]\rangle_\pi \,. \tag{2.43}$$

It might look that the right hand side of (2.43) depends on z, but as the calculation below shows it is not the case. Indeed, using (2.23), we obtain

$$\lambda_1(x,k,z) = \int e^{ip\cdot z}\alpha(x,k,p)\hat{V}(dp).$$

Here, as we recall (see (2.9)), $\hat{V}(dp) := \sum_{n\ge 0}\xi_n(U)e_n(p)\mu(dp)$ and

$$\alpha(x,k,p) = i(2\pi)^{-d}[\gamma(p) - ip\cdot k]^{-1}\sum_{\sigma=\pm 1}\sigma\lambda(x, k+\sigma p/2). \tag{2.44}$$

As a result we obtain

$$\mathcal{L}\lambda(x,k) = -i\left\langle \sum_{\sigma=\pm 1}\sigma \int_{\mathbb{R}^{2d}} e^{i(p+p')\cdot z}\alpha\left(x, k+\sigma p/2, p'\right)\hat{V}(dp)\hat{V}(dp')\right\rangle_\pi .$$

A simple application of (2.4) shows that the right hand side equals

$$-i\sum_{\sigma=\pm 1}\sigma\int_{\mathbb{R}^d}\alpha\left(x,k+\sigma p/2, -p\right)\frac{\mu(dp)}{(2\pi)^d}. \tag{2.45}$$

and substituting for $\alpha(\cdot)$ from (2.44) we obtain that

$$\mathcal{L}\lambda(x,k) = -(2\pi)^{-2d}\int_{\mathbb{R}^d}\left\{\left[\gamma(p) - ip\cdot\left(k+\frac{p}{2}\right)\right]^{-1}\left[\lambda\left(x,k\right) - \lambda\left(x,k+p\right)\right]\right.$$
$$\left. + \left[\gamma(p) - ip\cdot\left(k-\frac{p}{2}\right)\right]^{-1}\left[\lambda\left(x,k\right) - \lambda\left(x,k-p\right)\right]\right\}\mu(dp).$$

Changing variables in the last expression, $p := k+p$ for the term corresponding to the first summand, $p := k-p$ for the other one we obtain that

$$\mathcal{L}\lambda(x,k) = (2\pi)^{-2d}\Sigma(k)\int_{\mathbb{R}^d}\sigma(k,p)\left[\lambda\left(x,p\right) - \lambda\left(x,k\right)\right]dp, \tag{2.46}$$

where

$$\sigma(k,p) := \frac{1}{\Sigma(k)}\hat{R}\left(\frac{|p|^2 - |k|^2}{2}, p-k\right),$$

and $\Sigma(k)$ - the scattering cross-section corresponding to a wavevector k – is chosen in such a way that $\int \sigma(k,p)dp = 1$. Here

$$\hat{R}(\omega,p) := \int_{\mathbb{R}^{d+1}} e^{-i(\omega t + p\cdot x)} R(t,x)dtdx = \frac{2\gamma(p)\hat{R}(p)}{\omega^2+\gamma^2(p)}.$$

A simple calculation shows that

$$\Sigma(k) = \int_{\mathbb{R}^d} \frac{2\gamma(p)\hat{R}(p)dp}{\left(p\cdot k + |p|^2/2\right)^2 + \gamma^2(p)}. \tag{2.47}$$

2.6 Probabilistic representation of the radiative transport equation

Define by $\{\bar{W}(t),\, t\ge 0\}$ the solution of the linear kinetic equation

$$\begin{array}{l}\partial_t \bar{W}(t,x,k) + k\cdot\nabla_x \bar{W}(t,x,k) = \mathcal{L}\bar{W}(t,x,k),\\ \bar{W}(0,x,k) = W_0(x,k),\end{array} \tag{2.48}$$

where $W_0 \in H^{s,0}$. Let

$$T_0(t)f(x,k) := e^{-t\Sigma(k)} f(x-kt,k), \quad \forall\, f\in H^{s,u},\, t\ge 0$$

for an arbitrary $s,u\in\mathbb{R}$. By the solution of (2.48) we mean here a function $\{\bar{W}(t),\, t\ge 0\}$ that belongs to $C([0,+\infty),H^{s,0})$ and such that

$$\bar{W}(t) = T_0(t)W_0 + \int_0^t T_0(t-s)\mathcal{L}\bar{W}(s)ds. \tag{2.49}$$

One can show by a standard application of Gronwall's inequality that such a solution is unique.

Below we give a probabilistic formula for the solution to (2.48) treating it as the solution of Kolmogorov's equation for a certain Markov jump process. The results of this section are standard and their proofs can be found e.g. in Apppendix 2 of [12]. Let $t(k) := \Sigma^{-1}(k)$. The scattering kernel $\sigma(k,p)$ corresponds to the transition probability density of a certain Markov chain $K_0, K_1, \ldots$. Let $\mathbb{P}_k$ and $\mathbb{E}_k$ be respectively the path measure and its expectation corresponding to the chain satisfying $K_0 = k$. Let $\sigma_0, \sigma_1, \ldots$ be i.i.d. exponential random variables with intensity 1. Let $t_0 := 0$ and $t_n := \sum_{i=0}^{n-1} \sigma_i t(K_i)$ for $n\ge 1$. Define then $K(t) := K_n$, $t\in[t_n, t_{n+1})$. Since $\hat{R}(-\omega,-p) = \hat{R}(\omega,p)$ the Lebesgue measure on $\mathbb{R}^d$ is invariant for the process $\{K(t),\, t\ge 0\}$.

The solution of (2.48) has a representation given by

$$\bar{W}(t,x,k) = \mathbb{E}_k\left\{W_0(x - \int_0^t K(s)ds, K(t))\right\}. \tag{2.50}$$

We can rewrite (2.50) more explictly. Iterating (2.49) we obtain

$$\bar{W}(t,x,k) = \sum_{n=0}^{+\infty} W_n(t,x,k), \tag{2.51}$$

where $W_0(t) := T_0(t)W_0$ and

$$W_n(t,x,k) := \int_0^{+\infty}\!\!\ldots\int_0^{+\infty} d\tau_{0,n} \int\!\ldots\!\int dk_{1,n} \prod_{i=1}^n \sigma(k_{i-1},k_i)$$
$$\times\left\{\exp\left\{-\sum_{i=0}^n \tau_i\right\} W_0\left(x-\mathcal{X}_n, k_n\right) 1[\sum_{i=0}^{n-1} t(k_i)\tau_i \le t < \sum_{i=0}^n t(k_i)\tau_i]\right\}$$

for $n \geq 1$. Here $k_0 := k$,

$$\tilde{\mathcal{X}}_n := \sum_{i=0}^{n-1} k_i t(k_i)\tau_i + k_n \left(t - \sum_{i=0}^{n-1} t(k_i)\tau_i\right)$$

and $d\tau_{0,n} := d\tau_0 \dots d\tau_n$, $dk_{1,n} := dk_1 \dots dk_n$. Integrating over τ_n and changing remaining variables according to $\tau_i' := t(k_i)\tau_i$, $i = 0, \dots, n-1$ we get

$$W_n(t,x,k) = \int_0^{+\infty} \dots \int_0^{+\infty} d\tau_{0,n} \int \dots \int dk_{1,n} \prod_{i=1}^{n} [\sigma(k_{i-1},k_i)\Sigma(k_{i-1})] \tag{2.52}$$
$$\times \left\{ \exp\left\{ -\sum_{i=0}^{n} \Sigma(k_i)\tau_i \right\} W_0\left(x - \mathcal{X}_n, k_n\right) \delta\left(t - \sum_{i=0}^{n} \tau_i\right) \right\}$$

Here $\mathcal{X}_n := \sum_{i=0}^{n} k_i \tau_i$.

Define $\mathcal{B}_{p_1,p_2}$ as the Banach spaces that is the completion of $\mathcal{S}(\mathbb{R}^{2d})$ under the norm

$$\|\phi\|_{p_1,p_2}^{p_1} := \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} |\hat{\phi}(q,y)|^{p_2} dy\right]^{p_1/p_2} dq.$$

The definition can be easily extended to cover the case when one, or both of the indices equal $+\infty$. Denote $\widehat{W}(t,q,k) := \mathcal{F}_1(W(t))(q,k)$.

Proposition 2.7 *Suppose that $W_0(x,k) = \delta \otimes f(x,k)$, where $f \in \mathcal{S}(\mathbb{R}^d)$. Then, for any $p \in [1,+\infty]$*

$$W_{*,p} := \sup_{t \geq 0} \|\hat{W}(t)\|_{\infty,p} < +\infty. \tag{2.53}$$

Proof. Estimate (2.53) follows immediately from the invariance of the Lebesgue measure under the process $\{K(t),\, t \geq 0\}$ and the formula

$$\widehat{W}(t,q,k) = \mathbb{E}_k \left\{ \exp\left\{ -iq \cdot \int_0^t K(s)ds \right\} f(K(t)) \right\}. \tag{2.54}$$

that is a consequence of (2.50). $\square$

Using (2.52) we can write that

$$\widehat{W}(\tau,q,k) = \sum_{m \geq 0} \widehat{W}_m(\tau,q,k), \tag{2.55}$$

where $\widehat{W}_0(\tau,q,k) := \mathrm{e}^{-i\tau q\cdot k}\mathrm{e}^{-\Sigma(k)\tau} f(k)$ and

$$\widehat{W}_m(\tau,q,k) := \int_0^{+\infty} \dots \int_0^{+\infty} d\tau_{0,m} \int \dots \int dk_{1,m} \prod_{i=0}^{m-1} [\sigma(k_i,k_{i+1})\Sigma(k_i)] \tag{2.56}$$
$$\times \exp\left\{ -\sum_{j=0}^{m} \Sigma(k_j)\tau_j \right\} \exp\left\{ -i \sum_{j=0}^{m} q \cdot k_j \tau_j \right\} f(k_m)\delta\left(\tau - \sum_{j=0}^{m} \tau_j\right).$$

Here $k_0 := k$.

2.7 The formulation of the main result

Let us recall that the initial data for the Wigner transform is assumed to be of the form $W_0(x,k) = \delta(x)f(k)$, where $f \in C_0^\infty(\mathbb{R}^d)$ is a smooth energy distribution in directions k. Consider the rescaled fluctuations of the solution of the Wigner equation around its mean, that is,

$$Z_\varepsilon(t) := \varepsilon^{-1/2}[W_\varepsilon(t) - \bar{W}(t)].$$

It satisfies the equation

$$\begin{aligned} &\partial_t Z_\varepsilon(t) = AZ_\varepsilon(t) + \varepsilon^{-1}\mathcal{K}_\varepsilon[\bar{W}(t); V_{t/\varepsilon}] + \varepsilon^{-1/2}\left\{\mathcal{K}_\varepsilon[Z_\varepsilon(t); V_{t/\varepsilon}] - \mathcal{L}\bar{W}(t)\right\}, \\ &Z_\varepsilon(0) = 0. \end{aligned} \tag{2.57}$$

Suppose also that $s, u > d/2$ and $\bar{Z}(t)$ is the solution in $H^{-s,-u}$ of (2.48) with the initial data $\bar{Z}(0,x,k) = \delta(x)X(k)$. Here X is a real valued Gaussian H^{-u}-valued element given by

$$X(k) := i\sum_{\sigma=\pm 1}\sigma\int_0^{+\infty}\int \exp\{iK_\sigma\cdot ps\}\left[\gamma(p) - ip\cdot k\right]^{-1} f(K_\sigma)\frac{\hat{B}(ds,dp)}{(2\pi)^d} \tag{2.58}$$

that is, for any collection $\psi_j \in \mathcal{S}(\mathbb{R}^d)$, $j = 1,\ldots,N$, the pairings $\langle X, \psi_j\rangle$, $j = 1,\ldots,N$ are jointly Gaussian of mean zero and with the covariance given by

$$\mathcal{C}(\psi_i,\psi_j) := \mathbb{E}[\langle X,\psi_i\rangle\langle X,\psi_j\rangle] = \sum_{\sigma,\sigma'=\pm 1}\sigma\sigma'\int\frac{\nu(dp)}{(2\pi)^{2d}}\int_0^{+\infty} \mathrm{e}^{is(\sigma-\sigma')|p|^2/2} g_{\sigma,\sigma'}(ps,p)ds, \tag{2.59}$$

where

$$g_{\sigma,\sigma'}(q,p) := \int_{\mathbb{R}^{2d}} \mathrm{e}^{iq\cdot(k-k')}\psi_i(k)\psi_j(k')\left[(\gamma(p) - ip\cdot k)(\gamma(p) + ip\cdot k')\right]^{-1} f(K_\sigma)f(K'_{\sigma'})dkdk'.$$

We use above the notation

$$K_\sigma(k,p) := k + \sigma p/2, \tag{2.60}$$

and $K'_{\sigma'} := k' + \sigma' p'/2$. Using (2.51) and (2.52) we obtain that for any $\theta \in \mathcal{S}(\mathbb{R}^{2d})$ we have a Duhamel series for $\bar{Z}$:

$$\langle\bar{Z}(t),\theta\rangle = \sum_{n=0}^{+\infty}\langle\bar{Z}_n(t),\theta\rangle, \tag{2.61}$$

where

$$\begin{aligned} \langle\bar{Z}_n(t),\theta\rangle = &\int_0^{+\infty}\!\!\ldots\int_0^{+\infty} d\tau_{0,n}\int\!\ldots\!\int dk_{0,n}\prod_{i=1}^n\sigma(k_{i-1},k_i)\prod_{i=1}^n\Sigma(k_{i-1}) \\ &\times\left\{\exp\left\{-\sum_{i=0}^n\Sigma(k_i)\tau_i\right\}\theta\left(\mathcal{X}_n,k_0\right)X\left(k_n\right)\delta\left(t - \sum_{i=0}^n\tau_i\right)\right\}, \end{aligned} \tag{2.62}$$

The following theorem is the main result of the article.

Theorem 2.8 *Suppose that*

$$\hat{R} \in W^{\infty,[d/2]+1}(\mathbb{R}^d) \tag{2.63}$$

and

$$\sup_q\int|p+q|^{-2}\hat{R}(p)dp < +\infty. \tag{2.64}$$

Assume also that $\theta \in \mathcal{S}(\mathbb{R}^d)$. Under the above assumptions the finite dimensional laws of the processes $\{\langle Z_\varepsilon(t),\theta\rangle,\ t>0\}$ converge in law, as $\varepsilon\downarrow 0$, to those of $\{\langle\bar{Z}(t),\theta\rangle,\ t>0\}$.

Outline of the proof

The strategy of the proof is to gradually simplify the terms on the right side of the equation (2.57) for Z_ε. The first step is to eliminate the term of the apparent order $O(\varepsilon^{-1})$ in (2.57) in Section 3. This is done by adding a corrector $\Lambda_\varepsilon(t,x,k)$, in the spirit of the perturbed test function method. The crucial step here is Proposition 3.1 that shows that the corrector needed to eliminate the "apparently largest" term in (2.57) is actually small (as a distribution). This is done using the Duhamel expansion for the deterministic kinetic equation. This step reduces the asymptotics of Z_ε to those of $\tilde{Z}_\varepsilon$, solution of (3.1) below. The latter equation has three forcing terms, of which one is a martingale, and the other two are not but all come about because of the corrector Λ_ε. The next step is to eliminate the non-martingale forcing terms from (3.1). This is done in Section 4, see Theorem 4.1. The asymptotics of Z_ε is, therefore, reduced to those of Z^o_ε, the solution of (5.1), which is the "standard" Wigner equation (in particular, with the "apparently largest" term of the order $O(\varepsilon^{-1/2})$, not $O(\varepsilon^{-1})$ as in (2.57)), with a martingale forcing. In Section 5 we formulate Theorem 5.2 that allows us to replace the "Wigner equation" part of (5.1) with a kinetic equation and the same martingale forcing, see (5.5). The proof of this theorem is contained in Section 6. All that remains to do in Section 7 in order to finish the proof of Theorem 2.8 is to show that the solution of (5.5) converges to the solution of the kinetic equation with a random initial datum.

3 Eliminating the largest term

Definition of the corrector

We represent the solution to (2.57) in the form

$$Z_\varepsilon(t) = \tilde{Z}_\varepsilon(t) + \Lambda_\varepsilon(t),$$

where $\Lambda_\varepsilon(t,x,k) = \Lambda(t, V_{t/\varepsilon}; x, x/\varepsilon, k)$ and the corrector $\Lambda : [0,+\infty) \times \mathcal{E} \to C(\mathbb{R}^{2d}; L^2_{(s)}(\mu))$ is linear in the $\mathcal{E}$-variable and shall be specified in (3.5) below. We can write

$$\begin{aligned} d\tilde{Z}_\varepsilon(t) &= \left\{ A\tilde{Z}_\varepsilon(t) + \varepsilon^{-1/2}\mathcal{K}_\varepsilon[\tilde{Z}_\varepsilon(t); V_{t/\varepsilon}] + \sum_{i=0}^{2} \varepsilon^{-i/2}\tilde{\mathcal{W}}^\varepsilon_i(t) \right\} dt + d\mathcal{M}_\varepsilon(t) \\ \tilde{Z}_\varepsilon(0) &= -\Lambda_\varepsilon(0) \end{aligned} \tag{3.1}$$

Here, we have defined

$$\tilde{\mathcal{W}}^\varepsilon_i(t,x,k) = \tilde{\mathcal{W}}_i(t,x,k,x/\varepsilon; V_{t/\varepsilon}) \tag{3.2}$$

with

$$\begin{aligned} \tilde{\mathcal{W}}_0(t,x,k,z;U) &:= -[\partial_t + k\cdot\nabla_x]\Lambda(t,U;x,k,z), \\ \tilde{\mathcal{W}}_1(t,x,k,z;U) &:= -\mathcal{L}\bar{W}(t,x,k) + \mathcal{K}[\Lambda(t,U),z;U](x,k), \\ \tilde{\mathcal{W}}_2(t,x,k,z;U) &:= -[k\cdot\nabla_z + \mathfrak{Q}]\Lambda(t,U;x,z,k) + \mathcal{K}[\bar{W}(t),z;U](x,k) \end{aligned} \tag{3.3}$$

while $\{\mathcal{M}_\varepsilon(t),\, t\geq 0\}$ is a certain $H^{s,0}$-valued martingale, which we describe below. The function $\Lambda(t)$ is chosen in such a way that $\tilde{\mathcal{W}}_2(t) = 0$, or, equivalently,

$$[k\cdot\nabla_z + \mathfrak{Q}]\Lambda(t,x,z,k;U) = \mathcal{K}[\bar{W}(t),z;U](x,k). \tag{3.4}$$

This of course implies that $\mathcal{W}^\varepsilon_2(t) \equiv 0$, eliminating the largest term in (3.1).

