Long time energy transfer in the random Schrödinger equation

Tomasz Komorowski* Lenya Ryzhik †

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Abstract

We consider the long time behavior of solutions of the d-dimensional linear Boltzmann equation that arises in the weak coupling limit for the Schrödinger equation with a time-dependent random potential. We show that the intermediate mesoscopic time limit satisfies a Fokker-Planck type equation with the wave vector performing a Brownian motion on the $(d-1)$-dimensional sphere of constant energy, as in the case of a time-independent Schrödinger equation. However, the long time limit of the solution with an isotropic initial data satisfies an equation corresponding to the energy being the square root of a Bessel process of dimension $d/2$.

1 Introduction

The linear Boltzmann equation of the form

$$\frac{\partial W}{\partial t} + k \cdot \nabla_x W = \int_{\mathbb{R}^d} R\left(|k'|^2 - |k|^2, k' - k\right)\left[W(x,k') - W(x,k)\right] dk' \tag{1.1}$$

appears as the weak coupling limit of the random Schrödinger equation as follows. Consider the Schrödinger equation with a weak time-dependent random potential

$$i\partial_t \psi + \frac{1}{2}\Delta\psi - \sqrt{\sigma} V\left(t,x\right)\psi = 0. \tag{1.2}$$

The Wigner transform of the field ψ is defined as

$$W_\sigma(t,x,k) = \int_{\mathbb{R}^d} e^{ik\cdot y} \psi_\sigma\left(t, x - \frac{\sigma y}{2}\right) \psi_\sigma^*\left(t, x + \frac{\sigma y}{2}\right) \frac{dy}{(2\pi)^d}, \tag{1.3}$$

where $\psi_\sigma(t,x) = \psi(t/\sigma, x/\sigma)$. Under appropriate assumptions on mixing properties of the potential field, the expected value of the Wigner transform $\mathbb{E}W_\sigma(t)$ converges, as $\sigma \downarrow 0$, as a Schwartz distribution, to the solution of the linear transport equation (1.1). The function $R(\omega,p)$ is the power spectrum of the random fluctuations $V(t,x)$ (see the review paper [1] and references therein for more background). In the particular case when $V(t,x)$ is Markovian and isotropic we have

$$R(\omega,p) = \frac{2\gamma(|p|)\hat{R}(|p|)}{\omega^2 + \gamma^2(|p|)}. \tag{1.4}$$

Here, $\gamma(|p|)$ is the mixing rate, and $\hat{R}(|p|)$ is the spatial power spectrum of the random fluctuations. Two extreme cases correspond to either $V(t,x)$ being a white noise process in time when $R(\omega,p) =$

*IMPAN, ul. Śniadeckich 8, 00-956 Warsaw, Poland, e-mail: komorow@hektor.umcs.lublin.pl
†Department of Mathematics, Stanford University, Stanford, CA 94305, USA, e-mail: ryzhik@math.stanford.edu

$\hat{R}(|p|)$, or time-independent potential $V(x)$, when $R(\omega,p)=\hat{R}(|p|)\delta(\omega)$. In the former case one can show that the long time limit of $W(t,x,k)$ is the solution of the Fokker-Planck equation

$$\frac{\partial W}{\partial t}+k\cdot\nabla_x W=D\Delta_k W,$$

with an explicitly computable diffusion coefficient D. In the latter case, the limit satisfies the Fokker-Planck equation

$$\frac{\partial W}{\partial t}+k\cdot\nabla_x W=D\nabla_k\cdot[(I-\hat{k}\otimes\hat{k})\nabla_k W],$$

which is the Kolmogorov equation for a diffusion whose k-component is a Brownian motion on the sphere $\{|k|=\text{const}\}$, the x-component is its time integral, and $\hat{k}=k/|k|$. These regimes are discussed in detail in [3].

In the present paper, we are interested in the long time limit of the solutions of the linear Boltzmann equation (1.1) away from these two extremes. We assume that

$$\hat{R}(\cdot)\in\mathcal{S}(\mathbb{R})\quad\text{and}\quad\gamma(\cdot)\in C^1(\mathbb{R})\quad\text{are both even and non-negative,}\tag{1.5}$$

and suppose furthermore that

$$0<\gamma_0\le\gamma(\ell)\le\frac{1}{\gamma_0},\quad\forall\,\ell\in\mathbb{R}\tag{1.6}$$

for some $\gamma_0>0$. The uniform bounds on $\gamma(|p|)$ separate this regime both from "white-noise in time potentials" ($\gamma\to+\infty$) and the "frozen in time potentials" ($\gamma\downarrow 0$).

The linear Boltzmann equation (1.1) is the Kolmogorov equation for the process $(X_t(x,k),K_t(k))$, where $K_t(k)$ is a jump process, starting at k, with the generator given by

$$\mathcal{L}f(k)=\int_{\mathbb{R}^d}\frac{2(\gamma\hat{R})(|k'-k|)}{(k'^2-k^2)^2+\gamma^2(|k'-k|)}\left[f(k')-f(k)\right]dk,\tag{1.7}$$

defined on Borel measurable functions $f(\cdot)$ having at most polynomial growth, and

$$X_t(x,k)=x+\int_0^t K_s(k)ds.\tag{1.8}$$

The questions about the scaled limits of the solutions of the equation (1.1) can be phrased in terms of the long time limits of such processes, and we will use this tool extensively.

As the above jump process does not preserve $|K_t(k)|$ we expect large k's to become important, and accordingly consider k of size ε^{-1}, where $\varepsilon\ll 1$ is a small parameter. In our first result, see Theorem 2.5 below, we show that the scaled processes $\varepsilon K_{t/\varepsilon^3}(k/\varepsilon)$ converge in law to a diffusion with generator L, which is, up to a constant factor, the Laplace-Beltrami operator on the sphere of radius $|k|$ – the same result as for a time-independent potential. That is, on the time scale $O(\varepsilon^{-3})$ solutions of the linear Boltzmann equation that arises from the Scrödinger equation with a time-dependent random potential behave as those coming from time-independent potentials in the Schrödinger equation. As a consequence, we have the following.