Using the same argument as the one below (2.41) we obtain that

$$\Lambda(t,x,z,k;U) = \int e^{ip\cdot z}\tilde{\alpha}(t,x,k,p)\hat{V}(dp), \tag{3.5}$$

where

$$\tilde{\alpha}(t,x,k,p) = i(2\pi)^{-d}[\gamma(p) - ip\cdot k]^{-1}\sum_{\sigma=\pm 1}\sigma\bar{W}(t,x,k+\sigma p/2).$$

Note that we have, cf. (2.41) and (2.43),

$$\left\langle \mathcal{K}[\Lambda(t,U),z;U](x,k) - \mathcal{L}\bar{W}(t,x,k)\right\rangle_{\pi} = 0.$$

From (2.57) and Proposition 2.3, cf. also (2.19), we conclude that

$$\mathcal{M}_{\varepsilon}(t,x,k) := i\sum_{\sigma=\pm 1}\sigma\int_0^{t/\varepsilon}\int \mathrm{e}^{ip\cdot x/\varepsilon}[-\gamma(p)+ip\cdot k]^{-1}\bar{W}(\varepsilon s,x,k+\sigma p/2)\frac{\hat{B}(ds,dp)}{(2\pi)^d}. \tag{3.6}$$

The martingale is, therefore, Gaussian, adapted to the natural filtration corresponding to the Brownian motion.

The following crucial estimate shows that the weak limits of Z_ε and $\tilde{Z}_\varepsilon$ (if they exist) are the same – recall, once again, that working with $\tilde{Z}_\varepsilon$ is simpler since the leading order term in the equation for $\tilde{Z}_\varepsilon$ has an apparent order $O(\varepsilon^{-1/2})$ rather than $O(\varepsilon^{-1})$.

Proposition 3.1 *For any $t>0$ there exists a constant $C>0$ such that for all $\theta\in\mathcal{S}(\mathbb{R}^{2d})$ we have*

$$\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right] \le C\left(\varepsilon\log\frac{1}{\varepsilon}\right)^2 |||\theta|||^2_{1,1,B}.$$

Proof of Proposition 3.1

We have

$$\Lambda_\varepsilon(t,x,k) = i\sum_{\sigma=\pm 1}\sigma\int e^{ip\cdot x/\varepsilon}[-\gamma(p)+ip\cdot k]^{-1}\bar{W}(t,x,K_\sigma)\frac{\hat{V}(t/\varepsilon,dp)}{(2\pi)^d},$$

where K_σ is given by (2.60). The process $\Lambda_\varepsilon(t)$ is Gaussian, of zero mean. The variance of $\langle\Lambda_\varepsilon(t),\theta\rangle$ equals

$$\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right] = \sum_{\sigma,\sigma'=\pm 1}\sigma\sigma'\int e^{\frac{ip\cdot(x-x')}{\varepsilon}}\Gamma(p,k,k')\theta(x,k)\theta(x',k')\bar{W}(s,x,K_\sigma)\bar{W}(s,x',K'_{\sigma'})\frac{dxdx'dkdk'\mu(dp)}{(2\pi)^{2d}}, \tag{3.7}$$

where,

$$\Gamma(p,k,k') := \left\{[\gamma(p)-ip\cdot k][\gamma(p)+ip\cdot k']\right\}^{-1}. \tag{3.8}$$

The right side of (3.7) can be further transformed by using expansion (2.51) for $\bar{W}$ leading to

$$\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right] = \sum_{n,n'\ge 0}\Lambda_\varepsilon^{(n,n')},$$

with $\Lambda_\varepsilon^{(n,n')}$ given by formulas similar to (3.7), with the product of $\bar{W}(s,x,K_\sigma)$ and $W(s,x',K'_{\sigma'})$ replaced by the respective product of $W_n(s,x,K_\sigma)$ and $W_{n'}(s,x',K'_{\sigma'})$ defined in (2.52). Using those definitions and the representation $\delta(t)=(2\pi)^{-1}\int e^{i\beta t}d\beta$ we can write

$$\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right]=(2\pi)^{-2d-1}\sum_{n,n'\geq 0}\sum_{\sigma,\sigma'=-1,1}\sigma\sigma'\underbrace{\int_0^{+\infty}\dots\int_0^{+\infty}}_{n+1-\text{times}}d\tau_{0,n}\underbrace{\int_0^{+\infty}\dots\int_0^{+\infty}}_{n'+1-\text{times}}d\tau'_{0,n}$$
$$\times\int e^{i(\beta+\beta')t}\exp\left\{-i\left(\beta\sum_{i=0}^n\tau_i-\beta'\sum_{i=0}^{n'}\tau'_i\right)\right\}\exp\left\{i(\sum_{i=0}^n p\cdot k_i\tau_i-\sum_{i=0}^{n'}p\cdot k'_i\tau'_i)/\varepsilon\right\}$$
$$\times\exp\left\{-i\left(\sum_{i=0}^n q\cdot k_i\tau_i-\sum_{i=0}^{n'}q'\cdot k'_i\tau'_i\right)\right\}\mathcal{F}_1(\theta)(q,k)\mathcal{F}_1(\theta)(q',k') \tag{3.9}$$
$$\times\prod_{i=0}^n e^{-\Sigma(k_i)\tau_i}\prod_{i=0}^{n'}e^{-\Sigma(k'_i)\tau'_i}\mathcal{I}_{n,n'}(p,\mathbf{k}^{(n)},\mathbf{k}'^{(n')})\frac{d\beta d\beta' dqdq'd\mathbf{k}^{(n)}d\mathbf{k}'^{(n')}\mu(dp)}{(2\pi)^{d+2}}.$$

Here,

$$\mathcal{I}_{n,n'}(p,\mathbf{k}^{(n)},\mathbf{k}'^{(n')}):=\Gamma(p,k,k')f(k_n)f(k'_{n'})\prod_{i=1}^n\left[\Sigma(k_{i-1})\sigma(k_{i-1},k_i)\right]\prod_{i=0}^{n'}\left[\Sigma(k'_{i-1})\sigma(k'_{i-1},k'_i)\right],$$

$k_0:=K_\sigma$, $k'_0:=K'_{\sigma'}$ and for abbreviation sake we write $d\tau:=d\tau_0\dots d\tau_n$, $d\tau':=d\tau'_0\dots d\tau'_{n'}$, $d\mathbf{k}^{(n)}:=dkdk_1\dots dk_n$, $d\mathbf{k}'^{(n')}:=dk'dk'_1\dots dk'_{n'}$. Integrating out the τ variables we get

$$\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right]=(2\pi)^{-2d-2}\sum_{n,n'\geq 0}\sum_{\sigma,\sigma'=-1,1}\sigma\sigma'\int e^{i(\beta+\beta')t}\mathcal{F}_1(\theta)(q,k)\mathcal{F}_1(\theta)(q',k')\mathcal{I}_{n,n'}(p,\mathbf{k}^{(n)},\mathbf{k}'^{(n')})$$
$$\times\left[\prod_{i=0}^n[\Sigma(k_i)+i[\beta+(q+\frac{p}{\varepsilon})]\cdot k_i]\right]^{-1}\left[\prod_{i=0}^{n'}[\Sigma(k'_i)+i[\beta'+(q'+\frac{p}{\varepsilon})]\cdot k_i]\right]^{-1}d\beta d\beta' dqdq'd\mathbf{k}^{(n)}d\mathbf{k}'^{(n')}\mu(dp).$$

Using the fact that $\hat{e}_A(\beta)=(A+i\beta)^{-1}$ is the Fourier transform of the function

$$e_A(t):=\begin{cases}e^{-At}, & t>0\\ 0, & t<0\end{cases}$$

we can rewrite

$$\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right]=(2\pi)^{-2d-2}\sum_{n,n'\geq 0}\sum_{\sigma,\sigma'=-1,1}\sigma\sigma'\int\int e_{A_0}*\dots*e_{A_n}(t)e_{A'_0}*\dots*e_{A'_{n'}}(t),$$
$$\times\mathcal{F}_1(\theta)(q,k)\mathcal{F}_1(\theta)(q',k')\mathcal{I}_{n,n'}(p,\mathbf{k}^{(n)},\mathbf{k}'^{(n')})dqdq'd\mathbf{k}^{(n)}d\mathbf{k}'^{(n')}\mu(dp) \tag{3.10}$$

where $A_i:=\Sigma(k_i)+i(q+(p/\varepsilon))\cdot k_i$ and $A'_i:=\Sigma(k'_i)+i(q'+(p/\varepsilon))\cdot k'_i$. Computing the convolution on the right hand side of (3.10) we obtain that this expression equals

$$(2\pi)^{-2d-2}\sum_{n,n'\geq 0}\sum_{\sigma,\sigma'=-1,1}\sigma\sigma'\int_{\Delta_n(t)}d\tau_{1,n}\int_{\Delta_{n'}(t)}d\tau'_{1,n'}\int\int\exp\left\{-A_0\tau_1\right\}\exp\left\{-A'_0\tau'_1\right\}\mathcal{F}_1(\theta)(q,k)$$
$$\times\mathcal{F}_1(\theta)(q',k')\Gamma(p,k,k')f(k_n)f(k'_{n'})\times\prod_{i=1}^n\left[\exp\left\{-A_i(\tau_{i+1}-\tau_i)\right\}\Sigma(k_{i-1})\sigma(k_{i-1},k_i)\right] \tag{3.11}$$
$$\times\prod_{i=0}^{n'}\left[\exp\left\{-A'_i(\tau'_{i+1}-\tau'_i)\right\}\Sigma(k'_{i-1})\sigma(k'_{i-1},k'_i)\right]dqdq'd\mathbf{k}^{(n)}d\mathbf{k}'^{(n')}\mu(dp),$$

where $\tau_{n+1}=\tau'_{n'+1}:=t$. We can further rewrite (3.11) as being equal to

$$
\begin{aligned}
&(2\pi)^{-2d-2}\sum_{n,n'\ge 0}\sum_{\sigma,\sigma'=-1,1}\sigma\sigma'\int_{\Delta_n(t)}d\tau_{1,n}\int_{\Delta_{n'}(t)}d\tau'_{1,n'}\int\int dqdq'dkdk'\mu(dp) \qquad (3.12)\\
&\times\Sigma(k_0)\exp\left\{-\Sigma(k_0)\tau_1\right\}\Sigma(k'_0)\exp\left\{-\Sigma(k'_0)\tau_1\right\}\mathcal{F}_1(\theta)(q,k)\mathcal{F}_1(\theta)(q',k')\Gamma(p,k,k')\\
&\times\mathcal{F}(G_n(\cdot;k))\left((p/\varepsilon+q)(\tau_1-\tau_0),\ldots,(p/\varepsilon+q)(\tau_{n+1}-\tau_n)\right)\\
&\times\mathcal{F}(G_{n'}(\cdot;k'))\left((p/\varepsilon+q')(\tau'_1-\tau'_0),\ldots,(p/\varepsilon+q')(\tau'_{n+1}-\tau'_n)\right),
\end{aligned}
$$

where

$$
G_n(k_1,\ldots,k_n;k):=g(k_n)\prod_{i=1}^{n-1}\exp\left\{-\Sigma(k_i)(\tau_{i+1}-\tau_i)\right\}\prod_{i=1}^{n}\hat{R}\left(\frac{|k_i|^2-|k_{i-1}|^2}{2},k_i-k_{i-1}\right)
$$

and

$$
g(k_n):=\Sigma^{-1}(k_n)f(k_n)\exp\left\{-\Sigma(k_n)(\tau_{n+1}-\tau_n)\right\}.
$$

We have

$$
\mathcal{R}_1:=\sup_k\int\hat{R}\left(\frac{|l|^2-|k|^2}{2},l-k\right)dl=\sup_k\int\frac{\hat{R}\,(l-k)}{\gamma(l-k)}dl\le\frac{2}{\gamma_*}\int\hat{R}\,(l)\,dl<+\infty. \qquad (3.13)
$$

Since $\Sigma(k_n)$ is continuous and strictly positive for all k_n and $f(k_n)$ is compactly supported we have $\|g\|_\infty<+\infty$ and as a result

$$
\mathcal{G}_1:=\sup_k\int|G_n(k_1,\ldots,k_n;k)|d\mathbf{k}_{1,n}\le\|g\|_\infty\mathcal{R}_1^n<+\infty. \qquad (3.14)
$$

Here $d\mathbf{k}_{1,n}:=dk_1\ldots dk_n$. We can also easily estimate

$$
\sup_{y_1,\ldots y_n}|y_i\mathcal{F}(G_n(\cdot;k))\,(y_1,\ldots,y_n)\,|\le\int|\nabla_{k_i}G_n(k_1,\ldots,k_n;k)|d\mathbf{k}_{1,n}.
$$

Lemma 3.2 *We have*

$$
\mathcal{G}_*:=\sum_{i=1}^{n}\sup_k\int|\nabla_{k_i}G_n(k_1,\ldots,k_n;k)|d\mathbf{k}_{1,n}\le C_*\mathcal{R}_1^{n-1},
$$

where the constant C_* depends on $\mathcal{R}_*$, $\mathcal{G}_1$, t but not on $\mathcal{R}_1$ and n.

We postpone the proof of Lemma 3.2 for the moment. Returning to (3.12) we obtain $\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right]=\sum_{n,n'\ge 0}\Lambda_\varepsilon^{(n,n')}$ and

$$
\begin{aligned}
\Lambda_\varepsilon^{(n,n')}&=(2\pi)^{-2d-2}\sum_{\sigma,\sigma'=-1,1}\sigma\sigma'\int_{\Delta_n(t)}d\tau_{1,n}\int_{\Delta_{n'}(t)}d\tau'_{1,n'}\int\int dqdq'dkdk'\mu(dp)\\
&\times\Sigma(k_0)\exp\left\{-\Sigma(k_0)\tau_1\right\}\Sigma(k'_0)\exp\left\{-\Sigma(k'_0)\tau_1\right\}\mathcal{F}_1(\theta)(q,k)\mathcal{F}_1(\theta)(q',k')\Gamma(p,k,k')\\
&\times\left(1+\sum_{i=1}^{n}|(p/\varepsilon+q)|(\tau_{i+1}-\tau_i)\right)\mathcal{F}(G_n(\cdot;k))\left((p/\varepsilon+q)(\tau_1-\tau_0),\ldots,(p/\varepsilon+q)(\tau_{n+1}-\tau_n)\right)\\
&\times\left(1+\sum_{i=1}^{n'}|(p/\varepsilon+q')|(\tau'_{i+1}-\tau'_i)\right)\mathcal{F}(G_{n'}(\cdot;k'))\left(p/\varepsilon+q')(\tau'_1-\tau'_0),\ldots,(p/\varepsilon+q')(\tau'_{n+1}-\tau'_n)\right)\\
&\times\left[\left(1+\sum_{i=1}^{n}|(p/\varepsilon+q)|(\tau_{i+1}-\tau_i)\right)\left(1+\sum_{i=1}^{n'}|(p/\varepsilon+q')|(\tau'_{i+1}-\tau'_i)\right)\right]^{-1}
\end{aligned}
$$

and, by virtue of Lemma 3.2, the right hand side can be estimated by

$$C_*^2(n+1)(n'+1)\|\!|\theta|\!\|_{1,1,B}^2\frac{{\mathcal{R}_\infty}^{n+n'-2}}{n!(n')!}$$
$$\times\sup_{q,q'}\int_0^t\int_0^t\int\left[\left(1+\sum_{i=1}^{n}|(p/\varepsilon+q)|\tau\right)\left(1+\sum_{i=1}^{n'}|(p/\varepsilon+q')|\tau'\right)\right]^{-1}d\tau d\tau'\mu(dp).$$

Observe that

$$\int_0^t[1+|(p/\varepsilon+q)|\tau]^{-1}d\tau\le\varepsilon+\int_\varepsilon^t[1+|(p/\varepsilon+q)|\tau]^{-1}d\tau$$
$$\le\varepsilon+\varepsilon|p+\varepsilon q|^{-1}\int_\varepsilon^t\tau^{-1}d\tau\le C\left(\varepsilon\log\frac{1}{\varepsilon}\right)[1+|p+\varepsilon q|^{-1}]$$

for some constant $C>0$ and all $\varepsilon\in(0,1)$. Therefore,

$$\mathbb{E}\left[\langle\Lambda_\varepsilon(t),\theta\rangle^2\right]\le C\left(\varepsilon\log\frac{1}{\varepsilon}\right)^2\|\!|\theta|\!\|_{1,1,B}^2e^{C\Sigma_*}\left(1+\sup_q\int|p+q|^{-2}\mu(dp)\right)$$

for some constant $C>0$. The constant on the right hand side is finite thanks to (2.64). □

The proof of Lemma 3.2

In order to prove Lemma 3.2 we shall need the following estimate.

Lemma 3.3 *We have*

$$\mathcal{R}_*:=\sup_k\int|l|\left|\partial_\omega\hat{R}\left(\frac{|l|^2-|k|^2}{2},l-k\right)\right|dl<+\infty. \tag{3.15}$$

Proof of Lemma 3.3. Let us first explain the rough balance leading to (3.15). Let $f(k,\omega)$ be a bounded function supported inside the set $\{|k|\le1,|\omega|\le1\}$. Then the support of the function $f((k+p)^2-k^2,p)$ (as function of p) lies inside the set $\{|p|\le1,|(k\cdot p)|\le10\}$. It follows that

$$\int|l|f(k^2-l^2,k-l)dl\le C(1+|k|)\,|\{|p|\le1,|(k\cdot p)|\le10\}|\le C,$$

which is the spirit of (3.15).