Theorem 1.1 *Suppose that $W(t,x,k)$ satisfies* (1.1) *with the initial data of the form $W(0,x,k)=W_0(x,\varepsilon k)$, $(x,k)\in\mathbb{R}^{2d}$, where $\operatorname{supp}W_0\subset\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$ and $d\ge 2$. Then, we have*

$$\lim_{\varepsilon\downarrow 0}W\left(\frac{t}{\varepsilon^3},\frac{x}{\varepsilon^4},\frac{k}{\varepsilon}\right)=\bar{W}(t,x,k),$$

where

$$\partial_t \bar{W}(t,x,k) + k \cdot \nabla_x \bar{W}(t,x,k) = L\bar{W}(t,x,k), \qquad (1.9)$$
$$\bar{W}(0,x,k) := W_0(x,k),$$

and

$$Lf(x,k) := \nabla_k \cdot \Big[\frac{b}{|k|}\Big(I - \hat{k}\otimes\hat{k}\Big)\cdot\nabla_k f(x,k)\Big], \quad \forall f \in C^2(\mathbb{R}^{2d}), \qquad (1.10)$$

with

$$b := \frac{\pi}{2(d-1)}\int_{\mathbb{R}^{d-1}} |\bar{p}|^2 \hat{R}(|\bar{p}|)d\bar{p}. \qquad (1.11)$$

The fact that the process corresponding to L is restricted to a sphere $\{|k| = \text{const}\}$ reflects the fact that, on the time scale $t \sim O(\varepsilon^{-3})$, the process $\varepsilon K_t(k/\varepsilon)$ keeps its absolute value constant. However, unlike for time-independent potentials, on a longer time scale its magnitude does evolve, which is captured by the following limit theorem for times $t \sim O(\varepsilon^{-4})$.

Theorem 1.2 *Suppose that the assumptions of Theorem* 1.1 *are in force and the initial data is of the form* $W(0,x,k) = W_0(x,\varepsilon|k|)$. *Then*

$$\lim_{\varepsilon\downarrow 0} W\left(\frac{t}{\varepsilon^4}, \frac{x}{\varepsilon^{9/2}}, \frac{k}{\varepsilon}\right) = \bar{W}(t,x,|k|),$$

where $\bar{W}(t,x,\ell) = \bar{V}(t,x,\ell^{1/4})$ *and*

$$\partial_t \bar{V}(t,x,\ell) = \widehat{\mathfrak{L}}\bar{V}(t,x,\ell), \quad (t,x,\ell) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+,$$
$$\bar{V}(0,x,\ell) = W_0(x,\ell^4), \quad (x,\ell) \in \mathbb{R}^d \times \mathbb{R}_+. \qquad (1.12)$$

Here $\mathbb{R}_+ := (0,+\infty)$,

$$\widehat{\mathfrak{L}}f(x,\ell) := \mathfrak{M}f(x,\ell) + \frac{\ell^{5/4}}{2bd}\Delta_x f(x,\ell), \qquad (1.13)$$

and

$$\mathfrak{M}f(x,\ell) := dc\partial_\ell f(x,\ell) + 4c\ell\partial_\ell^2 f(x,\ell), \quad \forall\, f \in C^\infty(\mathbb{R}^{d+1}), \qquad (1.14)$$

where

$$c := \int_{\mathbb{R}^d} (\gamma\hat{R})(|p|)dp. \qquad (1.15)$$

It is immediate to see that the operator $\mathfrak{M}$ is the generator of the random process $\mathfrak{Z}_t(\ell)$ which is a simple time change of the square of a Bessel process [7]: $\mathfrak{Z}_t(\ell) = \mathfrak{b}_{2ct}(\ell)$. Recall that the square of a δ-dimensional Bessel process [7] is a unique pathwise solution of the stochastic differential equation (see e.g. Theorem IX.3.4 of [7])

$$\mathfrak{b}_t = \ell + \delta t + 2\int_0^t \sqrt{|\mathfrak{b}_s|}dw_s. \qquad (1.16)$$

In our case $\delta := d/2$. It is well known, see, for instance, (ii), p. 442 of [7], that when $\delta \geq 2$, which in our situation corresponds to $d \geq 4$, the square of the Bessel process does not attain the value 0 if it starts at $\ell > 0$. When $d = 2$, or 3 we have $\delta = 1$, or $3/2$, respectively, so the limiting diffusion is recurrent and enters 0 infinitely many times. However $L_t(0)$ – its occupation time at 0, up to t, – vanishes, see [7], Proposition XI.1.5, p. 442.

The full generator $\widehat{\mathfrak{L}}$ corresponds to a diffusion $(\mathfrak{b}_{2ct}(\ell), \mathfrak{X}_t(x,\ell))$, where

$$\mathfrak{X}_t(x,\ell) = x + \left(\frac{1}{bd}\right)^{1/2} \int_0^t \mathfrak{b}_{2cs}^{5/8}(\ell) dW(s) \tag{1.17}$$

and $W(t)$ is a d-dimensional, standard Brownian motion. Thanks to uniqueness of solutions of the S.D.E. system given by (1.16) and (1.17) we conclude that the corresponding martingale problem is well-posed in the sense of [9], see Corollary 8.1.6, p. 202 of ibid. As a result the Cauchy problem (1.12) has at most one solution in $C_b^{1,2,2}(\bar{\mathbb{R}}_+ \times \mathbb{R}^d \times \bar{\mathbb{R}}_+)$, and we do not need to impose a boundary condition at $\ell = 0$.

By a direct calculation it is easy to see that $\bar{W}(t,x,\ell)$ itself satisfies the Cauchy problem

$$\begin{aligned} &\partial_t \bar{W}(t,x,\ell) = \mathfrak{L}\bar{W}(t,x,\ell), \quad (t,x,\ell) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+, \\ &\bar{W}(0,x,\ell) = W_0(x,\ell), \quad (x,\ell) \in \mathbb{R}^d \times \mathbb{R}_+, \end{aligned} \tag{1.18}$$

where

$$\mathfrak{L}f(x,\ell) := Mf(x,\ell) + \frac{\ell^5}{2bd}\Delta_x f(x,\ell), \tag{1.19}$$

and M is the generator of the limiting process for $\varepsilon|K_{t/\varepsilon^4}(k/\varepsilon)|$, see Theorem 3.1 below, given by

$$Mf(\ell) := \frac{(d-3)c}{4\ell^3}\partial_\ell f(\ell) + \frac{c}{4\ell^2}\partial_\ell^2 f(\ell). \tag{1.20}$$

Since the generator $\mathcal{L}$ of $K_t(k)$ is symmetric, i.e.

$$\int_{\mathbb{R}^d} \mathcal{L}f(k)g(k)dk = \int_{\mathbb{R}^d} f(k)\mathcal{L}g(k)dk,$$

for all f,g bounded, the generator of the limiting process needs also to satisfy symmetry condition, i.e. given an f from the domain of M, we should have

$$\int_0^{+\infty} Mf(\ell)g(\ell)\ell^{d-1}d\ell = \int_0^{+\infty} f(\ell)Mg(\ell)\ell^{d-1}d\ell, \quad \forall\, g, \in C^2[0,+\infty).$$

This forces the boundary condition $f'(0) = 0$ in dimensions $d = 2, 3$, otherwise there is no need to impose the boundary condition. For this reason in case $d = 2$, or 3 the Cauchy problem (1.18) should be supplemented with the Neumann boundary condition

$$\partial_\ell \bar{W}(t,x,0) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2,$$

unlike in the Cauchy problem for $\bar{V}(t,x,\ell)$ for which no boundary condition is required at $\ell = 0$. In dimensions $d \geq 4$ we do not need to impose any boundary condition in (1.18). The advantage of stating the result in terms of $\bar{V}(t,x,\ell)$, rather than for $\bar{W}(t,x,\ell)$ directly, is that the generator for $\bar{V}$ is not singular at $\ell = 0$ and many of the technical issues one would need to address working with $\bar{W}$ simply do not arise.

Let us also comment on the spatial diffusion coefficient in (1.19). Recall that the original unscaled process satisfies (1.8): $\dot{X}_t = K_t$. As K_t becomes equidistributed over the sphere, the directional information in the speed of X_t is lost and X_t converges to a Brownian motion. However, $|K_t|$, which is the magnitude of $|\dot{X}_t|$, is a non-trivial process on the time scales $O(\varepsilon^{-4})$, which reflects itself in the fact that the diffusion coefficient for the Brownian motion in x grows in $\ell = |K|$.

The paper is organized as follows. As we have already hinted in the foregoing the proofs of Theorems 1.1 and 1.2 are based on probabilistic arguments. The proof of the first result, which is much simpler, is contained in Section 2. In Section 3 we consider the energy evolution on time scale ε^{-4} arriving at the proof of the spatially homogeneous version of Theorem 1.2. The extension to the inhomogeneous situation is done in Section 4. Section 5 contains technical results that are used in the proofs of both theorems.

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2 Proof of Theorem 1.1

2.1 Outline of the proof of Theorem 1.1

Let us first briefly sketch the proof. Recall that (1.1) is the Kolmogorov equation for a process (X_t, K_t), where $\{K_t,\, t \ge 0\}$ is a jump process with the generator (1.7), and X_t given by (1.8) is its time integral. The jump process starts at k, with jumps occurring at renewal times $\{T_n,\, n \ge 0\}$. More precisely, the skeleton chain given by $B_n(k) := K_t$ for $t \in [T_n, T_{n+1})$ is Markov with the transition probability densities

$$P(k,k') := \frac{1}{\Sigma(|k|)} \cdot \frac{2\gamma(|k'-k|)\hat{R}(|k'-k|)}{\gamma^2(|k'-k|) + [(k'-k)\cdot(k'+k)]^2}.$$

Here the total scattering cross-section is

$$\Sigma(r) := \int_{\mathbb{R}^d} \frac{2\gamma(|p|)\hat{R}(|p|)dp}{\gamma^2(|p|) + [p\cdot(p+2r\hat{k})]^2}, \tag{2.1}$$

and $\hat{k} := k/|k|$. We omit writing the starting point k in the notation when its value is obvious from the context. Given the chain $\{B_n(k),\, n \ge 0\}$ the times $T_{n+1} - T_n$, are independent and exponentially distributed according to $\exp(\Sigma(B_n))$, that is,

$$P(T_{n+1} - T_n > s | B_n) = e^{-\Sigma(B_n)s}.$$

The generator of the rescaled process

$$K_t^{(\varepsilon)}(k) := \varepsilon K_{t/\varepsilon^3}(k/\varepsilon) \tag{2.2}$$

is given by

$$\mathcal{L}_\varepsilon f(k) := \int_{\mathbb{R}^d} Q^{(\varepsilon)}(k,k')[f(k') - f(k)]dk', \tag{2.3}$$

with

$$Q^{(\varepsilon)}(k,k') := \frac{1}{\varepsilon^{d+3}} \cdot \frac{2\gamma(\varepsilon^{-1}|k'-k|)\hat{R}(\varepsilon^{-1}|k'-k|)}{\gamma^2(\varepsilon^{-1}|k'-k|) + \varepsilon^{-4}[(k'-k)\cdot(k'+k)]^2}. \tag{2.4}$$

It is defined for all measurable functions $f(\cdot)$ with polynomial growth. The corresponding Markov chain is given by $\{\varepsilon B_n,\, n \ge 0\}$. Its transition probability densities are equal

$$P^{(\varepsilon)}(k,k') := \frac{1}{\varepsilon^d \Sigma\left(\varepsilon^{-1}|k|\right)} \cdot \frac{2\gamma(\varepsilon^{-1}|k'-k|)\hat{R}(\varepsilon^{-1}|k'-k|)}{\gamma^2(\varepsilon^{-1}|k'-k|) + \varepsilon^{-4}[(k'-k)\cdot(k'+k)]^2}. \tag{2.5}$$