We now prove (3.15) more carefully. Assume that $|k|\ge1$. We have

$$\partial_\omega\hat{R}(\omega,p)=-\frac{4\gamma(p)\omega\hat{R}(p)}{(\gamma^2(p)+\omega^2)^2}.$$

For $n\ge0$ we let $A_n(k):=[n\le|k-l|\le n+1]$. Since, for any $M>0$ there exists $C>0$ such that $\hat{R}(p)\le C\langle p\rangle^{-M}$, we can easily see that

$$\int_{A_n(k)}|l|\left|\partial_\omega\hat{R}\left(\frac{|l|^2-|k|^2}{2},l-k\right)\right|dl\le\frac{C(|k|+n)^2}{\langle n\rangle^M}\int_{A_n(k)}\frac{||k|-|l||dl}{1+|k|^4||k|-|l||^4}.$$

We change variables $\ell:=l/|k|$. The right hand side of the above estimate equals

$$\frac{C(|k|+n)^2|k|^{d+1}}{\langle n\rangle^M}\int_{\tilde{A}_n(k)}\frac{|1-|\ell||d\ell}{1+|k|^8|1-|\ell||^4}, \tag{3.16}$$

where $\tilde{A}_n(k) := [n|k|^{-1} \le |\hat{k} - \ell| \le (n+1)|k|^{-1}]$ and $\hat{k} := k|k|^{-1}$. Note that $D_n(k) \supset \tilde{A}_n(k)$ for $|k| \ge C_1(C,\rho) \vee 1$, where $C_1(C,\rho)$ is a certain constant depending only on C, ρ and

$$D_n(k) := [\ell : |\hat{\ell} - \hat{k}| \le (n+1)|k|^{-1},\ |\ell| \in ([1-(n+1)|k|^{-1}] \vee 0, 1+(n+1)|k|^{-1})].$$

The expression in (3.16) can be estimated by

$$\begin{aligned}
&\frac{C(|k|+n)^2|k|^{d+1}}{\langle n\rangle^M}\int_{D_n(k)}\frac{|1-|\ell||d\ell}{1+|k|^8|1-|\ell||^4}\\
&\le \frac{C_2(|k|+n)^2|k|^{d+1}|k|^{-d+1}n^{d-1}}{\langle n\rangle^M}\int_0^{1+(n+1)|k|^{-1}}\frac{x^{d-1}|1-x|dx}{1+|k|^8|1-x|^4}\\
&\le \frac{C_3(|k|+n)^2|k|^2(1+n|k|^{-1})^{d-1}}{\langle n\rangle^{M-d+1}}\int_0^{1+(n+1)|k|^{-1}}\frac{|1-x|dx}{1+|k|^8|1-x|^4}. \qquad (3.17)
\end{aligned}$$

After the change of variables $x' := |k|^2 x$ the utmost right hand side of (3.17) can be estimated by

$$\frac{C_3(1+n|k|^{-1})^{d+1}}{\langle n\rangle^{M-d+1}}\int_0^{+\infty}\frac{|1-x|dx}{1+|k|^8|1-x|^4}. \qquad (3.18)$$

Therefore we can estimate

$$\sup_{|k|\ge 1}\int |l|\left|\partial_\omega \hat{R}\left(\frac{|l|^2-|k|^2}{2}, l-k\right)\right| dl \le \sup_{|k|\ge 1}\sum_{n\ge 0}\frac{C_4(1+n|k|^{-1})^{d+1}}{\langle n\rangle^{M-d+1}} \le \sum_{n\ge 0}\frac{C_4}{\langle n\rangle^{M-2d}} < +\infty,$$

provided $M > 2d$. $\square$

Proof of Lemma 3.2. Suppose first that $i \ne n$. Then,

$$\begin{aligned}
&\nabla_{k_i} G_n(k_1,\ldots,k_n;k) = -\nabla\Sigma(k_i)(\tau_{i+1}-\tau_i)G_n(k_1,\ldots,k_n;k)\\
&+g(k_n)\prod_{j=1}^{n-1}\exp\{-\Sigma(k_j)(\tau_{j+1}-\tau_j)\}\prod_{j\ne i,i+1}\hat{R}\left(\frac{|k_j|^2-|k_{j-1}|^2}{2}, k_j-k_{j-1}\right)\\
&\times\left\{\sum_{j=0}^{1}(-1)^{j-1}\left[\left(k_{i+j}\partial_\omega\hat{R}+\nabla\hat{R}\right)\left(\frac{|k_{i+j}|^2-|k_{i+j-1}|^2}{2}, k_{i+j}-k_{i+j-1}\right)\right]\right.\\
&\left.\times\hat{R}\left(\frac{|k_{i+1-j}|^2-|k_{i-j}|^2}{2}, k_{i+1-j}-k_{i-j}\right)\right\}.
\end{aligned}$$

We obtain therefore

$$\int |\nabla_{k_i} G_n(k_1,\ldots,k_n;k)|d\mathbf{k}_{1,n} \le \mathcal{R}_*\mathcal{G}_1 t + \|g\|_\infty \mathcal{R}_1^{n-1}\mathcal{R}_*.$$

On the other hand when $i = n$ we get

$$\begin{aligned}
&\nabla_{k_n} G_n(k_1,\ldots,k_n;k) = \nabla_{k_n} g(k_n)\prod_{i=1}^{n-1}\exp\{-\Sigma(k_i)(\tau_{i+1}-\tau_i)\}\prod_{i=1}^{n}\hat{R}\left(\frac{|k_i|^2-|k_{i-1}|^2}{2}, k_i-k_{i-1}\right)\\
&+g(k_n)\prod_{j=1}^{n-1}\left[\exp\{-\Sigma(k_j)(\tau_{j+1}-\tau_j)\}\hat{R}\left(\frac{|k_j|^2-|k_{j-1}|^2}{2}, k_j-k_{j-1}\right)\right]\\
&\times\left[\left(k_n\partial_\omega\hat{R}+\nabla\hat{R}\right)\left(\frac{|k_n|^2-|k_{n-1}|^2}{2}, k_n-k_{n-1}\right)\right].
\end{aligned}$$

Note that

$$\nabla_{k_n} g(k_n) := -\Sigma^{-1}(k_n)\nabla_{k_n}\Sigma(k_n) f(k_n)\exp\left\{-\Sigma(k_n)(\tau_{n+1}-\tau_n)\right\}\left[\Sigma^{-1}(k_n)+\tau_{n+1}-\tau_n\right]$$
$$+\Sigma^{-1}(k_n)\nabla_{k_n} f(k_n)\exp\left\{-\Sigma(k_n)(\tau_{n+1}-\tau_n)\right\}.$$

and

$$\|\nabla_{k_n} g\|_\infty \le (\mathcal{R}_* + 1)\sup_{k_n}\left\{\Sigma^{-1}(k_n)|f(k_n)|[\Sigma^{-1}(k_n)+t] + \Sigma^{-1}(k_n)|\nabla_{k_n} f(k_n)|\right\} < +\infty.$$

Hence,

$$\int |\nabla_{k_n} G_n(k_1,\ldots,k_n;k)| d\mathbf{k}_{1,n} \le \|\nabla_{k_n} g\|_\infty \mathcal{R}_1^n + \|g\|_\infty \mathcal{R}_1^{n-1}\mathcal{R}_*$$

and the conclusion of the lemma follows. □

4 Elimination of the non-martingale forcing

Proposition 3.1 shows that the weak limits of Z_ε and $\tilde{Z}_\varepsilon$ are the same. We will now further write $\tilde{Z}_\varepsilon = U_\varepsilon + Z_\varepsilon^o$, where U_ε satisfies (3.1) without the martingale term $d\mathcal{M}_\varepsilon$ and the same initial data as $\tilde{Z}_\varepsilon$, while Z_ε^o satisfies (3.1) without the terms involving $\mathcal{W}_i$ (but with $d\mathcal{M}_\varepsilon$), and with zero initial data. It will turn out that the weak limit of U_ε vanishes, while Z_ε^o converges to $\bar{Z}$.

More precisely, let $\{U_\varepsilon(t),\, t\ge 0\}$ be the solution of the equation

$$\partial_t U_\varepsilon(t) = AU_\varepsilon(t) + \varepsilon^{-1/2}\mathcal{K}_\varepsilon[U_\varepsilon(t);V_{t/\varepsilon}] + \sum_{i=0}^{1}\varepsilon^{-i/2}\tilde{\mathcal{W}}_i^\varepsilon(t), \tag{4.1}$$
$$U_\varepsilon(0,x,k) = -\Lambda_\varepsilon(0).$$

Equation (4.1) can be rewritten in the mild form.

$$U_\varepsilon(t) = \sum_{i=0}^{2}\mathcal{G}_\varepsilon^{(i)}(t) + \varepsilon^{-1/2}\int_0^t S_0(t-s)\mathcal{K}[U_\varepsilon(s);V_{t/\varepsilon}]ds, \tag{4.2}$$

where

$$\mathcal{G}_\varepsilon^{(0)}(t) := -S_0(t)\Lambda_\varepsilon(0), \tag{4.3}$$
$$\mathcal{G}_\varepsilon^{(1)}(t) := \int_0^t S_0(t-s)\tilde{\mathcal{W}}_0^\varepsilon(s)ds,$$
$$\mathcal{G}_\varepsilon^{(2)}(t) := \varepsilon^{-1/2}\int_0^t S_0(t-s)\tilde{\mathcal{W}}_1^\varepsilon(s)ds.$$

Here $\tilde{\mathcal{W}}_i^\varepsilon(s)$ and $\Lambda_\varepsilon(t)$ are given by (3.2) and (3.5), respectively.

Let $\hat{U}_\varepsilon(t,q,k) = \mathcal{F}_1(U_\varepsilon(t))(q,k)$. Performing the Fourier transform in the x variable and writing the Duhamel series as in Section 2.4 we obtain

$$\hat{U}_\varepsilon(t,q,k) = \sum_{i=0}^{2}\sum_{n\ge 0}\widehat{\mathcal{G}}_\varepsilon^{(i,n)}(t,q,k), \tag{4.4}$$

where $\widehat{\mathcal{G}}^{(i,0)}_\varepsilon(t)=\widehat{\mathcal{G}}^{(i)}_\varepsilon(t):=\mathcal{F}_1(\mathcal{G}^{(i)}_\varepsilon(t))$ and

$$\widehat{\mathcal{G}}^{(i,n)}_\varepsilon(t,q,k):=(\varepsilon^{-1/2}i)^n\sum_{\sigma_1,\ldots,\sigma_n=\pm1}\sigma_1\ldots\sigma_n\int_{\Delta_n(t)}\int\exp\left\{i\sum_{j=1}^nQ_j\cdot K_j(s_{j-1}-s_j)\right\} \tag{4.5}$$
$$\times\prod_{j=1}^n\hat V\left(\frac{s_j}{\varepsilon},dp_j\right)\widehat{\mathcal{G}}^{(i)}_\varepsilon\left(s_n,Q_n,K_n\right)d\mathbf{s}^{(n)}d\mathbf{p}^{(n)},$$

with $i=0,1,2,$ and $n\ge1$. Here $K_n,Q_n,\mathbf{s}^{(n)},\mathbf{p}^{(n)}$ are given by (2.39). The main result of this section concerns the behavior of $U_\varepsilon(t)$, as $\varepsilon\downarrow0$.

Theorem 4.1 *For any $t>0$ and $\theta\in\mathcal{S}(\mathbb{R}^{2d})$ we have*

$$\lim_{\varepsilon\to0+}\mathbb{E}\left[\langle U_\varepsilon(t),\theta\rangle^2\right]=0.$$

Proof of Theorem 4.1

Let

$$G^{(i,n_1,n_2)}_\varepsilon(t,\theta):=\mathbb{E}\left[\langle\mathcal{G}^{(i,n_1)}_\varepsilon(t),\theta\rangle\langle\mathcal{G}^{(i,n_2)}_\varepsilon(t),\theta\rangle\right].$$

From (4.4) we obtain that

$$\mathbb{E}\left[\langle U_\varepsilon(t),\theta\rangle\right]^2\le3\sum_{i=0}^2\sum_{n_1,n_2\ge0}G^{(i,n_1,n_2)}_\varepsilon(t,\theta). \tag{4.6}$$

The theorem in question is a simple conclusion of the following.

Proposition 4.2 *There exist constants $G^{(i)}_{n_1,n_2}(T)$ such that*

$$\sup_{t\in[0,T]}|G^{(i,n_1,n_2)}_\varepsilon(t,\theta)|\le\varepsilon G^{(i)}_{n_1,n_2}(T)\|\theta\|^2_{1,1},\quad\forall\,\varepsilon\in(0,1] \tag{4.7}$$

and

$$\sum_{n_1,n_2=0}^{+\infty}G^{(i)}_{n_1,n_2}(T)<+\infty.$$

for $i=0,1,2$.

Proof. We consider only the case when $i=2$. The other cases, i.e. $i=0,1$, can be deals similarly (in fact they are simpler). Observe that then estimate (4.7) needs to be checked only for $n=n_1+n_2$ even (otherwise its left hand side vanishes). Using (4.5) (for $i=2$) and (2.53) we conclude that the left hand side of (4.7) can be estimated by

$$C^n\varepsilon^{-n/2-1}\|\theta\|^2_{1,1}W^2_{*,1} \tag{4.8}$$
$$\times\int\int_{D^t_{n_1+1,n_2+1}}d\mathbf{s}_1d\mathbf{s}_2\int\left|\mathbb{E}\left\{\prod_{i=1}^2\left[\prod_{j=1}^{n_i}\hat V(\frac{s_{ij}}{\varepsilon},dp_{ij})\right]\hat V_2(\frac{s_{in_i+1}}{\varepsilon},dp_{in_i+1},dp_{in_i+2})\right\}\right|$$

Here $D^t_{n,k}:=\Delta_n(t)\times\Delta_k(t)$,

$$\hat V_2(t,dp,dq)=\hat V(t,dp)\hat V(t,dq)-\hat R(p)\delta(p+q)dpdq \tag{4.9}$$

We shall also denote $D^{t,s}_{n,k} := \Delta_n(t,s) \times \Delta_k(t,s)$.

Symmetry consideration concerning the first n_i variables s_{ij} for $i = 1, 2$ respectively allow us to rewrite the right hand side of (4.8) as being equal to

$$\frac{C^n \|\theta\|^2_{1,1} W^2_{*,1}}{n_1! n_2! \varepsilon^{n/2+1}} \int_0^t \int_0^t ds_{1,n_1+1} ds_{2,n_2+1} \int_{s_{1,n_1+1}}^t \int_{s_{2,n_2+1}}^t d\mathbf{s}_1 d\mathbf{s}_2 \tag{4.10}$$
$$\times \int \left| \mathbb{E}\left\{ \prod_{i=1}^2 \left[\prod_{j=1}^{n_i} \hat{V}(\frac{s_{ij}}{\varepsilon}, dp_{ij}) \right] \hat{V}_2(\frac{s_{in_i+1}}{\varepsilon}, dp_{in_i+1}, dp_{in_i+2}) \right\} \right|,$$

where $\mathbf{s}_i = (s_{i1}, \ldots, s_{in_i})$. Using the rules of computing joint moments of mean zero Gaussian random variables we conclude that the right hand side of (4.8) equals

$$\frac{C^n \|\theta\|^2_{1,1} W^2_{*,1}}{n_1! n_2! \varepsilon^{n/2+1}} \Bigg| \sum_{\mathcal{F}} \int_0^t \int_0^t ds_{1,n_1+1} ds_{2,n_2+1} \int_{s_{1,n_1+1}}^t \int_{s_{2,n_2+1}}^t d\mathbf{s}_1 d\mathbf{s}_2 \int \tag{4.11}$$
$$\times \prod_{(jk;lm)\in\mathcal{F}} \mathbb{E}\left[\hat{V}(\frac{s_{jk}}{\varepsilon}, dp_{jk}) \hat{V}(\frac{s_{lm}}{\varepsilon}, dp_{lm}) \right] d\mathbf{s}_1 d\mathbf{s}_2 \Bigg|,$$

where the summation extends over pairings formed over the pairs $((i_1 j_1); (i_2, j_2))$, $(i_k, j_k) \in \mathcal{V} := \{(1,1), \ldots, (1, n_1+1), (2,1), \ldots, (2, n_2+1)\}$ that contain at least one bond of the form $((i,j), (1, n_1+1))$ (then it has to contain a bond $((i,j), (2, n_2+1))$), for some $(i,j) \in \mathcal{V} \setminus \{(1, n_1+1), (2, n_2+1)\}$ (then it has to contain a bond $((i',j'), (2, n_2+1))$ for some $(i',j') \in \mathcal{V} \setminus \{(1, n_1+1), (2, n_2+1)\}$). Applying the relation

$$\mathbb{E}\left[\hat{V}(t, dp) \hat{V}(s, dq) \right] = (2\pi)^d e^{-\gamma(p)|t-s|} \delta(p+q) \hat{R}(p) dp dq, \tag{4.12}$$

we can estimate the expression in (4.11) by

$$\frac{C^n \|\theta\|^2_{1,1} W^2_{*,1}}{n_1! n_2! \varepsilon^{n/2+1}} \sum_{\mathcal{F}} \int_0^t \cdots \int_0^t \int \prod_{(jk;lm)\in\mathcal{F}} e^{-\gamma(p_{jk})|s_{jk}-s_{lm}|/\varepsilon} \delta(p_{jk}+p_{lm}) \hat{R}(p_{jk}) d\mathbf{s}_1 d\mathbf{s}_2 d\mathbf{p}_1 d\mathbf{p}_2, \tag{4.13}$$

Here $\mathbf{p}_j = (p_{j1}, \ldots, p_{jn_j})$ and the range of summation extends over all pairings between elements of $\mathcal{V}$.

Changing variables $s'_{jk} := s_{jk}/\varepsilon$ we obtain that expression (4.13) equals

$$\frac{C^n \|\theta\|^2_{1,1} W^2_{*,1} \varepsilon}{n_1! n_2!} \sum_{\mathcal{F}} \int \prod_{(k,l)\in\mathcal{F}} \left[\varepsilon \int_0^{t/\varepsilon} \int_0^{t/\varepsilon} e^{-\gamma(p_{jk})|s_{jk}-s_{lm}|} ds_{jk} ds_{lm} \right] \tag{4.14}$$
$$\times \delta(p_{jk}+p_{lm}) \hat{R}(p_{jk}) d\mathbf{p}_1 d\mathbf{p}_2$$
$$\le \frac{C^n t^{n/2+1} \|\theta\|^2_{1,1} W^2_{*,1} \varepsilon}{n_1! n_2!} \sum_{\mathcal{F}} \int \prod_{(k,l)\in\mathcal{F}} \delta(p_{jk}+p_{lm}) \frac{\hat{R}(p_{jk})}{\gamma(p_{jk})} d\mathbf{p}_1 d\mathbf{p}_2$$
$$= \frac{C^n t^{n/2} \|\theta\|^2_{1,1} W^2_{*,1} \varepsilon (n+1)!!}{n_1! n_2!} \left[\int \frac{\hat{R}(p)}{\gamma(p)} dp \right]^{n/2+1}.$$

In the last step above we used the fact that the total number of pairings for a set of $n+2 = n_1+n_2+2$ elements equals $(n+1)!!$. $\square$

5 The term with the martingale forcing

We have now got rid of the largest apparent order term in (2.57) with the help of the corrector, as well as of the non-martingale forcing terms that arose after the addition of the corrector. Therefore, the problem is now reduced to the Wigner equation with a martingale forcing, see (5.1) below. Our next task is to replace the Wigner equation with the kinetic equation with the same martingale forcing, and we formulate that result in this section.