The main ingredient in the proof of Theorem 1.1 is Lemma 2.4 below which shows that the generator $\mathcal{L}_\varepsilon f(k)$ tends to $Lf(k)$ (given by (1.10)) uniformly on annuli

$$\mathcal{A}(\rho) := [\rho \le |k| \le \rho^{-1}] \text{ for } \rho \in (0,1). \tag{2.6}$$

This convergence of generators allows us both to conclude tightness of the laws of $\{K_t^{(\varepsilon)}(k),\, t \geq 0\}$, as $\varepsilon \downarrow 0$ by using Theorem VI.4.13, p. 358 of [4], as well as identify the limit since the limiting martingale problem for the operator L is well-posed. Convergence of X_t to the correct limit is an easy consequence.

2.2 Estimates on the cross-section and moments

In order to prove convergence of generators we need, in particular, to compute the limit of the total scattering cross-section and estimate some moments of the jump process – those will be used to bound the remainder error terms, and this is what we do in this section.

The limiting cross-section

We first compute the asymptotics of the cross-section for large values of k. For sake of brevity, for any $p \in \mathbb{R}^d$ we write $p = p_1\hat{k} + \bar{p}$ where $\bar{p} = (I - \hat{k} \otimes \hat{k})p$. Let us denote also

$$\mathcal{I}(r) := \sup_{q \geq r} \int_{\mathbb{R}^{d-1}} (1 + |\bar{p}|^2)(\gamma\hat{R})\left((q^2 + |\bar{p}|^2)^{1/2}\right) d\bar{p}. \tag{2.7}$$

From (1.6) and the assumption that $\hat{R} \in \mathcal{S}(\mathbb{R})$ we conclude that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-N} \mathcal{I}(1/\varepsilon) = 0, \tag{2.8}$$

for any $N > 0$.

Lemma 2.1 *There exists $C, \delta > 0$ such that*

$$|\varepsilon^{-1}\Sigma\left(\varepsilon^{-1}\right) - \pi\Sigma_0| \leq C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0, 1], \tag{2.9}$$

with

$$\Sigma_0 := \int_{\mathbb{R}^{d-1}} \hat{R}(|\bar{p}|)d\bar{p}. \tag{2.10}$$

Proof. Let us substitute $p_1' := 2p_1/\varepsilon$ and $\bar{p}' := \bar{p}$. Then,

$$\varepsilon^{-1}\Sigma\left(\varepsilon^{-1}\right) = \int_{\mathbb{R}^d} \frac{(\gamma\hat{R})(\{(\varepsilon p_1)^2 + |\bar{p}|^2\}^{1/2})dp}{\gamma^2(\{(\varepsilon p_1)^2 + |\bar{p}|^2\}^{1/2}) + (|\bar{p}|^2 + (\varepsilon p_1)^2 + p_1)^2}. \tag{2.11}$$

Suppose that $\kappa \in (0, 1)$. The right hand side can be written as $\sum_{i=1}^3 J_\varepsilon^{(i)}$, with each $J_\varepsilon^{(i)}$, $i = 1, 2, 3$ corresponding to the regions of integration $\{|p_1| \leq \varepsilon^{-1-\kappa}\}$, $\{\varepsilon^{-1-\kappa} < |p_1| \leq 2\varepsilon^{-2}\}$ and $\{2\varepsilon^{-2} < |p_1|\}$, respectively.

In the first region we change variables $p_1' = \psi_\varepsilon(p_1) := p_1 + \varepsilon^2 p_1^2$ and $p_i' := p_i$, $i = 2, \ldots, d$. Note that $|\psi_\varepsilon'(p_1) - 1| \leq 2\varepsilon^{1-\kappa}$ for $|p_1| \leq \varepsilon^{-1-\kappa}$. Then

$$J_\varepsilon^{(1)} = \int_{\mathbb{R}^{d-1}} \int_{-\varepsilon^{-1-\kappa}+\varepsilon^{-2\kappa}}^{\varepsilon^{-1-\kappa}+\varepsilon^{-2\kappa}} \frac{(\gamma\hat{R})(\{[\varepsilon\psi_\varepsilon^{-1}(p_1)]^2 + |\bar{p}|^2\}^{1/2})|(\psi_\varepsilon^{-1})'(p_1)|dp_1}{\gamma^2(\{[\varepsilon\psi_\varepsilon^{-1}(p_1)]^2 + |\bar{p}|^2\}^{1/2}) + (|\bar{p}|^2 + p_1)^2} d\bar{p}.$$

Let also

$$J^{(1)} := \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{(\gamma\hat{R})(|\bar{p}|)dp_1 d\bar{p}}{\gamma^2(|\bar{p}|) + (|\bar{p}|^2 + p_1)^2}. \tag{2.12}$$

Since $\kappa \in (0,1)$ and $\hat{R} \in \mathcal{S}(\mathbb{R})$ we obtain

$$|J_\varepsilon^{(1)} - J^{(1)}| \le C\varepsilon^{1+\delta} \tag{2.13}$$

for some $C, \delta > 0$ and all $\varepsilon \in (0,1]$. We may now integrate in p_1 in (2.12) using an elementary formula

$$\int_{\mathbb{R}} \frac{A dx}{A^2 + (B+x)^2} = \pi, \quad \forall A > 0,\ B \in \mathbb{R}$$

and obtain that the right hand side of (2.12) equals $\pi\Sigma_0$. Note that $|J_\varepsilon^{(2)}|$ can be bounded from above by $C\varepsilon^{-2}\mathcal{I}(1/\varepsilon^\kappa)$ that vanishes, as $\varepsilon \downarrow 0$, due to (2.8). Finally, we have

$$|J_\varepsilon^{(3)}| \le C\mathcal{I}(1/\varepsilon^\kappa) \int_{|p_1| \ge 2\varepsilon^2} \frac{dp_1}{1 + (\varepsilon p_1)^2}.$$

Both of these terms can be estimated by $C\varepsilon^{1+\delta}$ for some $C, \delta > 0$. Summarizing, we have proved (2.9).□

Combining (2.1) with (2.9) we conclude the following.

Corollary 2.2 *There exists $C > 0$ such that*

$$\Sigma(r) \le \frac{C}{1+r}, \quad \forall r \ge 0. \tag{2.14}$$

Estimates of the moments of jumps

Next, we obtain the asymptotics of the following moments:

$$\hat{p}_\varepsilon^{(n)}(k) := \int_{\mathbb{R}^d} |k - k'|^n P^{(\varepsilon)}(k, k') dk'$$

and

$$\hat{q}_\varepsilon^{(n)}(k) := \int_{\mathbb{R}^d} |k - k'|^n Q^{(\varepsilon)}(k, k') dk',$$

for any $n \ge 0$, $\varepsilon > 0$. As we have mentioned, they will be needed to estimate the error terms while proving the convergence of the respective Dynkin's martingales. The following lemma identifies the order of magnitude for these moments and will be used also in the proof of Theorem 1.2. The uniform bounds (2.15)-(2.16) will be mostly useful in the proof of Theorem 1.2, and even there primarily in dimension two, when it is possible for the particle to reach zero. On the other hand, (2.17)-(2.18) are the key to the error estimates for both Theorem 1.1 and Theorem 1.2.

Lemma 2.3 *For any $n \ge 3$ we have*

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{n-1}} \sup_k \hat{p}_\varepsilon^{(n)}(k) < +\infty, \tag{2.15}$$

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{n-3}} \sup_k \hat{q}_\varepsilon^{(n)}(k) < +\infty, \tag{2.16}$$

and, in addition, for any $\rho \in (0,1)$ we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{n-1}} \sup_{k \in \mathcal{A}(\rho)} \hat{p}_\varepsilon^{(n)}(k) = 0 \tag{2.17}$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{n-3}} \sup_{k \in \mathcal{A}(\rho)} \hat{q}_\varepsilon^{(n)}(k) = 0. \tag{2.18}$$

Here $\mathcal{A}(\rho)$ is the annulus (2.6).

Proof. First we show (2.16) and (2.18). Using (2.5) and Lemma 2.1 we get, for any $\rho \in (0,1)$ fixed

$$\frac{1}{\varepsilon^{n-3}} \sup_{k \in \mathcal{A}(\rho)} \hat{q}_\varepsilon^{(n)}(k) = \sup_{k \in \mathcal{A}(\rho)} \frac{1}{\varepsilon^{n-3}} \int_{\mathbb{R}^d} |k'-k|^n Q^{(\varepsilon)}(k,k')dk' \le C \sup_{k \in \mathcal{A}(\rho)} \int_{\mathbb{R}^d} \frac{|p|^n (\gamma \hat{R})(|p|)dp}{\gamma_0^2 + (|p|^2 + 2|k|p_1/\varepsilon)^2}.$$

Formula (2.18) then follows from the Lebesgue dominated convergence theorem.

On the other hand, we also have, now for any $k \in \mathbb{R}^d$,

$$\frac{1}{\varepsilon^{n-3}} \sup_{k \in \mathbb{R}^d} \hat{q}_\varepsilon^{(n)}(k) \le \frac{C}{\gamma_0^2} \int_{\mathbb{R}^d} |p|^n (\gamma \hat{R})(|p|)dp$$

and (2.16) follows.

From Corollary 2.2 we conclude that

$$\hat{p}_\varepsilon^{(n)}(k) \le C\varepsilon^2(|k|+1)\hat{q}_\varepsilon^{(n)}(k), \quad \forall\, k \in \mathbb{R}^d,\, \varepsilon \in (0,1]. \tag{2.19}$$

Hence (2.17) follows immediately from (2.18). Next, we deduce from (2.19) and (2.16) that

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{n-1}} \sup_{|k| \le 1} \hat{p}_\varepsilon^{(n)}(k) < +\infty \tag{2.20}$$

To finish the proof of (2.15) it suffices to show that we have

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{n-1}} \sup_{|k| \ge 1} \hat{p}_\varepsilon^{(n)}(k) \le \limsup_{\varepsilon \downarrow 0} \sup_{|k| \ge 1} \int_{\mathbb{R}^d} \frac{C|k|g(|p|)dp}{\gamma_0^2 + (|p|^2 + 2|k|p_1/\varepsilon)^2} = 0. \tag{2.21}$$

Here we have defined the non-negative function $g(p) = |p|^n(\gamma\hat{R})(|p|)$. This function satisfies, for each $N \in \mathbb{N}$,

$$C_N := \sup_{r \ge 0}(1 + r^N)g(r) < +\infty, \tag{2.22}$$

which is the reason why (2.21) holds. To see that (2.22) implies (2.21), divide the domain of integration into two regions $R_1 := \{|p_1| \ge p(k,\varepsilon)\}$, where $p(k,\varepsilon) := |k|/(2\varepsilon)$, and its complement R_2. The expression under supremum in (2.21) corresponding to the integration over R_1 can be estimated by

$$C|k| \int_{|p_1| \ge p(k,\varepsilon)} g(|p|)dp \le \frac{C|k|}{p(k,\varepsilon)+1}, \quad \forall\, \varepsilon \in (0,1],\, k \in \mathbb{R}^d,$$

for some $C > 0$, thanks to (2.22), provided we use this condition with a sufficiently large N. Therefore, we have

$$\limsup_{\varepsilon \downarrow 0} \sup_{|k| \ge 1} \int_{R_1} \frac{|k|g(|p|)dp}{\gamma_0^2 + (|p|^2 + 2|k|p_1/\varepsilon)^2} = 0. \tag{2.23}$$