The stochastic equation with the martingale forcing

Let us define $Z^o_\varepsilon(t) := \tilde{Z}_\varepsilon(t) - U_\varepsilon(t)$. It satisfies the stochastic equation

$$\begin{aligned} &dZ^o_\varepsilon(t) = \left\{AZ^o_\varepsilon(t) + \varepsilon^{-1/2}\mathcal{K}_\varepsilon[Z^o_\varepsilon(t); V_{t/\varepsilon}]\right\} dt + d\mathcal{M}_\varepsilon(t), \\ &Z^o_\varepsilon(0) = 0, \end{aligned} \tag{5.1}$$

where the additive noise $\{\mathcal{M}_\varepsilon(t),\, t \ge 0\}$ is given by (3.6). We can perform the Fourier transform in the first variable on both sides of (5.1) and obtain, as in Section 4, that $\hat{Z}^o_\varepsilon(t,q,k) = \mathcal{F}_1(Z^o_\varepsilon(t))(q,k)$ is given by

$$\hat{Z}^o_\varepsilon(t,q,k) = \sum_{n\ge 0} \hat{Z}_{n,\varepsilon}(t,q,k). \tag{5.2}$$

Here,

$$\begin{aligned} &\hat{Z}_{0,\varepsilon}(t,q,k) := \int_0^t \mathrm{e}^{iq\cdot k(t-s)} d\widehat{\mathcal{M}}_\varepsilon(s,q,k), \\ &\widehat{Z}_{n,\varepsilon}(t,q,k) = (\varepsilon^{-1/2}i)^n \int_0^t \int \widehat{\mathcal{V}}_n(t,s_{n+1},q,k,d\mathbf{p}^{(n)}) d\widehat{\mathcal{M}}_\varepsilon(s_{n+1},Q_n,K_n), \end{aligned} \tag{5.3}$$

for $n \ge 1$. Here,

$$\widehat{\mathcal{V}}_n(t,s,q,k,d\mathbf{p}^{(n)}) := \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \sigma_1 \ldots \sigma_n \int\limits_{\Delta_n(t,s)} \exp\left\{i\sum_{j=0}^n Q_j \cdot K_j(s_j - s_{j+1})\right\} \prod_{j=1}^n \hat{V}\left(\frac{s_j}{\varepsilon}, dp_j\right) d\mathbf{s}^{(n)},$$

where $\Delta_n(t,s) := [t \ge s_1 \ge \ldots \ge s_n \ge s]$, $s_0 := t$, $s_{n+1} = s$ and $\widehat{\mathcal{M}}_\varepsilon(t,q,k)$ is a Gaussian maringale given by

$$\widehat{\mathcal{M}}_\varepsilon(t,q,k) := \mathcal{F}_1(\mathcal{M}_\varepsilon(t))(q,k) = i\sum_{\sigma=\pm 1} \sigma \int_0^{t/\varepsilon} \int [-\gamma(p) + ip\cdot k]^{-1} \hat{W}\left(\varepsilon s, q - \frac{p}{\varepsilon}, K_\sigma\right) \hat{B}(ds,dp) \tag{5.4}$$

and $\hat{W}(t,q,k) := \mathcal{F}_1(\bar{W})(t,q,k)$. The next proposition says that, for a fixed $\varepsilon > 0$, we can, indeed, represent the solution of (5.1) as a convergent series (5.2).

Proposition 5.1 *For any $s \in \mathbb{R}$, $p \ge 1$ the series in* (5.2) *is convergent in the $L^p(\mathbb{P})$ sense in $H^{s,0}$. In addition, we have $\hat{Z}^o_\varepsilon(t,q,k) = \mathcal{F}_1(Z^o_\varepsilon(t))(q,k)$.*

The proof is standard and is, therefore, omitted. The simple reason for convergence is that integration over the simplex $\Delta_n(t)$ provides sufficient decay for the terms of the series.

The kinetic equation with the martingale forcing

We shall define $\{\bar{Z}_\varepsilon(t),\, t\ge 0\}$ as the Duhamel solution of

$$\begin{aligned} &d\bar{Z}_\varepsilon(t)=\left\{A\bar{Z}_\varepsilon(t)+\mathcal{L}\bar{Z}_\varepsilon(t)\right\}dt+d\mathcal{M}_\varepsilon(t),\\ &\bar{Z}_\varepsilon(0)=0. \end{aligned} \tag{5.5}$$

Using (2.52) we can write

$$\hat{Z}_\varepsilon(t,q,k):=\mathcal{F}_1(\bar{Z}_\varepsilon(t))(q,k)=\sum_{n\ge 0}\bar{Z}_{n,\varepsilon}(t,q,k). \tag{5.6}$$

Here,

$$\begin{aligned} \bar{Z}_{0,\varepsilon}(t,q,k)&:=\int_0^t \mathrm{e}^{iq\cdot k(t-s)}d\widehat{\mathcal{M}}_\varepsilon(s,q,k),\\ \bar{Z}_{n,\varepsilon}(t,q,k)&=\int_0^t\int \bar{\mathcal{V}}_n(t,s,q,\mathbf{k}_{0,n})d\widehat{\mathcal{M}}_\varepsilon(s_{n+1},q,k_n)d\mathbf{k}_{1,n}, \end{aligned} \tag{5.7}$$

for $n\ge 1$ and

$$\begin{aligned} \bar{\mathcal{V}}_n(t,s,q,k_{0,n})&:=\int_{\Delta_n(t,s)}\exp\left\{i\sum_{j=0}^n q\cdot k_j(s_j-s_{j+1})\right\}\\ &\times\prod_{i=1}^n\left[\sigma(k_{i-1},k_i)\Sigma(k_{i-1})\right]\exp\left\{-\sum_{i=0}^n\Sigma(k_i)(s_j-s_{j+1})\right\}d\mathbf{s}^{(n)}d\mathbf{k}_{1,n}. \end{aligned}$$

Here $\mathbf{k}_{j,n}=(k_j,\dots,k_n)$ and $d\mathbf{k}_{j,n}=dk_j\dots dk_n$ for any $j\le n$.

The next theorem shows that solutions of (5.1) an (5.5) are asymptotically close to each other. This is the most difficult step in the proof of Theorem 2.8.

Theorem 5.2 *Suppose that $\{Z^o_\varepsilon(t),\, t\ge 0\}$ and $\{\bar{Z}_\varepsilon(t),\, t\ge 0\}$ are the solutions of (5.1) and (5.5) respectively. Then,*

$$\lim_{\varepsilon\to 0+}\mathbb{E}\left[\langle Z^o_\varepsilon(t),\theta\rangle-\langle\bar{Z}_\varepsilon(t),\theta\rangle\right]^2=0$$

for any $t>0$ and $\theta\in\mathcal{S}(\mathbb{R}^{2d})$.

6 The proof of Theorem 5.2

6.1 Some preliminary results and terminology

We start with the following.

Proposition 6.1 *For any $\varepsilon>0$ we have*

$$\mathbb{E}\left[\langle Z^o_\varepsilon(t),\theta\rangle-\langle\bar{Z}_\varepsilon(t),\theta\rangle\right]^2=\sum_{n\ge 1}\mathcal{A}_n(\varepsilon)+\sum_{n\ge 1}\mathcal{B}_n(\varepsilon), \tag{6.1}$$

where

$$\mathcal{A}_n(\varepsilon):=\mathbb{E}\langle\hat{Z}^o_{2n,\varepsilon}(t),\hat{\theta}\rangle^2-\mathbb{E}\langle\bar{Z}_{n,\varepsilon}(t),\hat{\theta}\rangle^2$$

and

$$\mathcal{B}_n(\varepsilon):=-2\left\{\mathbb{E}\left[\left(\langle\hat{Z}^o_{2n,\varepsilon}(t),\hat{\theta}\rangle-\langle\bar{Z}_{n,\varepsilon}(t),\hat{\theta}\rangle\right)\langle\bar{Z}_{n,\varepsilon}(t),\hat{\theta}\rangle\right]\right\}.$$

Once again, for $\varepsilon > 0$ this propostion is quite standard so we do not present it here.

A direct calculation shows that

$$\mathbb{E}\langle \hat{Z}^o_{n,\varepsilon}(t), \hat{\theta}\rangle^2 = \frac{(-1)^{n+1}}{\varepsilon^{n+1}} \sum_{\sigma_1,\sigma_2,\rho} \sum_{\mathcal{F}} \sigma_1\sigma_2 \int_0^t d\tau \int \mathcal{V}(\sigma, \mathcal{F}, \mathbf{p}_1^{(n)}, \mathbf{p}_2^{(n)}; t, \tau) \tag{6.2}$$
$$\times \hat{\theta}^*(q_1, k_1)\hat{\theta}^*(q_2, k_2)\Gamma(p, K_n^{(1)}, K_n^{(2)})\hat{W}\left(\tau, Q_n^{(1)} - \frac{p}{\varepsilon}, K_{n,\sigma_1}^{(1)}\right) \hat{W}\left(\tau, Q_n^{(2)} - \frac{p}{\varepsilon}, K_{n,\sigma_2}^{(2)}\right) d\mathbf{p}dkdq$$

cf. formula (3.8) for the definition of $\Gamma(\cdot)$. Here $\mathbf{p}_i^{(n)} := (p_{i1}, \ldots, p_{in})$, $d\mathbf{p}dkdq$ is an abbreviation for the volume element $\nu(dp)dq_1dq_2dk_1dk_2d\mathbf{p}_1^{(n)}d\mathbf{p}_2^{(n)}$, and we set

$$Q_n^{(i)} := q_i - \frac{1}{\varepsilon}\sum_{m=1}^n p_{i,m}, \quad K_n^{(i)} := k_i + \frac{1}{2}\sum_{m=1}^n \sigma_{im}p_{im}, \quad K_{n,\sigma}^{(i)} := K_n^{(i)} + \frac{\sigma}{2}p \quad \text{for } i = 1, 2, \tag{6.3}$$

and

$$\mathcal{V}(\rho, \mathcal{F}, \mathbf{p}_1^{(n)}, \mathbf{p}_2^{(n)}; t, \tau) := \bar{\sigma} \int_{D_n(t,\tau)} \exp\left\{ i \sum_{ij} Q_{ij} \cdot K_{ij}(s_{ij} - s_{ij+1}) \right\}$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \left[e^{-\gamma(p_{jk})|s_{jk} - s_{j'm}|/\varepsilon} \hat{R}(p_{jk})\delta(p_{jk} + p_{j'm}) \right] d\mathbf{s}_1^{(n)} d\mathbf{s}_2^{(n)}. \tag{6.4}$$

The first summation in (6.2) extends over $\sigma_1, \sigma_2, \sigma_{ij} = \pm 1$, $\bar{\sigma} := \prod_{(ij)} \sigma_{ij}$, $D_n(t,\tau) := \Delta_n(t,\tau) \times \Delta_n(t,\tau)$, and

$$\mathbf{p}_i^{(n)} = (p_{i1}, \ldots, p_{in}), \quad Q_{ij} := q_i - \frac{1}{\varepsilon}\sum_{m=1}^j p_{im},$$
$$K_{ij} := k_i + \frac{1}{2}\sum_{m=1}^j \sigma_{im}p_{im} \quad \text{for } i = 1, 2.$$

The second summation there extends over all pairings formed over pairs of integers (ij), with $i = 1, 2$, and $j = 1, \ldots, n$. The pairs are ordered lexicographically, that is, we say that $(ij) \prec (i'j')$ if $i < i'$, or if $i = i'$ then $j \le j'$. If (e, f) is an edge we say that e, f are left and right vertices respectively if $e \prec f$. Also, given a vertex $e = (ij)$ we will use the notation $s(e) = s_{ij}$, $p(e) = p_{ij}$. We say that an edge $v = (e, f)$ *straddles over* $v' = (e', f')$ if $v \ne v'$ and $e \prec e' \prec f' \prec f$. Edges $v = (e, f)$ and v' are said to *intersect* each other if they are different, not straddled by each other and one of the vertices, say e', satisfies $e \prec e' \prec f$. *A mixed edge* is of the form $((1j_1), (2j_2))$.

For a given pairing $\mathcal{F}$ let $h_1(\cdot; \mathcal{F})$ be a function defined over its edges assigning to each $v \in \mathcal{F}$ its vertex in such a way that the number of v, for which $h_1(v; \mathcal{F}) = (1j)$ equals $[n/2]$. Let $h_2(v; \mathcal{F})$ be the other edge of v. We shall omit writing $\mathcal{F}$ in the notation of these functions if it is obvious from the context.

A pairing is called *time-ordered* if all edges are of the form $((i; 2j-1), (i; 2j))$ for some $i = 1, 2$ and $j = 1, \ldots, n$. A pairing $\mathcal{F}$ is said to be *negligible* if it does not contain mixed edges and belongs to either of three classes of pairings: 1) $\mathcal{E}_1$ consisting of pairings containing an edge $((ij), (ij'))$ such that $|j' - j| \ge 4$, 2) $\mathcal{E}_2$ pairings with at least two edges $((i_kj_k), (i_kj_k')) \in \mathcal{F}$, $k = 1, 2$ with $|j_1' - j_1| \ge 3$ and $|j_2' - j_2| \ge 2$, or 3) $\mathcal{E}_3$ pairings with at least three edges $((ij), (ij')) \in \mathcal{F}$ such that $|j' - j| \ge 2$. *An almost time-ordered pairing* is defined as a pairing that contains no mixed edges and is neither ladder, nor negligible. Observe that the pairings considered in this case can be divided into two classes: 1) $\mathcal{E}_4$ - those that contain an edge of the form $((i,j),(i,j+3))$ and all other edges are of the form $((i,j),(i,j+1))$, 2) $\mathcal{E}_5$ - containing edges $((i,j),(i,j+2))$, $((i,j+1),(i,j+3))$ and all other edges are of the form $((i,j),(i,j+1))$. By $\mathcal{E}_6$ we denote the class of pairings containing mixed edges. Such pairings are called *mixed*.

Similarly, with $\bar{K}_{n,\sigma} := k_n + \sigma p/2$, we can write

$$\mathbb{E}\langle \bar{Z}_{n,\varepsilon}(t), \hat{\theta}\rangle^2 = -\frac{1}{\varepsilon}\sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^t d\tau \int\int \bar{\mathcal{V}}_n(t,\tau,q,\mathbf{k}_{0,n})\bar{\mathcal{V}}_n(t,\tau,q',\mathbf{k}'_{0,n})\hat{\theta}^*(q,k)\hat{\theta}^*(q',k')$$
$$\times\Gamma(p,k,k')\hat{W}\left(\tau, q-\frac{p}{\varepsilon}, \bar{K}_{n,\sigma}\right)\hat{W}\left(\tau, q'-\frac{p}{\varepsilon}, \bar{K}'_{n,\sigma'}\right) d\mathbf{p}dqdk, \tag{6.5}$$

with $d\mathbf{p}dqdk := \nu(dp)dqdq'd\mathbf{k}_{0,n}d\mathbf{k}'_{0,n}$ and

$$\mathbb{E}\left[\langle \hat{Z}^o_{2n,\varepsilon}(t), \hat{\theta}\rangle\langle \bar{Z}_{n,\varepsilon}(t), \hat{\theta}\rangle\right] = \frac{(-1)^{n+1}}{\varepsilon^{1+n}}\sum_{\sigma,\sigma',\rho=\pm 1}\sum_{\mathcal{F}} \sigma\sigma' \int_0^t d\tau \int\int \mathcal{V}(\rho,\mathcal{F},\mathbf{p}^{(n)};t,\tau)\bar{\mathcal{V}}_n(t,\tau,q',\mathbf{k}'_{0,n})$$
$$\times\hat{\theta}^*(q,k)\hat{\theta}^*(q',k')\Gamma(p,K_{2n},k')\hat{W}\left(\tau, Q_{2n}-\frac{p}{\varepsilon}, K_{2n,\sigma}\right)\hat{W}\left(\tau, q'-\frac{p}{\varepsilon}, \bar{K}'_{n,\sigma'}\right) d\mathbf{p}dqdk. \tag{6.6}$$

The second summation on the right hand side extends over all pairings formed over vertices $\{1,\ldots,2n\}$,

$$\mathcal{V}(\sigma,\mathcal{F},\mathbf{p}^{(n)};t,\tau) := \bar{\sigma}\int\limits_{\Delta_{2n}(t,\tau)} \exp\left\{i\sum_{j=1}^{2n} Q_j\cdot K_j(s_j - s_{j+1})\right\}\prod_{(j,j')\in\mathcal{F}}\left[e^{-\gamma(p_j)|s_j-s_{j'}|/\varepsilon}\hat{R}(p_j)\delta(p_j+p_{j'})\right]d\mathbf{s}^{(n)}.$$

6.2 Estimates of $\mathcal{A}_n(\varepsilon)$

We define $\mathcal{A}_n^{(1)}(\varepsilon)$, $\mathcal{A}_n^{(2)}(\varepsilon)$, $\mathcal{A}_n^{(4)}(\varepsilon)$ by expressions analogous to (6.2) except for the fact that the summation extends only over negligible, almost time-ordered and mixed pairings correspondingly. By $\tilde{\mathcal{A}}_{2n}^{(3)}(\varepsilon)$ we denote the respective expression corresponding to the time-ordered pairing. We let

$$\mathcal{A}_n^{(3)}(\varepsilon) := \tilde{\mathcal{A}}_{2n}^{(3)}(\varepsilon) - \mathbb{E}\langle \bar{Z}_{n,\varepsilon}(t), \hat{\theta}\rangle^2.$$

The following result holds.