For the integral over R_2 we use the change of variables $p_1' := \psi(p_1)$, where $\psi(p_1) = p_1^2 + 2|k|p_1/\varepsilon$. Note that $|\psi'(p_1)| \ge |k|/\varepsilon$, for $|p_1| < p(k,\varepsilon)$, hence the respective integral equals

$$\int_{\psi(-p(k,\varepsilon))}^{\psi(p(k,\varepsilon))} \frac{|k|dp_1}{|\psi'(\psi^{-1}(p_1))|} \int_{\mathbb{R}^{d-1}} \frac{g\left(\{[\psi^{-1}(p_1)]^2 + |\bar{p}|^2\}^{1/2}\right)d\bar{p}}{\gamma_0^2 + (|\bar{p}|^2 + p_1)^2}.$$

From (2.22) the expression above can be estimated by

$$\varepsilon \int_{\mathbb{R}^d} \frac{C_{d+1}dp_1 d\bar{p}}{[\gamma_0^2 + (|\bar{p}|^2 + p_1)^2](1 + |\bar{p}|^{d+1})}.$$

Therefore, we also have

$$\lim_{\varepsilon \downarrow 0} \sup_{|k| \ge 1} \int_{R_2} \frac{|k|g(|p|)dp}{\gamma_0^2 + (|p|^2 + 2|k|p_1/\varepsilon)^2} = 0, \tag{2.24}$$

and (2.21) follows. □

2.3 Asymptotics of the generator $\mathcal{L}_\varepsilon$

Lemma 2.3 shows that the higher moments with respect to the transition probability measure $Q^{(\varepsilon)}(k,k')$ vanish as $\varepsilon \to 0$, at least on the time scale of Theorem 1.1. We will now compute the asymptotics of the lower moments that are important in showing the convergence of generators, both for the proof of Theorem 1.1 and Theorem 1.2: set

$$b_\varepsilon(k) = \int_{\mathbb{R}^d} (k'-k)Q^{(\varepsilon)}(k,k')dk', \tag{2.25}$$

$$a_\varepsilon(k) = \int_{\mathbb{R}^d} (k'-k)^{\otimes 2}Q^{(\varepsilon)}(k,k')dk', \tag{2.26}$$

and

$$d_\varepsilon(k) = \int_{\mathbb{R}^d} (k'-k)^{\otimes 3}Q^{(\varepsilon)}(k,k')dk'. \tag{2.27}$$

We also define

$$b(k) = -\frac{(d-1)b}{|k|^2}\hat{k}, \quad \bar{b}(\ell) = \frac{(d-2)c}{2\ell^3} \tag{2.28}$$

and

$$a(k) = \frac{2b}{|k|}(I - \hat{k}\otimes\hat{k}), \quad \bar{a}_1(\ell) := \frac{c}{2\ell^2}, \quad \bar{a}_2(\ell) := -\frac{c}{2\ell^2}, \quad \forall k \in \mathbb{R}^d,\ \ell \in (0,+\infty).$$

The crucial ingredient in the proof of both our main results is the following asymptotics for the moments. Since its demonstration relies on rather tedious computations, that might distract the attention of a reader from the main points of the remaining part of the argument, we postpone its presentation till Section 5.

Lemma 2.4 *For any $\rho \in (0,1)$ we have*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \sup_{k\in\mathcal{A}(\rho)} \left| b_\varepsilon(k) - b(k) - \varepsilon\bar{b}(|k|)\hat{k} \right| = 0, \tag{2.29}$$

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \sup_{k\in\mathcal{A}(\rho)} \left| a_\varepsilon(k) - a(k) - \varepsilon\left\{ \bar{a}_1(|k|)\hat{k}\otimes\hat{k} + \bar{a}_2(|k|)\left[I - \hat{k}\otimes\hat{k}\right]\right\} \right| = 0, \tag{2.30}$$

and

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \sup_{k\in\mathcal{A}(\rho)} |d_\varepsilon(k)| = 0. \tag{2.31}$$

The asymptotics of the first moment $Q^{(\varepsilon)}(k,k')$ and its covariance matrix will allow us to identify the drift coefficient and diffusivity matrix of the diffusion which is the limit of $K_t^{(\varepsilon)}(k)$. The operator L appearing in Theorem 1.1 is the generator of this diffusion and has the form

$$Lf(k) = \frac{1}{2}a(k)\cdot\nabla_k^2 f + b(k)\cdot\nabla_k f. \tag{2.32}$$

2.4 Convergence of the momentum process

We now use Lemma 2.4, the results contained of Chapter VI of [4] and Chapter 7 of [9] to prove convergence of the rescaled jump process $K_t^{(\varepsilon)}(k)$ defined by (2.2).

Theorem 2.5 *The process $\{K_t^{(\varepsilon)}(k),\, t\geq 0\}$ converges in law over $D[0,+\infty)$, as $\varepsilon\downarrow 0$ to a diffusion $\{k_t(k),\, t\geq 0\}$ starting at k and with the generator L given by (1.10).*

Proof. Fix $\rho \in (0,1)$ and, in order to remove the singularity at $k=0$ of the generator (1.10), let

$$L^{(\rho)} f(k) := \phi^{(\rho)}(|k|) Lf(k), \quad f \in C_0^2(\mathbb{R}^d). \tag{2.33}$$

Here $\phi^{(\rho)} \in C_0^\infty(0,+\infty)$ is a C^∞ function that satisfies

$$0 \le \phi^{(\rho)} \le 1,\ \phi^{(\rho)}(x) \equiv 1 \text{ for } \rho^{-1} \ge x \ge \rho,\ \phi^{(\rho)}(x) \equiv 0 \text{ for } x \in [0,\rho/2] \cup [2/\rho,+\infty). \tag{2.34}$$