Proposition 6.2 *There exist constants $C_1, C_2 > 0$ and $\kappa\in(0,1)$ such that*

$$|\mathcal{A}_n^{(1)}(\varepsilon)| \le \frac{C_1^n}{n!}\varepsilon + (C_2\varepsilon^\kappa)^n, \tag{6.7}$$

$$|\mathcal{A}_n^{(3)}(\varepsilon)| \le \frac{C_1^n}{n!}\varepsilon^\kappa, \tag{6.8}$$

$$|\mathcal{A}_n^{(i)}(\varepsilon)| \le \frac{C_1^n}{n!}, \tag{6.9}$$

for all $n\ge 1$, $\varepsilon > 0$ and $i = 2, 4$.

The proof of (6.7)**.** Observe that

$$\mathcal{A}_n^{(1)}(\varepsilon) \le \frac{C^n}{\varepsilon^{n+1}}\sum_{i=1}^{3}\sum_{\mathcal{F}\in\mathcal{E}_i}\int_{D_n(t,0)}\prod_{(e,f)\in\mathcal{F}} \mathrm{e}^{-\gamma_*|s_f-s_e|/\varepsilon}d\mathbf{s}_1^{(n)}d\mathbf{s}_2^{(n)}, \tag{6.10}$$

where the constant $C > 0$ depends on θ, W, γ_* and the measure μ. Suppose that n is even, consideration for n odd is almost identical. Fix a negligible pairing $\mathcal{F} \in \mathcal{E}_1$. The term corresponding to $\mathcal{E}_1$ in (6.10) can be estimated by

$$\frac{C^{n+1}}{\varepsilon^{n+1}} \sum_{i=1,2} \sum_{|j-j'|\geq 4} \int_{D_n(t,0)} \mathrm{e}^{-\gamma_*|s_{ij}-s_{ij'}|/\varepsilon} \mathbb{E}\left[\prod_{e\neq (ij),(ij')} w\left(\frac{s_e}{\varepsilon}\right)\right] d\mathbf{s}_1^{(n)} d\mathbf{s}_2^{(n)}, \tag{6.11}$$

where $C > 0$ is some constant indpendent of n and $\varepsilon > 0$, $\{w(s),\, s \geq 0\}$ is a stationary, one dimensional, linear diffusion described by

$$dw(s) = -\gamma_* w(s)ds + \sqrt{2\gamma_*}dB(s)$$

and $\{B(s),\, s \geq 0\}$ is a one dimensional, standard Brownian motion. Let us choose an arbitrary $\kappa \in (0,1)$ and denote $v_0 := ((ij),(ij'))$. Divide the domain of integration into two sets D_1 and D_2 depending on whether $|s_{ij} - s_{ij'}| \geq \varepsilon^\kappa$, or not. The expression in (6.11) can be written as $I_1 + I_2$ corresponding to each domain of integrtion. We have then

$$I_1 \leq \frac{C^{n+1}}{(n!)^2\varepsilon^{n+1}} \sum_{i=1,2} \sum_{|j-j'|\geq 4} \underbrace{\int_0^t \cdots \int_0^t}_{2n-\text{times}} e^{-\gamma_*|s_{ij}-s_{ij'}|/\varepsilon} \mathbb{E}\left[\prod_{e\neq (ij),(ij')} w\left(\frac{s_e}{\varepsilon}\right)\right] d\mathbf{s}_1^{(n)} d\mathbf{s}_2^{(n)} \tag{6.12}$$

$$= \frac{C^{n+1}}{(n!)^2\varepsilon^{n+1}} \sum_{i=1,2} \sum_{|j-j'|\geq 4} \sum_{\mathcal{F}} \underbrace{\int_0^t \cdots \int_0^t}_{n-\text{times}} e^{-\gamma_*|s_{ij}-s_{ij'}|/\varepsilon} \mathbb{E}\left[\prod_{e\neq (ij),(ij')} w\left(\frac{s_e}{\varepsilon}\right)\right] d\mathbf{s}_1^{(n)} d\mathbf{s}_2^{(n)},$$

where the summation $\sum_{\mathcal{F}}$ extends over all pairings formed over all vertices $(kl) \notin \{(ij),(ij')\}$.

In the case of integration over D_1 for $v \neq v_0$ we let $\tilde{s}_{h_1(v)} := s_{h_1(v)}/\varepsilon$. We can estimate then

$$I_1 \leq \frac{C^n}{(n!)^2\varepsilon^2} e^{-\gamma_*\varepsilon^{\kappa-1}} \sum_{\mathcal{F}} \underbrace{\int_0^t \cdots \int_0^t}_{n-1-\text{times}} \int_{\mathbb{R}^{n-1}} \prod_{v\in\mathcal{F}, v\neq v_0} e^{-\gamma_*|s_{h_2(v)}/\varepsilon - \tilde{s}_{h_1(v)}|} ds_{h_2(v)} d\tilde{s}_{h_1(v)} \tag{6.13}$$

$$\leq \frac{C_1^n t^{n-1}(2n-3)!!}{(n!)^2\varepsilon^2} e^{-\gamma_*\varepsilon^{\kappa-1}} \leq \frac{C_1^n}{n!}\varepsilon.$$

The expression corresponding to integration over D_2 can be estimated using the fact that $|s_{ij} - s_{ij'}| \leq \varepsilon^\kappa$. Let r, u denote the respective numbers of edges of $\mathcal{F}$ that are straddled by $v_0 := ((ij),(ij'))$ or intersect v_0. Obviously, $|j - j'| = 1 + 2r + u$. Denote by $\mathcal{V}$ the set of vertices of those edges that neither intersect, straddle, nor coincide with v_0 and $\mathcal{V}^c$ the remaining ones. Let $m_1 := \#\mathcal{V}$ and $m_2 := 2n - m_1 - 2 = 2(r + u)$. We have then,

$$I_2 \leq \frac{1}{m_1!\varepsilon^{n+1}} \sum_{\substack{i=1,2,\\ |j-j'|\geq 4}} \sum_{\mathcal{F}_1} \int_{\substack{\Delta_{m_2+2}(t,0),\\ |s_{ij}-s_{ij'}|\leq\varepsilon^\kappa}} \cdots \int \prod_{(e,f)\in\mathcal{F}_1} \mathrm{e}^{-\gamma_*|s_e-s_f|/\varepsilon} d\mathbf{s}_{\mathcal{V}^c} \underbrace{\int_0^t \cdots \int_0^t}_{m_1-\text{times}} \mathbb{E}\left[\prod_{e\in\mathcal{V}} w\left(\frac{s_e}{\varepsilon}\right)\right] d\mathbf{s}_{\mathcal{V}}. \tag{6.14}$$

Here $d\mathbf{s}_{\mathcal{V}} := \prod_{(kl)\in\mathcal{V}} ds_{kl}$ and likewise $d\mathbf{s}_{\mathcal{V}^c}$. The summation $\sum_{\mathcal{F}_1}$ extends over all pairings formed over all vertices belonging to $\mathcal{V}^c \setminus \{(ij),(ij')\}$. For $v \neq v_0$ that does not intersect v_0 we change variable $\tilde{s}_{h_1(v)} := s_{h_1(v)}/\varepsilon$. Also for an edge $v = (e,f)$ that intersects v_0 and e lies between vertices of v_0 we let $\tilde{s}_f := s_f/\varepsilon$. We can write then

$$I_2 \leq \sum_{i=1,2,|j-j'|\geq 4} \frac{C^n \varepsilon^{\kappa|j'-j|} \varepsilon^{m_1/2} \varepsilon^u}{(m_1/2)!\varepsilon^{n+1}}$$

The exponent of ε appearing in the expression above equals

$$\kappa|j'-j|+u+m_1/2-n-1=\kappa|j'-j|-r-2\geq(\kappa-1/2)|j'-j|-3/2.$$

We can choose $\kappa\in(0,1)$ such that the above expression is positive since $|j'-j|>3$. In fact, since $m_1+2|j'-j|\geq 2n$ we have

$$I_2\leq(C_2\varepsilon^{\kappa})^n \tag{6.15}$$

for some constant $C_2>0$. Considerations in the remaining two cases, i.e. pairings belonging to $\mathcal{E}_i$, $i=2,3$ are similar and we conclude in this way that (6.7) follows.

The proof of (6.9) **for** $i=2$**.** One can obtain then a bound for $\mathcal{A}_n^{(2)}(\varepsilon)$ analogous to (6.10) with $\mathcal{E}_i$, $i=4,5$. Since the expressions $(s_{i,j}-s_{i,j+3})+(s_{i,j+1}-s_{i,j+2})$ and $(s_{i,j}-s_{i,j+2})+(s_{i,j+1}-s_{i,j+3})$ are comparable with $\sum_{K=0}^{2}(s_{i,j+K}-s_{i,j+K+1})$ on the set $s_{i,j}\geq s_{i,j+1}\geq s_{i,j+2}\geq s_{i,j+3}$ it suffices only to consider the pairings belonging to the classes $\mathcal{E}_4$. The bound obtained for the this class can be used also to estimate the expression containing pairings from $\mathcal{E}_5$. The term corresponding to $\mathcal{E}_4$ can be estimated by

$$\frac{C^{n+1}}{\varepsilon^{n+1}}\sum_{i=1,2}\sum_{j=1}^{n-3}\int_{D_n(t,0)}\prod_{K=0}^{2}\exp\left\{-\gamma_1(s_{i,j+K}-s_{i,j+K+1})/\varepsilon\right\}\mathbb{E}\left[{\prod_e}'w\left(\frac{s_e}{\varepsilon}\right)\right]d\mathbf{s}_1^{(n)}d\mathbf{s}_2^{(n)}, \tag{6.16}$$

where $C,\gamma_1>0$ are some constants indpendent of n and $\varepsilon>0$ and $\prod_e'$ denotes the product over all vertices except $(i,j),(i,j+1),(i,j+2),(i,j+3)$. Changing variables $\tilde{s}_{K+j}:=s_{K+j}/\varepsilon$, $K=1,2,3$ and dealing with the expectation term as above we obtain that the expression in (6.16) is bounded from above by $C_1^n/n!$.

Bound of (6.9) **for** $i=4$**. The case of mixed pairings.** Consider now the case when $\mathcal{F}$ is of class $\mathcal{E}_6$. Suppose that the edge $(e,f):=((1,j_1),(2,j_2))$ corresponds to the smallest values of such "mixed" s, that is, all smaller times come from the same simplex, say from the one corresponding to the first index 2: $s(e)\geq s(f)\geq s_{2,j_2+1}\geq\ldots\geq s_{2,n}$. The other case, i.e. when the first index equals 1, can be argued in the same way. Let $\mathcal{V}_{j_1,j_2}:=\{(1,1),\ldots,(1,n),(2,1),\ldots,(2,j_2-1)\}\setminus\{(1,j_1)\}$. In case $j_2=1$ we suppose that $\mathcal{V}_{j_1,1}:=\{(1,1),\ldots,(1,n)\}\setminus\{(1,j_1)\}$. Let also $\Delta(t,s_{1,j_1},\tau):=\Delta_{j_1-1}(t,s_{1,j_1})\times\Delta_{n-j_1}(s_{1,j_1},\tau)$ and $D_{t,\tau}:=\Delta_{n-1}(t,\tau)\times\Delta_{j_2-1}(t,s_{2,j_2-1})$. The term corresponding to $\mathcal{E}_6$ can be estimated by

$$\begin{aligned}&\frac{C^{n+1}}{\varepsilon^{n+1}}\sum_{j_1,j_2=1}^{n}\int_0^t d\tau\int_{D_{t,\tau}}d\mathbf{s}_{\mathcal{V}_{j_1,j_2}}\int\mathbb{E}\left[\prod_{e\in\mathcal{V}_{j_1,j_2}}\hat{V}\left(\frac{s_e}{\varepsilon},dp_e\right)\right]|\hat{\theta}(q_2,k_2)|\hat{R}(p_{1,j_1})\delta(p_{1,j_1}+p_{2,j_2})\\&\times\prod_{k=1}^{(n-j_2)/2}\left[\hat{R}(p_{2,j_2+2k-1})\delta(p_{2,j_2+2k-1}+p_{2,j_2+2k})\right]|\mathcal{G}_\varepsilon(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1})|d\mathbf{p}dq_2dk_2,\end{aligned} \tag{6.17}$$

where $d\mathbf{p}$ is the volume element corresponding to the integration over all relevant p variables,

$$\begin{aligned}\mathcal{G}_\varepsilon(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1}):=&\int_{s_{1,j_1+1}}^{s_{1,j_1-1}}\int_{\tau}^{s_{2,j_2-1}}\prod_{i=1}^{2}\left[\exp\left\{i\mathcal{C}_\varepsilon^{(i)}(q_i,\mathbf{p}_i^{(n)})\frac{s_{i,j_i}}{\varepsilon}\right\}\mathcal{I}_\varepsilon^{(i)}(s_{i,j_i})\right]\\&\times\exp\left\{-\gamma(p_{1,j_1})\frac{|s_{1,j_1}-s_{2,j_2}|}{\varepsilon}\right\}ds_{1,j_1}ds_{2,j_2}\end{aligned} \tag{6.18}$$

with

$$\mathcal{C}_\varepsilon^{(1)}(q_1,\mathbf{p}_1^{(n)}) := \frac{1}{2}\left\{\left(\varepsilon q_1 - \sum_{m=1}^{j_1} p_{1m}\right)\cdot\left(\sum_{m=1}^{j_1}\rho_{1m}p_{1m}\right) - \left(\varepsilon q_1 - \sum_{m=1}^{j_1-1} p_{1m}\right)\cdot\left(\sum_{m=1}^{j_1-1}\rho_{1m}p_{1m}\right)\right\},$$
$$\mathcal{C}_\varepsilon^{(2)}(q_2,\mathbf{p}_1^{(n)}) := -\varepsilon Q_{2,j_2-1}\cdot K_{2,j_2-1},$$

and

$$\mathcal{I}_\varepsilon^{(1)}(s_{1,j_1}) := \int \hat{\theta}^*(q_1,k_1)\exp\left\{i\mathcal{C}_\varepsilon^{(3)}(q_1,\mathbf{p}_1^{(n)})\cdot k_1\right\}\hat{W}\left(\tau, Q_n^{(1)}-\frac{p}{\varepsilon}, K_{n,\sigma_1}^{(1)}\right)[\gamma(p)-ip\cdot K_{1,n}]^{-1}dk_1,$$
$$\mathcal{I}_\varepsilon^{(2)}(s_{2,j_2}) := \int_{\Delta_{n-j_2}(s_{2,j_2},\tau)}\exp\left\{i\sum_{j=j_2}^{n} Q_{2,j}\cdot K_{2,j}(s_{2,j}-s_{2,j+1})\right\}$$
$$\times\prod_{k=1}^{(n-j_2)/2}\exp\left\{-\gamma(p_{2,j_2+2k-1})(s_{2,j_2+2k-1}-s_{2,j_2+2k})/\varepsilon\right\}d\mathbf{s}_{j_2+1,n}^{(2)},$$

where $d\mathbf{s}_{j_2+1,n}^{(2)} := ds_{2,j_2+1}\dots ds_{2,n}$ and

$$\mathcal{C}_\varepsilon^{(3)}(q_1,\mathbf{p}_1^{(n)}) := q_1(t-\tau)/\varepsilon - \sum_{m=1}^{n} p_{1m}(s_{1m}-\tau)/\varepsilon.$$

By the Plancherel formula we can write

$$\mathcal{I}_\varepsilon^{(1)}(s_{1,j_1}) := \int \mathcal{F}(\theta)^*(q_1, z+\mathcal{C}_\varepsilon^{(3)}(q_1,\mathbf{p}_1^{(n)}))\exp\{iz\cdot\mathcal{P}\}F\left(\tau, Q_n^{(1)}-\frac{p}{\varepsilon}, z\right)dz, \tag{6.19}$$

where

$$\mathcal{P} := \frac{1}{2}\sum_{j=1}^{n}\sigma_{1,m}p_{1,m} + \frac{\sigma_1}{2}p,$$
$$F(\tau,q,z) := \int e^{-iz\cdot k}\hat{W}(\tau,q,k+\sigma_1 p/2)[\gamma(p)-ip\cdot k]^{-1}dk.$$

Using (2.52) we can write $\mathcal{I}_\varepsilon^{(1)}(s_{1,j_1}) = \sum_{l\ge 0}\mathcal{I}_{l,\varepsilon}^{(1)}(s_{1,j_1})$, where

$$\mathcal{I}_{l,\varepsilon}^{(1)}(s_{1,j_1}) := \int_0^{+\infty}\dots\int_0^{+\infty}d\tau_{0,l}\int\dots\int dk_{1,l}^{(1)}\int dz\prod_{i=1}^{l-1}\left[\sigma(k_i^{(1)},k_{i+1}^{(1)})\Sigma(k_i^{(1)})\right] \tag{6.20}$$
$$\times\exp\left\{-\sum_{m=1}^{m}\Sigma(k_m^{(1)})\tau_m\right\}\widehat{\mathcal{K}}\left(\left(Q_n^{(1)}-\frac{p}{\varepsilon}\right)\tau_0 - z\right)\mathcal{F}(\theta)^*(q_1, z+\mathcal{C}_\varepsilon^{(3)}(q_1,\mathbf{p}_1^{(n)}))\exp\{iz\cdot\mathcal{P}\}$$
$$\times\exp\left\{-i\sum_{m=1}^{l}\left(Q_n^{(1)}-\frac{p}{\varepsilon}\right)\cdot k_m^{(1)}\tau_m\right\}f(k_l^{(1)})\delta\left(\tau-\sum_{m=0}^{l}\tau_m\right).$$

Here $k_0^{(1)} := k_1$. We apply here the convention of writing $dx_{k,m} := dx_k\dots dx_m$ for any indexed variable x_k. Here $\widehat{\mathcal{K}}(z)$ is the Fourier transform of

$$\mathcal{K}(k) := \hat{R}\left((|k_1^{(1)}|^2-|k|^2)/2, k_1^{(1)}-k\right)\exp\{-\Sigma(k)\tau_0\}[\gamma(p)-ip\cdot(k-\sigma_1 p/2)]^{-1}.$$

Using the expansion of $\mathcal{I}_\varepsilon^{(1)}(s_{1,j_1})$ we can write $\mathcal{G}_\varepsilon = \sum_{l\geq 0}\mathcal{G}_{l,\varepsilon}$, where $\mathcal{G}_{l,\varepsilon}$ is given by (6.18) in which $\mathcal{I}_\varepsilon^{(1)}$ is replaced by $\mathcal{I}_{l,\varepsilon}^{(1)}$.

From formula (2.47) one can conclude that

$$(1+|z|^2)^{[d/2]+1}|\widehat{\mathcal{K}}(z)| \leq C,$$

where the constant C depends only γ_*, d and $\sum_{m=0}^{[d/2]+1}\sup_p|\nabla^m\hat{R}(p)|$, cf assumption (2.63). Thus,

$$\int |\widehat{\mathcal{K}}(z)|dz =: \mathcal{K}_* < +\infty. \tag{6.21}$$

Changing variables $\tilde{s}_{2,j_2+2k} := s_{2,j_2+2k}/\varepsilon$, $k=0,\ldots,(n-j_2)/2$ and $\tilde{s}_{1,j_1} := s_{1,j_1}/\varepsilon$ and using (6.21) we obtain

$$|\mathcal{G}_{l,\varepsilon}(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1})| \leq \frac{C^{n-j_2+l}}{l!}\varepsilon^{(n-j_2)/2+2}\mathcal{K}_*\left(\sup_{y\in\mathbb{R}^d,\omega\in\mathbb{S}^{d-1}}\int_{-\infty}^{+\infty}|\mathcal{F}(\theta)(q_1,y+|p_{1,j_1}|\omega s)|ds\right), \tag{6.22}$$

which, after summing up over l-s, leads to an upper bound

$$|\mathcal{G}_\varepsilon(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1})| \leq C^{n-j_2}\varepsilon^{(n-j_2)/2+2}, \tag{6.23}$$

which in turn leads to an estimate of (6.17) by

$$\begin{aligned}&\frac{C^{n+1}}{\varepsilon^{n+1}}\sum_{j_1,j_2=1}^{n}\int_0^t d\tau\int_{D_{t,\tau}}d\mathbf{s}_{\mathcal{V}_{j_1,j_2}}\int\mathbb{E}\left[\prod_{e\in\mathcal{V}_{j_1,j_2}}V\left(\frac{s_e}{\varepsilon},dp_e\right)\right]|\hat{\theta}(q_2,k_2)|\hat{R}(p_{1,j_1})\delta(p_{1,j_1}+p_{2,j_2})\\ &\times\varepsilon^{(n-j_2)/2+2}\mathcal{K}_*\left(\sup_{y\in\mathbb{R}^d,\omega\in\mathbb{S}^{d-1}}\int_{-\infty}^{+\infty}|\mathcal{F}(\theta)(q_1,y+|p_{1,j_1}|\omega s)|ds\right)\\ &\prod_{k=1}^{(n-j_2)/2}\left[\hat{R}(p_{2,j_2+2k-1})\delta(p_{2,j_2+2k-1}+p_{2,j_2+2k})\right]d\mathbf{p}dqdk_2.\end{aligned} \tag{6.24}$$

Estimating $\mathbb{E}\left[\prod_{e\in\mathcal{V}_{j_1,j_2}}V(s_e/\varepsilon,dp_e)\right]$ in the same way as in previous cases we conclude that expression (6.24) can be bounded from above by

$$C^{n+1}\left(\int\frac{\hat{R}(p)}{|p|}dp\right)\sum_{j_2=1}^{n}\frac{1}{[(n+j_2)/2]!}$$

for some constant $C>0$ and (6.9) follows, cf (2.64).

The proof of (6.8)**.** Using(6.2) we can write that

$$\begin{aligned}\tilde{\mathcal{A}}_{2n}^{(3)}(\varepsilon) = -\varepsilon^{-(2n+1)}\int_0^t d\tau&\left\{\sum_{\sigma,\rho}\sigma\int\int\hat{\theta}^*(q_1,k_1)\hat{W}\left(\tau,q_1-\frac{p}{\varepsilon},\bar{K}_{2n,\sigma}(\rho)\right)dq_1dk_1d\mathbf{p}\right.\\ &\times[\gamma(p)+i\bar{K}_{2n}(\rho)\cdot p]^{-1}\bar{R}(\mathbf{p})\bar{\rho}\int_{\Delta_{2n}(t,\tau)}\left[\prod_{k=0}^{n}\exp\left\{iq_1\cdot\bar{K}_{2k}(\rho)(s_{2k}-s_{2k+1})\right\}\right.\\ &\left.\left.\times\prod_{k=1}^{n}\exp\left\{-[\gamma(p_{2k-1})-iq_{\varepsilon,k}\cdot\bar{K}_{2k-1}(\rho)](s_{2k-1}-s_{2k})/\varepsilon\right\}\right]d\mathbf{s}_{1,2n}\right\}^2\end{aligned} \tag{6.25}$$

where $\bar\rho := \prod_{k=1}^{2n}\rho_k$, $\bar R(\mathbf{p}) := \prod_{k=1}^{n}\hat R(p_{2k-1})$, $d\mathbf{p} := \nu(dp)dp_1\dots dp_{2n-1}$ and $\bar K_0(\rho) := k_1$,

$$
\begin{aligned}
&q_{\varepsilon,k} := \varepsilon q_1 - p_{2k-1},\\
&\bar K_{2k-1}(\rho) := \bar K_{2k-2}(\rho) + \frac{1}{2}\rho_{2k-1}p_{2k-1},\\
&\bar K_{2k-2}(\rho) := k_1 + \frac{1}{2}\sum_{m=1}^{k-1}(\rho_{2m-1}-\rho_{2m})p_{2m-1}, \quad k=1,2,\dots,n+1,\\
&\bar K_{2n,\sigma}(\rho) := \bar K_{2n}(\rho) + \frac{\sigma}{2}p, \quad i=1,2.
\end{aligned}
$$

We change variables $\tilde s_k := s_k/\varepsilon$, $i=1,2$, $k=1,\dots,2n$. As a result

$$
\begin{aligned}
\tilde{\mathcal{A}}_{2n}^{(3)}(\varepsilon) = -\varepsilon^{2n-1}\int_0^t d\tau &\left\{\sum_{\sigma,\rho}\sigma\int\int\hat\theta^*(q_1,k_1)\hat W\left(\tau, q_1-\frac{p}{\varepsilon}, \bar K_{2n,\sigma}(\rho)\right)dq_1dk_1d\mathbf{p}\right.\\
&\left.\times\,[\gamma(p)+i\bar K_{2n}(\rho)\cdot p]^{-1}\bar R(\mathbf{p})\bar\rho\,\mathcal{D}_\varepsilon(t/\varepsilon,\tau/\varepsilon)\right\}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{D}_\varepsilon(t/\varepsilon,\tau/\varepsilon) := \int_{\Delta_{2n}(t/\varepsilon,\tau/\varepsilon)}&\left[\prod_{k=0}^{n}\exp\left\{i\varepsilon q_1\cdot\bar K_{2k}(\rho)(s_{2k}-s_{2k+1})\right\}\right.\\
&\left.\times\prod_{k=1}^{n}\exp\left\{-[\gamma(p_{2k-1})-iq_{\varepsilon,k}\cdot\bar K_{2k-1}(\rho)](s_{2k-1}-s_{2k})\right\}\right]d\mathbf{s}_{1,2n}.
\end{aligned}
$$

We can integrate out the s-variables with odd indices using an elementary formula

$$
\int_{s_0}^{s_2}\mathrm{e}^{iA(s_0-s_1)}\mathrm{e}^{-(B+iC)(s_1-s_2)}ds_1 = [B+i(A+C)]^{-1}[\mathrm{e}^{iA(s_0-s_2)}-\mathrm{e}^{-(B+iC)(s_0-s_2)}]
$$

valid for all $A,C\in\mathbb{R}$, $B>0$, $s_2>s_1$ we obtain, after changing variables $s_{2k} := \varepsilon s_{2k}$, that

$$
\begin{aligned}
\mathcal{D}_\varepsilon(t/\varepsilon,\tau/\varepsilon) := \varepsilon^{-n}\int_{\Delta_n(t,\tau)}\prod_{k=1}^{n}&\left\{\left[\gamma(p_{2k-1})+i\left(\varepsilon q_1\cdot\bar K_{2k-2}(\rho)-q_{\varepsilon,k}\cdot\bar K_{2k-1}(\rho)\right)\right]^{-1}\right.\\
\times\left[\exp\left\{iq_1\cdot\bar K_{2k-2}(\rho)(s_{2k-2}-s_{2k})\right\}\right.&\left.\left.-\exp\left\{-[\gamma(p_{2k-1})-iq_{\varepsilon,k}\cdot\bar K_{2k-1}(\rho)](s_{2k-2}-s_{2k})/\varepsilon\right\}\right]\right\}d\mathbf{s}_{1,2n}^{(e)},
\end{aligned}
$$

where $d\mathbf{s}_{1,2n}^{(e)} = ds_2\dots ds_{2n}$. Choose $\kappa\in(1/2,1)$. Considering the cases $s_{2k-2}-s_{2k}\ge\varepsilon^\kappa$ and $0<s_{2k-2}-s_{2k}<\varepsilon^\kappa$ we conclude that

$$
\begin{aligned}
\tilde{\mathcal{A}}_{2n}^{(3)}(\varepsilon) = -\varepsilon^{-1}\int_0^t d\tau\int\nu(dp)&\left\{\sum_{\sigma,\rho}\sigma\int\int\hat\theta^*(q_1,k_1)\hat W\left(\tau,q_1-\frac{p}{\varepsilon},\bar K_{2n,\sigma}(\rho)\right)dq_1dk_1d\mathbf{p}\right.\\
&\left.\times\,[\gamma(p)+i\bar K_{2n}(\rho)\cdot p]^{-1}\bar R(\mathbf{p})\bar\rho\,\bar{\mathcal{D}}(t,\tau)\right\}^2 + C_n(\varepsilon), \qquad (6.26)
\end{aligned}
$$

where

$$
\bar{\mathcal{D}}(t,\tau) := \int_{\Delta_n(t,\tau)}\prod_{k=1}^{n}\left\{\left[\gamma(p_{2k-1})+ip_{2k-1}\cdot\bar K_{2k-1}(\rho)\right]^{-1}\exp\left\{iq_1\cdot\bar K_{2k-2}(\rho)(s_{2k-2}-s_{2k})\right\}\right\}d\mathbf{s}_{1,2n}^{(e)},
$$

$$
C_n(\varepsilon)\le\frac{C^n}{n!}\varepsilon^{2\kappa-1}
$$

for some constant $C>0$. A simple calculation shows that the term of order ε^{-1} on the right hand side of (6.26) coincides with $\mathbb{E}\langle\bar Z_{n,\varepsilon}(t),\hat\theta\rangle^2$ and the conclusion of the proposition follows. □

6.3 Estimates of $\mathcal{B}_n(\varepsilon)$

For the most part the argument one can use to estimate $\mathcal{B}_n(\varepsilon)$ is a simplified version of the argument from the previous section. From (6.6) we obtain that

$$\mathbb{E}\left[\langle \hat{Z}^o_{2n,\varepsilon}(t),\hat{\theta}\rangle\langle \bar{Z}_{n,\varepsilon}(t),\hat{\theta}\rangle\right] = \frac{(-1)^{n+1}}{\varepsilon^{1+n}}\sum_{\sigma,\sigma',\rho=\pm 1}\sum_{\mathcal{F}}\sigma\sigma'\bar{\rho}\int_0^t d\tau\int\int \hat{\theta}^*(q,k)\hat{\theta}^*(q',k')\bar{\mathcal{V}}_n(t,s_{n+1},q',\mathbf{k}'_{0,n})$$
$$\times\int_{\Delta_{2n}(t,\tau)}\exp\left\{i\sum_{j=0}^n Q_j\cdot K_j(s_j-s_{j+1})\right\}\prod_{(j,j')\in\mathcal{F}}\left[e^{-\gamma(p_j)|s_j-s_{j'}|/\varepsilon}\hat{R}(p_j)\delta(p_j+p_{j'})\right]d\mathbf{s}_{1,2n} \quad (6.27)$$
$$\times\Gamma(p,K_{2n},k')\hat{W}\left(s_{n+1},Q_{2n}-\frac{p}{\varepsilon},K_{2n,\sigma}\right)\hat{W}\left(s_{n+1},q'-\frac{p}{\varepsilon},\bar{K}'_{n,\sigma'}\right)\nu(dp)dqdq'd\mathbf{k}'_{0,n}.$$

Here for a given sequence $\rho=(\rho_1,\ldots,\rho_{2n})\in\{-1,1\}^{2n}$ we let $\bar{\rho}:=\prod_{j=1}^{2n}\sigma_j$

$$\mathbf{p}_i^{(2n)}=(p_1,\ldots,p_{2n}),\quad Q_j:=q-\frac{1}{\varepsilon}\sum_{m=1}^j p_m,\quad K_j:=k+\frac{1}{2}\sum_{m=1}^j\rho_m p_m.$$

The summation $\sum_{\mathcal{F}}$ extends over all pairings formed over integers $j=1,\ldots,2n$. In analogy with the previous notation we say that an edge $v=(i,j)$ *straddles over* $v'=(i',j')$ if $v\neq v'$ and $i\le i'\le j'\le j$. Edges $v=(i,j)$ and v' are said to *intersect* each other if they are different, not straddled by each other and one of the vertices, say i', satisfies $i\prec i'\prec j$.

A pairing is called *time-ordered* if all edges are of the form $(j-1,j)$ and $j=1,\ldots,2n$. A pairing $\mathcal{F}$ is said to be *negligible* if it belongs to either of three classes of pairings: 1) $\mathcal{E}_1$ consisting of pairings containing an edge (j,j') such that $|j'-j|\ge 4$, 2) $\mathcal{E}_2$ pairings with at least two edges $(j_k,j'_k)\in\mathcal{F}$, $k=1,2$ with $|j'_1-j_1|\ge 3$ and $|j'_2-j_2|\ge 2$, or 3) $\mathcal{E}_3$ pairings with at least three edges $(j,j')\in\mathcal{F}$ such that $|j'-j|\ge 2$. *An almost time-ordered pairing* is defined as a pairing that is neither time-ordered, nor negligible. We can divide the summation over pairings appearing in (6.27) into three sums $\mathcal{B}_n^{(i)}(\varepsilon)$, $i=1,2$ and $\tilde{\mathcal{B}}_n^{(3)}(\varepsilon)$ according to the classes $\mathcal{E}_i$, $i=1,2,3$ described above. We let $\mathcal{B}_n^{(3)}(\varepsilon):=\tilde{\mathcal{B}}_n^{(3)}(\varepsilon)-\mathbb{E}\langle\bar{Z}_{n,\varepsilon}(t),\hat{\theta}\rangle^2$. Repeating almost literally the argument used in the previous section we obtain the following.

Proposition 6.3 *There exist constants $C_1,C_2>0$ and $\kappa\in(0,1)$ such that*

$$|\mathcal{B}_n^{(1)}(\varepsilon)|\le\frac{C_1^n}{n!}\varepsilon+(C_2\varepsilon^\kappa)^n, \quad (6.28)$$

$$|\mathcal{B}_n^{(3)}(\varepsilon)|\le\frac{C_1^n}{n!}\varepsilon^\kappa, \quad (6.29)$$

$$|\mathcal{B}_n^{(2)}(\varepsilon)|\le\frac{C_1^n}{n!}, \quad (6.30)$$

for all $n\ge 1$, $\varepsilon>0$.

6.4 The end of the proof of Theorem 5.2

In light of the results of Propositions 6.2 and 6.3 to finish the proof of the theorem we need to show that for each $n\ge 1$ we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-(1+n)}\int_0^t d\tau\int\int\mathcal{V}(\rho,\mathcal{F},\mathbf{p}_1^{(n)},\mathbf{p}_2^{(n)};t,\tau)\hat{\theta}^*(q_1,k_1)\hat{\theta}^*(q_2,k_2)\Gamma(p,K_{1,n},K_{2,n}) \quad (6.31)$$
$$\times\hat{W}\left(\tau,Q_n^{(1)}-\frac{p}{\varepsilon},K^{(1)}_{n,\sigma_1}\right)\hat{W}\left(\tau,Q_n^{(2)}-\frac{p}{\varepsilon},K^{(2)}_{n,\sigma_2}\right)d\mathbf{p}dqdk=0,$$

for an almost time-ordered, or mixed type pairing $\mathcal{F}$, according to the terminology of Section 6.1. Here $d\mathbf{p}dqdk$ is an abbreviation for the volume element $\nu(dp)dq_1dq_2dk_1dk_2d\mathbf{p}_1^{(n)}d\mathbf{p}_2^{(n)}$. The definition of the terms appearing in expression (6.31) are the same as those given in Section 6.2.

In addition we also need to prove that

$$\lim_{\varepsilon\to 0+}\varepsilon^{-(1+n)}\int_0^t d\tau\int\int\hat{\theta}^*(q,k)\hat{\theta}^*(q',k')\bar{\mathcal{V}}_n(t,\tau,q',\mathbf{k}'_{0,n}) \tag{6.32}$$
$$\times\int_{\Delta_{2n}(t,\tau)}\exp\left\{i\sum_{j=0}^{2n}Q_j\cdot K_j(s_j-s_{j+1})\right\}\prod_{(j,j')\in\mathcal{F}}\left[e^{-\gamma(p_j)|s_j-s_{j'}|/\varepsilon}\hat{R}(p_j)\delta(p_j+p_{j'})\right]d\mathbf{s}_{1,2n}$$
$$\times\Gamma(p,K_{2n},k')\hat{W}\left(\tau,Q_{2n}-\frac{p}{\varepsilon},K_{2n,\sigma}\right)\hat{W}\left(\tau,q'-\frac{p}{\varepsilon},\bar{K}'_{n,\sigma'}\right)\nu(dp)dqdq'd\mathbf{k}'_{0,n}=0$$

for an almost time-ordered pairing, according to the terminology of Section 6.3. We start with the proof of (6.31).