The operator $L^{(\rho)}$ is a generator of a regular diffusion on $\mathbb{R}^d$. We denote by $k_t^{(\rho)}(k)$ the respective diffusion starting at k. We also denote by $K_t^{(\varepsilon,\rho)}(k)$ the jump process, starting at k with the generator

$$\mathcal{L}_{\varepsilon,\rho} f(k) = \int_{\mathbb{R}^d} Q^{(\varepsilon,\rho)}(k,k')[f(k') - f(k)]dk' \tag{2.35}$$

defined for all measurable $f(\cdot)$ that are of a polynomial growth. The scattering kernel

$$Q^{(\varepsilon,\rho)}(k,k') = \phi^{(\rho)}(|k|) Q^{(\varepsilon)}(k,k'), \tag{2.36}$$

satisfies the analogues of (2.29)–(2.31) with coefficients $b_\varepsilon^{(\rho)}(k)$, $a_\varepsilon^{(\rho)}(k)$ and $d_\varepsilon^{(\rho)}(k)$ defined by formulas corresponding to (2.25)–(2.27) in which kernel $Q^{(\varepsilon)}(k,k')$ is replaced by $Q^{(\varepsilon,\rho)}(k,k')$, that is,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \sup_k \left| b_\varepsilon^{(\rho)} - \phi^{(\rho)}(|k|)[b(k) + \varepsilon \bar{b}(|k|)\hat{k}] \right| = 0, \tag{2.37}$$

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \sup_k \left| a_\varepsilon^{(\rho)} - \phi^{(\rho)}(|k|) \left\{ a(k) + \varepsilon \left\{ \bar{a}_1(|k|)\hat{k}\otimes\hat{k} + \bar{a}_2(|k|)\left[I - \hat{k}\otimes\hat{k} \right] \right\} \right\} \right| = 0 \tag{2.38}$$

and

$$\lim_{\varepsilon\downarrow 0} \sup_k \left| d_\varepsilon^{(\rho)} \right| = 0. \tag{2.39}$$

Using Taylor's expansion we conclude from Lemma 2.4 the following.

Proposition 2.6 *For a fixed $\rho > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$ we have*

$$\lim_{\varepsilon\downarrow 0} \|\mathcal{L}_{\varepsilon,\rho} f - L^{(\rho)} f\|_\infty = 0.$$

Convergence of generators implies the following result on convergence in law in the space $D[0,+\infty)$ of cadlag functions.

Proposition 2.7 *For a fixed $\rho > 0$ the process $\{K_t^{(\varepsilon,\rho)}(k),\ t \ge 0\}$ converges in law over $D[0,+\infty)$, as $\varepsilon \downarrow 0$, to the diffusion $\{k_t^{(\rho)}(k),\ t \ge 0\}$.*

Proof. An $\mathbb{R}^d$-valued process $\mathcal{M}^{(\varepsilon,\rho)}(t) := (\mathcal{M}_1^{(\varepsilon,\rho)}(t), \ldots, \mathcal{M}_d^{(\varepsilon,\rho)}(t))$, where

$$\mathcal{M}_j^{(\varepsilon,\rho)}(t) := K_{t,j}^{(\varepsilon,\rho)}(k) - k - \int_0^t b_{j,\varepsilon}^{(\rho)}(K_{s,j}^{(\varepsilon,\rho)}(k))ds, \quad j = 1,\ldots,d$$

is a martingale, whose quadratic variation equals

$$\sum_{j=1}^d \langle \mathcal{M}_j^{(\varepsilon,\rho)} \rangle(t) = \sum_{j=1}^d \int_0^t a_{jj,\varepsilon}^{(\rho)}(K_{s,j}^{(\varepsilon,\rho)}(k))ds.$$

From (2.37)– (2.39) we conclude that there exists $C>0$ such that

$$\sum_{j=1}^{d}\left(\int_s^t |b_{j,\varepsilon}^{(\rho)}(K_{s,j}^{(\varepsilon,\rho)}(k))|ds+\langle \mathcal{M}_j^{(\varepsilon,\rho)}\rangle(t)-\langle \mathcal{M}_j^{(\varepsilon,\rho)}\rangle(s)\right)\le C(t-s)$$

for all $s<t$ and $\varepsilon\in(0,1]$. Using Theorem VI.5.17, p. 365 of [4] with condition C2) we conclude tightness of the family of processes $\{K_t^{(\varepsilon,\rho)}(k),\, t\ge 0\}$, as $\varepsilon\downarrow 0$. The above means that for any sequence $\varepsilon_n\downarrow 0$ we can choose a subsequence such that the corresponding processes are convergent in law over $D[0,+\infty)$.

Next, we show that the limiting law is supported on $C[0,+\infty)$. For that purpose, let us linearly interpolate the (discontinuous) trajectory of the jump process between the nodal points $(T_n^{(\varepsilon,\rho)},B_n^{(\varepsilon,\rho)})$ given by the jump times and positions, to obtain a continuous trajectory process $\{\tilde K_t^{(\varepsilon,\rho)}(k),\, t\ge 0\}$. We have the following.

Lemma 2.8 *For any $\delta,T>0$ we have*

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}\left[\sup_{t\in[0,T]}|\tilde K_t^{(\varepsilon,\rho)}(k)-K_t^{(\varepsilon,\rho)}(k)|\ge\delta\right]=0.$$

Proof. Denote by $\mathcal{N}_T^{(\varepsilon,\rho)}$ the number of jumps of $K_t^{(\varepsilon,\rho)}(k)$ up to time T. We have

$$\mathbb{E}\mathcal{N}_T^{(\varepsilon,\rho)}=\frac{1}{\varepsilon^3}\int_0^T\mathbb{E}\left\{\phi^{(\rho)}(|K_s^{(\varepsilon,\rho)}(k)|)\Sigma\left(\frac{|K_s^{(\varepsilon,\rho)}(k)|}{\varepsilon}\right)\right\}ds\le\frac{\|\Sigma\|_\infty T}{\varepsilon^3},\quad\forall\,\varepsilon\in(0,1].$$