The case of an almost time-ordered pairing. In this case $n=2\ell$. Suppose first that $\mathcal{F}\in\mathcal{E}_4$. It contains an edge of the form $((i_0,2\ell_0-1),(i_0,2\ell_0+2))$ and all other edges are of the form $((i,j),(i,j+1))$. With no loss of generality we may assume that $i_0=1$. It suffices therefore to prove that

$$\lim_{\varepsilon\to 0+}\varepsilon^{-(1+n)}\int_0^t d\tau\int\int\hat{\theta}^*(q_1,k_1)\widehat{W}\left(\tau,q_1-\frac{p}{\varepsilon},K^{(1)}_{n,\sigma_1}\right)\prod_{i=1}^2\mathcal{D}_i(\tau)\prod_{\substack{m=1\\ i=1,2}}^{\ell}\hat{R}(p_{i,2m-1})dk_1dqdp=0, \tag{6.33}$$

where $dqdp$ is the abbreviation for the volume element $\nu(dp)dq_1dq_2dk_1d\mathbf{p}_1^{(n)}d\mathbf{p}_2^{(n)}$ and

$$\mathcal{D}_1(\tau):=\int_{\Delta_n(t,\tau)}\prod_{m\neq\ell_0+1}\exp\left\{\left[i(\varepsilon q_1-p_{1,2m-1})\cdot K_{1,2m-1}-\gamma(p_{1,2m-1})\right](s_{1,2m-1}-s_{1,2m})/\varepsilon\right\}$$
$$\times\exp\left\{i\left[(\varepsilon q_1+p_{1,2\ell_0+1}-p_{1,2\ell_0-1})\cdot K_{1,2\ell_0}-2\gamma(p_{1,2\ell_0-1})\right](s_{1,2\ell_0}-s_{1,2\ell_0+1})/\varepsilon\right\}$$
$$\times\exp\left\{\left[i(\varepsilon q_1-p_{1,2\ell_0-1})\cdot K_{1,2\ell_0+1}-\gamma(p_{1,2\ell_0+1})\right](s_{1,2\ell_0+1}-s_{1,2\ell_0+2})/\varepsilon\right\}$$
$$\times\prod_{m\neq\ell_0}\exp\left\{iq_1\cdot K_{1,2m}(s_{1,2m}-s_{1,2m+1})\right\}d\mathbf{s}_1^{(n)}$$

and

$$\mathcal{D}_2(\tau):=\int_{\Delta_{2\ell}(t,\tau)}\int\Gamma(p,K_{1,2\ell},K_{2,2\ell})\hat{\theta}^*(q_2,k_2)\exp\left\{i\sum_{j=0}^{2\ell}Q_{2,j}\cdot K_{2,j}(s_{2,j}-s_{2,j+1})\right\}$$
$$\times\widehat{W}\left(\tau,q_2-\frac{p}{\varepsilon},K^{(2)}_{2\ell,\sigma_2}\right)\prod_{m=1}^{\ell}\exp\left\{-\gamma(p_{2,2m-1})(s_{2,2m-1}-s_{2,2m})/\varepsilon\right\}d\mathbf{s}_2^{(2\ell)}dk_2$$

To estimate $\mathcal{D}_1(\tau)$ we rewrite it in the form

$$\mathcal{D}_1:=\int_{\Delta_{2\ell}(t,\tau)}\mathcal{J}_\varepsilon(\mathbf{s}_1^{(2\ell)})\prod_{m\neq\ell_0+1}\exp\left\{-\gamma(p_{1,2m-1})(s_{1,2m-1}-s_{1,2m})/\varepsilon\right\}$$
$$\times\exp\left\{-2\gamma(p_{1,2\ell_0-1})(s_{1,2\ell_0}-s_{1,2\ell_0+1})/\varepsilon\right\}\exp\left\{-\gamma(p_{1,2\ell_0+1})(s_{1,2\ell_0+1}-s_{1,2\ell_0+2})/\varepsilon\right\}d\mathbf{s}_1^{(2\ell)},$$

where $\sup|\mathcal{J}_\varepsilon(\mathbf{s}_1^{(n2\ell)})|<+\infty$ and $\mathcal{P}_m:=\frac{1}{2}\sum_{j=1}^m\rho_{1,j}p_{1,j}+\sigma_1p/2$. Changing variables $s_{1,2m}:=s_{1,2m}/\varepsilon$, $s_{1,2\ell_0-1}:=s_{1,2\ell_0-1}/\varepsilon$ we obtain $\sup_{\tau\in[0,t]}|\mathcal{D}_1(\tau)|\le C\varepsilon^{\ell+1}$.

We can use the change of variables $s_{2,2j} := s_{2,2j}/\varepsilon$, $j = 1, \ldots, \ell$ to obtain that $|\mathcal{D}_2(\tau)| \le C\varepsilon^\ell$ uniformly in all variables that are left after integrating out $s_{2,j}$-s.

Using (2.55) we can write that $\mathcal{D}_2 = \sum_{m\ge 0} \mathcal{D}_2^{(m)}$, where

$$\mathcal{D}_2^{(0)}(\tau) := \int_{\Delta_{2\ell}(t,\tau)} \int \mathcal{J}_0\left(\left(q_2 - \frac{p}{\varepsilon}\right)\tau_0\right) \prod_{j=1}^{\ell} \exp\left\{-\gamma(p_{2,2j-1})(s_{2,2j-1} - s_{2,2j})/\varepsilon\right\} d\mathbf{s}_2^{(2\ell)}$$

and

$$\begin{aligned}\mathcal{D}_2^{(m)}(\tau) := &\int_{\Delta_{2\ell}(t,\tau)} \int_0^{+\infty} \ldots \int_0^{+\infty} \int \mathcal{J}\left(\left(q_2 - \frac{p}{\varepsilon}\right)\tau_0\right) \\ &\times \prod_{i=1}^{m-1} \left[\sigma(k_i^{(2)}, k_{i+1}^{(2)})\Sigma(k_i^{(2)})\right] \exp\left\{-\sum_{j=1}^{m} \Sigma(k_j^{(2)})\tau_j\right\} \exp\left\{-i\sum_{j=1}^{m}\left(q_2 - \frac{p}{\varepsilon}\right)\cdot k_j^{(2)}\tau_j\right\} f(k_m^{(2)}) \\ &\times \prod_{j=1}^{\ell} \exp\left\{-\gamma(p_{2,2j-1})(s_{2,2j-1} - s_{2,2j})/\varepsilon\right\} \delta\left(\tau - \sum_{j=0}^{m}\tau_j\right) d\tau_{0,m} d\mathbf{s}_2^{(2\ell)}.\end{aligned} \tag{6.34}$$

Here

$$\begin{aligned}\mathcal{J}_0(z) := &\int \Gamma(p, K_{1,2\ell}, K_{2,2\ell})\hat{\theta}^*(q_2, k_2) \exp\left\{i\sum_{j=0}^{2\ell} Q_{2,j}\cdot K_{2,j}(s_{2,j} - s_{2,j+1})\right\} \\ &\times \exp\left\{-iz\cdot K_{2\ell,\sigma_2}^{(2)}\right\} f(K_{2\ell,\sigma_2}^{(2)}) \exp\left\{-\Sigma(K_{2\ell,\sigma_2}^{(2)})\tau_0\right\} dk_2\end{aligned} \tag{6.35}$$

and

$$\begin{aligned}\mathcal{J}(z) := &\int \Gamma(p, K_{1,2\ell}, K_{2,2\ell})\hat{\theta}^*(q_2, k_2) \exp\left\{i\sum_{j=0}^{2\ell} Q_{2,j}\cdot K_{2,j}(s_{2,j} - s_{2,j+1})\right\} \\ &\times \exp\left\{-iz\cdot K_{2\ell,\sigma_2}^{(2)}\right\} \hat{R}\left(\frac{|k_1^{(2)}|^2 - |K_{2\ell,\sigma_2}^{(2)}|^2}{2}, k_1^{(2)} - K_{2\ell,\sigma_2}^{(2)}\right) \exp\left\{-\Sigma(K_{2\ell,\sigma_2}^{(2)})\tau_0\right\} dk_2\end{aligned} \tag{6.36}$$

Then, with the assumptions made one can easily verify that

$$|\mathcal{J}_0(z)| + |\mathcal{J}(z)| \le \frac{C}{|z|^2 + 1}. \tag{6.37}$$

With that estimate we conclude easily that

$$\begin{aligned}|\mathcal{D}_2^{(m)}(\tau)| \le C^m &\int_{\Delta_{2\ell}(t,\tau)} \int \ldots \int_{\tau \ge \sum_{i=1}^m \tau_i, \tau_i \ge 0} \left[1 + \left(\frac{\tau - \sum_{i=1}^m \tau_i}{\varepsilon}\right)^2\right]^{-1} \\ &\times \prod_{j=1}^{\ell} \exp\left\{-\gamma(p_{2,2j-1})(s_{2,2j-1} - s_{2,2j})/\varepsilon\right\} d\tau_{0,m} d\mathbf{s}_2^{(2\ell)}\end{aligned} \tag{6.38}$$

for $m \ge 1$. Changing variables $\tau_m := \tau_m/\varepsilon + \varepsilon^{-1}(\sum_{i=1}^{m-1}\tau_i - \tau)$ and $s_{2j} := s_{2j}/\varepsilon$ we obtain

$$|\mathcal{D}_2^{(m)}(\tau)| \le C^{m+\ell}\varepsilon^{\ell+1} \int_{-\infty}^{+\infty} \frac{d\tau_m}{1 + \tau_m^2} \int \ldots \int_{\tau \ge \sum_{i=1}^{m-1} \tau_i, \tau_i \ge 0} d\tau_{1,m-1} \le \frac{C_1^{m+\ell}}{m!}\varepsilon^{\ell+1}$$

for some constant $C_1 > 0$. Since we also have

$$|\mathcal{D}_2^{(0)}(\tau)| \le \frac{C^{2\ell}}{1+(\tau/\varepsilon)^2}$$

we can estimate the expression under the limit in (6.33) by

$$\varepsilon^{-(1+n)}\left[C^{2\ell}\varepsilon^{2\ell+1}\int_0^t \frac{d\tau}{1+(\tau/\varepsilon)^2}+\sum_{m\ge 1}\frac{C^{m+\ell}}{m!}\varepsilon^{2\ell+2}\right]\le C_1\varepsilon$$

for some constant $C_1 > 0$. What remains yet to be shown is estimate (6.37). We perform substitution $k_2 := K^{(2)}_{2\ell,\sigma_2}$ in (6.35). The case when $\mathcal{F}\in\mathcal{E}_5$ can be dealt with similarly.

Mixed pairings. Consider a mixed pairing $\mathcal{F}$ whose last mixed bond is $((1,j_1),(2,j_2))$ (as in Section 6.2). According to (6.2) the term corresponding to $\mathcal{E}_6$ can be estimated by

$$\begin{aligned}\mathcal{H}_\varepsilon &:= \frac{1}{\varepsilon^{n+1}}\left|\int_0^t d\tau\int_{D^{j_1,j_2}_{t,\tau}} d\mathbf{s}_{\mathcal{V}_{j_1,j_2}}\int \exp\left\{i\sum_{(ij),(i,j+1)\in\mathcal{V}_{j_1,j_2}} Q_{ij}\cdot K_{ij}(s_{ij}-s_{ij+1})\right\}\right.\\ &\times\mathbb{E}\left[\prod_{e\in\mathcal{V}_{j_1,j_2}} V\left(\frac{s_e}{\varepsilon},dp_e\right)\right]\hat\theta^*(q_2,k_2)(\gamma(p)-ip\cdot K_{2,n})^{-1}\\ &\left.\times\hat W\left(\tau,Q^{(2)}_n-\frac{p}{\varepsilon},K^{(2)}_{n,\sigma_2}\right)\mathcal{G}_\varepsilon(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1})d\mathbf{p}dqdk_2\right|.\end{aligned}\tag{6.39}$$

Here $\mathcal{G}_\varepsilon(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1})$ is defined by (6.18) and

$$D^{j_1,j_2}_{t,\tau} := [t\ge s_{1,1}\dots s_{j_1-1}\ge s_{1,j_1+1}\dots\ge s_{1,n}\ge 0,\ t\ge s_{2,1}\dots s_{2,j_2-1}\ge 0].$$

Using (2.55) we obtain $\mathcal{H}_\varepsilon\le\sum_{m\ge 0}\mathcal{H}^{(m)}_\varepsilon$, where

$$\begin{aligned}\mathcal{H}^{(m)}_\varepsilon &:= \frac{1}{\varepsilon^{n+1}}\left|\int_0^t d\tau\int_{D^{j_1,j_2}_{t,\tau}} d\mathbf{s}_{\mathcal{V}_{j_1,j_2}}\int_0^{+\infty}\dots\int_0^{+\infty} d\tau_{0,m}\int dk^{(2)}_{1,m}\int\exp\left\{-\Sigma(K^{(2)}_{n,\sigma_2})\tau_0\right\}\right.\\ &\times\hat R\left(\frac{|k^{(2)}_1|^2-|K^{(2)}_{n,\sigma_2}|^2}{2},k^{(2)}_1-K^{(2)}_{n,\sigma_2}\right)\exp\left\{-i\left(Q^{(2)}_n-\frac{p}{\varepsilon}\right)\cdot K^{(2)}_{n,\sigma_2}\tau_j\right\}\\ &\prod_{i=1}^{m-1}\left[\sigma(k^{(2)}_i,k^{(2)}_{i+1})\Sigma(k^{(2)}_i)\right]\exp\left\{-\sum_{j=1}^m\Sigma(k^{(2)}_j)\tau_j\right\}\exp\left\{-i\sum_{j=1}^m\left(Q^{(2)}_n-\frac{p}{\varepsilon}\right)\cdot k^{(2)}_j\tau_j\right\}\\ &\times f(k^{(2)}_m)\delta\left(\tau-\sum_{j=0}^m\tau_j\right)\exp\left\{i\sum_{(ij),(i,j+1)\in\mathcal{V}_{j_1,j_2}}Q_{ij}\cdot K^{(2)}_{ij}(s_{ij}-s_{ij+1})\right\}\mathbb{E}\left[\prod_{e\in\mathcal{V}_{j_1,j_2}}V\left(\frac{s_e}{\varepsilon},dp_e\right)\right]\\ &\left.\times\hat\theta^*(q_2,k_2)(\gamma(p)-ip\cdot K_{2,n})^{-1}\,\mathcal{G}_\varepsilon(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1})d\mathbf{p}dqdk_2\right|.\end{aligned}\tag{6.40}$$

This expression can be estimated in the same way as in (6.34). We obtain then

$$\begin{aligned}|\mathcal{H}^{(m)}_\varepsilon| &\le \frac{C^m}{\varepsilon^{n+1}}\int_0^t d\tau\int_{D^{j_1,j_2}_{t,\tau}} d\mathbf{s}_{\mathcal{V}_{j_1,j_2}}\int\dots\int_{\tau\ge\sum_{i=1}^m\tau_i,\tau_i\ge 0}\left[1+\left(\frac{\tau-\sum_{i=1}^m\tau_i}{\varepsilon}\right)^2\right]^{-1}\\ &\times\prod_{\substack{(e,f)\in\mathcal{F}\\ e,f\in\mathcal{V}_{j_1,j_2}}}\exp\{-\gamma(p_e)|s_e-s_e|/\varepsilon\}\,|\mathcal{G}_\varepsilon(s_{1,j_1-1},s_{1,j_1+1},s_{2,j_2-1})|d\tau_{0,m}d\mathbf{s}^{(2\ell)}_2\end{aligned}$$

Changing variables $\tau_m := \tau_m/\varepsilon + \varepsilon^{-1}(\sum_{i=1}^{m-1}\tau_i - \tau)$ and $s_e := s_e/\varepsilon$ we conclude, using (6.23),

$$|\mathcal{H}_\varepsilon^{(m)}| \le \frac{C^m}{m!}\varepsilon^{(n+j_2-2)/2+1}\varepsilon^{(n-j_2)+2},$$

which in turn implies that $|\mathcal{H}_\varepsilon| \le C\varepsilon$. This ends the proof of (6.31). The proof of (6.32) is obtained essentially in the same way. Actually, in this case we do not have to consider the mixed type pairings, so it suffices only to use the same argument as the one applied in the proof of (6.33).

7 The proof of Theorem 2.8

Suppose that $\{\bar{Z}(t),\, t \ge 0\}$ that is the solution of (2.48) with the initial condition $\bar{Z}(0) = \delta \otimes X$. The result in uqestion follows from.

Theorem 7.1 *Suppose that $t_0 > 0$ and $\theta \in \mathcal{S}(\mathbb{R}^d)$. Then, the finite dimensional distributions of $\{\langle \bar{Z}_\varepsilon(t), \theta\rangle\, t \ge t_0\}$ converge in law, as $\varepsilon \to 0+$, to the respective distributions of $\{\langle \bar{Z}(t), \theta\rangle\, t \ge t_0\}$.*

Proof. To simplify notation we shall show only the convergence in law of one dimensional marginals. The proof in the general case is almost identical. From (5.4) and (2.55) we can write that $\widehat{\mathcal{M}}_\varepsilon(t,q,k) = \widehat{\mathcal{M}}_\varepsilon^{(0)}(t,q,k) + \mathcal{R}_\varepsilon(t,q,k)$, where

$$\widehat{\mathcal{M}}_\varepsilon^{(0)}(t,q,k) := \frac{i}{\varepsilon^{1/2}}\sum_{\sigma=\pm1}\sigma\int_0^t\int[-\gamma(p)+ip\cdot k]^{-1}\hat{W}_0\left(s, q-\frac{p}{\varepsilon}, K_\sigma\right)\hat{B}(ds,dp) \tag{7.1}$$

and

$$\widehat{\mathcal{R}}_\varepsilon(t,q,k) := \frac{i}{\varepsilon^{1/2}}\sum_{m\ge1}\sum_{\sigma=\pm1}\sigma\int_0^t\int[-\gamma(p)+ip\cdot k]^{-1}\hat{W}_m\left(s, q-\frac{p}{\varepsilon}, K_\sigma\right)\hat{B}(ds,dp). \tag{7.2}$$

Correspondingly, $\langle \bar{Z}_\varepsilon(t), \theta\rangle = I_\varepsilon(t) + R_\varepsilon(t)$, where $I_\varepsilon(t) = \sum_{n\ge0} I_\varepsilon^{(n)}(t)$ with

$$\begin{aligned} I_\varepsilon^{(0)}(t) &:= \int_0^t\int e^{iq\cdot k(t-s)}\hat{\theta}^*(q,k)d\widehat{\mathcal{M}}_\varepsilon^{(0)}(s,q,k)dqdk, \\ I_\varepsilon^{(n)}(t) &= \int_0^t\int \hat{\theta}^*(q,k_0)\bar{\mathcal{V}}_n(t,s_{n+1},q,\mathbf{k}_{0,n})d\widehat{\mathcal{M}}_\varepsilon^{(0)}(s_{n+1},q,k_n)d\mathbf{k}_{0,n}dq. \end{aligned} \tag{7.3}$$

Likewise we let $R_\varepsilon(t) = \sum_{n\ge0} R_\varepsilon^{(n)}(t)$, where $R_\varepsilon^{(n)}(t)$ is defined by equation analogous to (7.3), in which martingale $\mathcal{M}_\varepsilon^{(0)}(t)$ should be replaced by $\mathcal{R}_\varepsilon(t)$.

Lemma 7.1 *We have*

$$\mathbb{E}R_\varepsilon^2(t) \le C\varepsilon \tag{7.4}$$

for some $C > 0$ and all $\varepsilon \in (0,1]$.

Proof. We have $R_\varepsilon(t) = \sum_{n\ge0,m\ge1} R_\varepsilon^{(n,m)}(t)$, where

$$\begin{aligned} R_\varepsilon^{(n,m)}(t) &:= \frac{i}{\varepsilon^{1/2}}\sum_{\sigma=\pm1}\sigma\int_0^t\int_{\Delta_n(t,s_{n+1})}\int_0^{+\infty}\ldots\int_0^{+\infty}\int\mathcal{G}(p,q,\mathbf{k},\mathbf{l},\tau,\mathbf{s}) \\ &\times d\mathbf{s}^{(n)}\hat{B}(ds_{n+1},dp)d\tau_{0,m}dl_{1,m}d\mathbf{k}_{0,n}dq. \end{aligned}$$

Here $l_0 := k_n + \sigma p/2$ and

$$
\begin{aligned}
&\mathcal{G}(p,q,\mathbf{k},\mathbf{l},\tau,\mathbf{s}) := \hat{\theta}^*(q,k_0)\exp\left\{i\sum_{j=0}^{n} q\cdot k_j(s_j - s_{j+1})\right\}\\
&\times[-\gamma(p)+ip\cdot k_n]^{-1}\prod_{i=1}^{n}[\sigma(k_{i-1},k_i)\Sigma(k_{i-1})]\exp\left\{-\sum_{i=0}^{n}\Sigma(k_i)(s_j - s_{j+1})\right\}\\
&\times\prod_{i=0}^{m-1}[\sigma(l_i,l_{i+1})\Sigma(l_i)]\exp\left\{-\sum_{j=0}^{m}\Sigma(l_j)\tau_j\right\}\exp\left\{-i\sum_{j=0}^{m}\left(q-\frac{p}{\varepsilon}\right)\cdot l_j\tau_j\right\}f(l_m)\\
&\times\delta\left(s_{n+1}-\sum_{j=0}^{m}\tau_j\right).
\end{aligned}
$$

We define

$$
\begin{aligned}
&J_\varepsilon(z) := \int \exp\{iq\cdot k_n(s_n - s_{n+1})\}[-\gamma(p)+ip\cdot k_n]^{-1}\\
&\times\hat{R}\left(\frac{1}{2}(|k_{n-1}|^2-|k_n|^2, k_{n-1}-k_n\right)\exp\{-\Sigma(k_n)(s_n-s_{n+1})\}\\
&\times\hat{R}\left(\frac{1}{2}(|l_1|^2-|l_0|^2, l_1-l_1\right)\exp\{-\Sigma(l_0)\tau_0\}\exp\{-iz\cdot l_0\}\,dk_n.
\end{aligned}
$$

Mimicking the argument used to obtain (6.37) we conclude that

$$
|J_\varepsilon(z)| \le \frac{C}{|z|^2+1} \tag{7.5}
$$

for some constant $C>0$. The second moment of $R_\varepsilon^{(n,m)}(t)$ equals

$$
\begin{aligned}
&\mathbb{E}[R_\varepsilon^{(n,m)}(t)]^2 = \frac{-1}{\varepsilon}\sum_{\sigma,\sigma'=\pm1}\sigma\sigma'\int_0^t ds_{n+1}\int_{D_n(t,s_{n+1})}\int_0^{+\infty}\dots\int_0^{+\infty}\int \mathcal{G}(p,q,\mathbf{k},\mathbf{l},\tau,s)\\
&\times\mathcal{G}(p',q',\mathbf{k}',\mathbf{l}',\tau',\mathbf{s}')dp\,dp'\,d\tau_{0,m}\,d\tau'_{0,m}\,dl_{1,m}\,dl'_{1,m}\,d\mathbf{k}_{0,n}\,d\mathbf{k}'_{0,n}\,dq\,dq'.
\end{aligned}
$$

This leads to the following estimate

$$
\begin{aligned}
&\mathbb{E}[R_\varepsilon^{(n,m)}(t)]^2 \le \frac{C^{n+m}}{\varepsilon}\int_0^t ds_{n+1}\int_{D_n(t,s_{n+1})} ds_{1,n}\\
&\times\left\{\int_{s_{n+1}\ge\sum_{i=1}^m\tau_i,\tau_i\ge0}\left[1+\left(\frac{s_{n+1}-\sum_{i=1}^m\tau_i}{\varepsilon}\right)^2\right]^{-1}d\tau_{1,m}\right\}^2.
\end{aligned}
$$

Changing variables $\tau_m := \tau_m/\varepsilon$ we conclude that

$$
\left\{\mathbb{E}[R_\varepsilon^{(n,m)}(t)]^2\right\}^{1/2} \le \frac{C^{n+m}\varepsilon^{1/2}}{m!n!},
$$

which in turn implies (7.4) □

From (7.1) we obtain that

$$
\begin{aligned}
I_\varepsilon^{(0)}(t) &:= \frac{i}{\varepsilon^{1/2}} \sum_{\sigma=\pm1} \sigma \int_0^t \int \mathrm{e}^{iq\cdot k(t-s)} \hat{\theta}^*(q,k) \\
&\times[-\gamma(p)+ip\cdot k]^{-1} \exp\left\{-is\left(q-\frac{p}{\varepsilon}\right)\cdot K_\sigma\right\} \mathrm{e}^{-\Sigma(K_\sigma)s} f(K_\sigma)\hat{B}(ds,dp)dqdk, \\
I_\varepsilon^{(n)}(t) &= \frac{i}{\varepsilon^{1/2}} \sum_{\sigma=\pm1} \sigma \int_0^t \int_{\Delta_n(t,s)} \int \hat{\theta}^*(q,k_0) \exp\left\{ i \sum_{j=0}^n q\cdot k_j(s_j - s_{j+1})\right\} \\
&\times[-\gamma(p)+ip\cdot k_n]^{-1} \prod_{i=1}^n [\sigma(k_{i-1},k_i)\Sigma(k_{i-1})] \exp\left\{-\sum_{i=0}^n \Sigma(k_i)(s_j - s_{j+1})\right\} \\
&\times \exp\left\{-is\left(q-\frac{p}{\varepsilon}\right)\cdot K_{n,\sigma}\right\} \mathrm{e}^{-\Sigma(K_{n,\sigma})s} f(K_{n,\sigma}) d\mathbf{s}^{(n)} \hat{B}(ds,dp) d\mathbf{k}_{0,n} dq.
\end{aligned}
$$

The following result holds.

Lemma 7.2 *There exists a sequence of non-negative numbers $\{C_n,\, n\geq 0\}$ such that*

$$
\mathbb{E}[I_\varepsilon^{(n)}(t)]^2 \leq C_n^2, \quad \forall\, \varepsilon \in (0,1],\, n \geq 0,
$$

and $\sum_{n\geq 0} C_n < +\infty$.

Proof. We have

$$
\begin{aligned}
\mathbb{E}|I_\varepsilon^{(n)}(t)|^2 &\leq \frac{C}{\varepsilon} \sum_{\sigma=\pm1} \int_0^t \int \left| \int_{\Delta_n(t,s)} \int \hat{\theta}^*(q,k_0) \exp\left\{ i\sum_{j=0}^{n-1} q\cdot k_j(s_j - s_{j+1})\right\}\right. \\
&\times \prod_{i=1}^{n-1} [\sigma(k_{i-1},k_i)\Sigma(k_{i-1})] \exp\left\{-\sum_{i=0}^{n-1} \Sigma(k_i)(s_j-s_{j+1})\right\} \\
&\times \left. J\left(s\left(q-\frac{p}{\varepsilon}\right)\right) d\mathbf{s}^{(n)} d\mathbf{k}_{0,n-1} dq \right|^2 \nu(dp)ds,
\end{aligned}
\tag{7.6}
$$

where

$$
\begin{aligned}
J(z) &:= \int\int [-\gamma(p)+ip\cdot k_n]^{-1} \exp\{-[\Sigma(k_n)+iq\cdot k_n](s_n - s)\} \\
&\times \hat{R}\left(\frac{1}{2}(|k_{n-1}|^2 - |k_n|^2), k_{n-1}-k_n\right) \exp\{-iz\cdot K_{n,\sigma}\} \mathrm{e}^{-\Sigma(K_{n,\sigma})s} f(K_{n,\sigma}) dk_n
\end{aligned}
$$

Since $J(z) \leq C(1+|z|^2)^{-1}$ for some constant $C>0$ we can estimate the right hand side of (7.6) by

$$
\frac{C^n}{n!\varepsilon} \int_0^t \int \left[\int |\hat{\theta}(q,k)| [1+|s(p/\varepsilon - q)|^2]^{-1} dqdk\right]^2 \nu(dp)ds,
$$

which after an application of Jensen's inequality and a subsequent change of variables $s' := s|p-\varepsilon q|/\varepsilon$ can be estimated by

$$
\frac{C^n \|\theta\|_{1,1}}{n!} \int_0^{t|p-\varepsilon q|/\varepsilon} \int\int |\hat{\theta}(q,k)|(1+s^2)^{-2} |p-\varepsilon q|^{-1} dqdk\nu(dp)ds \leq \frac{C_1^n \|\theta\|_{1,1}^2}{n!}
$$

for some $C_1 > 0$, cf (2.64). $\square$

Let $B_\varepsilon(t,dp) := \varepsilon^{1/2} B(t/\varepsilon, dp)$ and

$$X_\varepsilon(t,k) := i \sum_{\sigma=\pm1} \sigma \int_0^{t/\varepsilon} \int [-\gamma(p) + ip\cdot k]^{-1} \exp\{isp\cdot K_\sigma\} f(K_\sigma) \hat{B}_\varepsilon(ds,dp).$$

Define also

$$\tilde{I}_\varepsilon^{(0)}(t) := \int \mathrm{e}^{iq\cdot kt} \theta^*(q,k) X_\varepsilon(t,k) dq dk,$$

$$\tilde{I}_\varepsilon^{(n)}(t) = \int_{\Delta_n(t,0)} \int \theta^*(q,k_0) \exp\left\{ i \sum_{j=0}^n q\cdot k_j (s_j - s_{j+1})\right\}$$
$$\times \prod_{i=1}^n [\sigma(k_{i-1},k_i)\Sigma(k_{i-1})] \exp\left\{ -\sum_{i=0}^n \Sigma(k_i)(s_j - s_{j+1})\right\} X_\varepsilon(t,k_n) d\mathbf{k}_{0,n} dq.$$

Using an argument very similar to the one used to demonstrate Lemma 7.2 we can also conclude that there exists a sequence of non-negative numbers $\{C_n,\, n\ge 0\}$ such that

$$\mathbb{E}[\tilde{I}_\varepsilon^{(n)}(t)]^2 \le C_n^2, \quad \forall\, \varepsilon \in (0,1],\, n \ge 0 \tag{7.7}$$

and $\sum_{n\ge0} C_n < +\infty$. Moreover, we also have.

Lemma 7.3

$$\lim_{\varepsilon\to0+} \mathbb{E}[I_\varepsilon^{(n)}(t) - \tilde{I}_\varepsilon^{(n)}(t)]^2 = 0, \quad \forall\, n \ge 0. \tag{7.8}$$

Proof. Define

$$\mathcal{L}(k,l,\varrho) := \exp\{\varrho\{-\Sigma(l) + i[\xi - q\cdot(l+k)]\}\}.$$

A simple calculation shows that

$$\mathbb{E}[I_\varepsilon^{(n)}(t) - \tilde{I}_\varepsilon^{(n)}(t)]^2 \le \frac{1}{\varepsilon} \sum_{\sigma=\pm1} \int_0^t \int \left| \int_0^{+\infty} \ldots \int_0^{+\infty} \int [\mathcal{J}(k_n, K_{n,\sigma}, s) - 1] \right. \tag{7.9}$$
$$\times \exp\left\{ -i\xi(t - \sum_{i=0}^n \tau_i)\right\} \hat{\theta}^*(q,k_0) \exp\left\{ \sum_{i=0}^n [-\Sigma(k_i) + iq\cdot k_i]\tau_i \right\} [-\gamma(p) + ip\cdot k_n]^{-1}$$
$$\times \left. \prod_{i=1}^{n-1} [\sigma(k_{i-1},k_i)\Sigma(k_{i-1})]\ \exp\left\{ -i\frac{s}{\varepsilon} p\cdot K_{n,\sigma}\right\} f(K_{n,\sigma}) d\tau_{0,n} d\mathbf{k}_{0,n} dq d\xi \right|^2 ds\nu(dp).$$

Writing

$$\mathcal{L}(k,l,s) - 1 = \{-\Sigma(l) + i[\xi - q\cdot(l+k)]\} \int_0^s \exp\{\varrho\{-\Sigma(l) + i[\xi - q\cdot(l+k)]\}\} d\varrho.$$

and changing variables $s := s/\varepsilon$ we can rewrite the right hand side of (7.9) as being equal to

$$\varepsilon \sum_{\sigma=\pm1} \int_0^{t/\varepsilon} \int \left| \int_0^{+\infty} \ldots \int_0^{+\infty} \int_0^s \int (L_1 + L_2 + L_3) \mathcal{J}(ps) \right. \tag{7.10}$$
$$\times \exp\left\{ -i\xi(t - \sum_{i=0}^n \tau_i - \varepsilon\varrho)\right\} \hat{\theta}^*(q,k_0) \exp\left\{ \sum_{i=0}^{n-1} [-\Sigma(k_i) + iq\cdot k_i]\tau_i \right\}$$
$$\times \left. \prod_{i=1}^{n-1} [\sigma(k_{i-1},k_i)\Sigma(k_{i-1})]\ d\tau_{0,n} d\mathbf{k}_{0,n-1} dq d\xi d\varrho \right|^2 ds\nu(dp),$$

where

$$\mathcal{J}(z) := \int \exp\{[-\Sigma(k_n) + iq \cdot k_n]\tau_n\} [-\gamma(p) + ip \cdot k_n]^{-1} \exp\{-i\varepsilon\varrho q \cdot (k_n + K_{n,\sigma})\}$$
$$\times \hat{R}\left(\frac{1}{2}(|k_{n-1}|^2 - |k_n|^2), k_{n-1} - k_n\right) \exp\{-iz \cdot K_{n,\sigma}\} \mathrm{e}^{-\varepsilon\Sigma(K_{n,\sigma})\varrho} f(K_{n,\sigma}) dk_n$$

and $L_1 := -\Sigma(l)$, $L_2 := i\xi$, $L_3 := -iq \cdot (l + k)$. As in (6.37) we can argue that $|\mathcal{J}(z)| \le C(1 + |z|^2)^{-1}$ uniformly in all parameters, i.e. p, ε, k_{n-1}. From that we obtain that the expression in (7.10) is of order of magnitude $O(\varepsilon)$ and (7.8) follows. □

Let

$$\tilde{I}^{(0)}(t) := \int \mathrm{e}^{iq\cdot kt} \theta^*(q, k) X(k) dq dk,$$
$$\tilde{I}^{(n)}(t) = \int_{\Delta_n(t,0)} \int \theta^*(q, k_0) \exp\left\{ i \sum_{j=0}^{n} q \cdot k_j (s_j - s_{j+1}) \right\}$$
$$\times \prod_{i=1}^{n} [\sigma(k_{i-1}, k_i)\Sigma(k_{i-1})] \exp\left\{ - \sum_{i=0}^{n} \Sigma(k_i)(s_j - s_{j+1}) \right\} X(k_n) d\mathbf{k}_{0,n} dq,$$

where $X(k)$ is given by (2.58), and

In light of (7.7) to finish the proof of convergence in law of $\langle \bar{Z}_\varepsilon(t), \theta\rangle$ it suffices only to show that

$$\lim_{\varepsilon\to 0+} \mathbb{E}[\tilde{I}^{(n)}(t) - \tilde{I}^{(n)}_\varepsilon(t)]^2 = 0 \tag{7.11}$$

for each $n \ge 0$. Let

$$g_n(k_n) := \int_{\Delta_n(t,0)} \int \hat{\theta}^*(q, k_0) \exp\left\{ i \sum_{j=0}^{n} q \cdot k_j (s_j - s_{j+1}) \right\}$$
$$\times \prod_{i=1}^{n} [\sigma(k_{i-1}, k_i)\Sigma(k_{i-1})] \exp\left\{ - \sum_{i=0}^{n} \Sigma(k_i)(s_j - s_{j+1}) \right\} d\mathbf{s}^{(n)} d\mathbf{k}_{0,n-1}$$

A direct calculation shows that the expression under the limit can be estimated by

$$C \sum_{\sigma,\sigma'=\pm 1} \int \nu(dp) \left| \int_{t/\varepsilon}^{+\infty} \mathrm{e}^{is(\sigma-\sigma')|p|^2/2} g_{\sigma,\sigma'}(ps, p) ds \right|,$$

where

$$g_{\sigma,\sigma'}(q, p) := \int_{\mathbb{R}^{2d}} \mathrm{e}^{iq\cdot(k-k')} g_n(k_n) g_n(k'_n)$$
$$\times \left[(\gamma(p) - ip \cdot k_n)(\gamma(p) + ip \cdot k'_n)\right]^{-1} f(K_{n,\sigma}) f(K'_{n,\sigma'}) dk_n dk'_n.$$

Expression in (7.12) tends to 0, as $\varepsilon \to 0+$. This ends the proof of convergence in law of $\langle \bar{Z}_\varepsilon(t), \theta\rangle$.

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