Suppose that $\kappa\in(0,1)$. Then, by Chebyshev's inequality

$$\mathbb{P}[\mathcal{N}_T^{(\varepsilon,\rho)}\ge\varepsilon^{-3-\kappa}]\le C\varepsilon^{\kappa},\quad\forall\,\varepsilon\in(0,1]$$

for some constant $C>0$. On the other hand, we have

$$\mathbb{P}\left[\sup_{t\in[0,T]}|\tilde K_t^{(\varepsilon,\rho)}(k)-K_t^{(\varepsilon,\rho)}(k)|\ge\delta,\,\mathcal{N}_T^{(\varepsilon,\rho)}\le\varepsilon^{-3-\kappa}\right]$$
$$\le\mathbb{P}\left[\sup_{0\le n\le[\varepsilon^{-3-\kappa}]+1}|B_{n+1}^{(\varepsilon,\rho)}-B_n^{(\varepsilon,\rho)}|\ge\delta\right]\le\frac{1}{\delta^5}\sum_{n=0}^{[\varepsilon^{-3-\kappa}]+1}\mathbb{E}|B_{n+1}^{(\varepsilon,\rho)}-B_n^{(\varepsilon,\rho)}|^5.$$

Using (2.17) in Lemma 2.3 with $n=5$, we estimate the right hand side by

$$\frac{C([\varepsilon^{-3-\kappa}]+1)\varepsilon^4}{\delta^5}\to 0,$$

as $\varepsilon\downarrow 0$. This finishes the proof of the lemma. □

Finally we need to identify the limiting law for $\left\{K_t^{(\varepsilon,\rho)}(k),\, t\ge 0\right\}$. Thanks to Proposition 2.6 it solves the martingale problem that corresponds to the differential operator L_ρ. Since the respective problem is well-posed (this is a conclusion of Corollary 8.1.6 and Theorem 5.2.3 of [9]) this identifies the weak limit of the laws of $\left\{K_t^{(\varepsilon,\rho)}(k),\, t\ge 0\right\}$ and ends the proof of Proposition 2.7. □

Removal of the cut-off parameter

We now remove the regularization by $\rho > 0$. Suppose that $\tau^{\varepsilon,\rho}(k)$ (resp. $\tau^{\rho}(k)$) is the exit time of $\{K_t^{(\varepsilon)}(k),\, t \ge 0\}$ (resp. $k_t^{(\rho)}(k)$) from $\mathcal{A}(\rho)$, understood as equal to $+\infty$ if the process does not leave the region. A simple consequence of Proposition 2.7 is the following, see e.g. Theorem 2.1 of [2]

$$\mathbb{P}[T \wedge \tau^{\rho'}(k) \le h] \le \liminf_{\varepsilon \downarrow 0} \mathbb{P}[T \wedge \tau^{\varepsilon,\rho}(k) \le h] \le \limsup_{\varepsilon \downarrow 0} \mathbb{P}[T \wedge \tau^{\varepsilon,\rho}(k) \le h] \le \mathbb{P}[T \wedge \tau^{\rho}(k) \le h] \quad (2.40)$$

for any $T, h > 0$ and $\rho > \rho' > 0$. We also have the following.

Lemma 2.9 *Suppose that $d \ge 2$. Then,*

$$\lim_{\rho \downarrow 0} \mathbb{P}[\tau^{\rho}(k) \ge T] = 1, \quad \forall\, T > 0,\, k \ne 0.$$

Proof. For any $\alpha \in \mathbb{R}$ we let $\mathfrak{f}_\alpha(k) := |k|^{-\alpha}$. A straightforward calculation shows that

$$L_\rho \mathfrak{f}_\alpha(k) \equiv 0, \quad k \in \mathcal{A}_\rho. \quad (2.41)$$

Let $\tau_L^{\rho}(k)$ (resp. $\tau_R^{\rho}(k)$) be the exit time of the diffusion from $\mathcal{A}(\rho)$ via the sphere $|k| = \rho$ (resp. $|k| = \rho^{-1}$). We have of course $\tau^{\rho}(k) = \tau_L^{\rho}(k) \wedge \tau_R^{\rho}(k)$. Let $\alpha = -2$. Using Itô formula we get

$$\rho^{-2}\mathbb{P}\left[\tau_R^{\rho}(k) \le \tau_L^{\rho}(k) \wedge T\right] \le \mathbb{E}|k^{(\rho)}_{T \wedge \tau^{\rho}(k)}(k)|^2 = |k|^2. \quad (2.42)$$

Likewise, for $\alpha < 0$ we get

$$\rho^{-\alpha}\mathbb{P}\left[\tau_L^{\rho}(k) \le \tau_R^{\rho}(k) \wedge T\right] \le |k|^{-\alpha}. \quad (2.43)$$

From (2.42) and (2.43) we conclude that

$$\lim_{\rho \downarrow 0} \mathbb{P}\left[\tau^{\rho}(k) \le T\right] = \lim_{\rho \downarrow 0} \mathbb{P}\left[\tau_L^{\rho}(k) \le \tau_R^{\rho}(k) \wedge T\right] + \lim_{\rho \downarrow 0} \mathbb{P}\left[\tau_R^{\rho}(k) \le \tau_L^{\rho}(k) \wedge T\right] = 0,$$

so we are done. □

Corollary 2.10 *For any $k \ne 0$ the laws $Q_k^{(\rho)}$ of $\{k_t^{(\rho)}(k),\, t \ge 0\}$ converge, as $\rho \downarrow 0$, to the law Q_k that solves of the martingale problem corresponding to the generator L given by* (1.10).

Proof. The convergence part of the theorem follows from Lemma 2.9 and the fact that

$$Q_k^{(\rho)}(A \cap [\tau^{\rho} \le T]) = Q_k^{(\rho')}(A \cap [\tau^{\rho} \le T]), \quad \forall\, A \in \mathcal{M}_T,\, 0 < \rho' < \rho.$$

The limiting law Q_k satisfies

$$Q_k(A \cap [\tau^{\rho} \le T]) = Q_k^{(\rho)}(A \cap [\tau^{\rho} \le T]), \quad \forall\, A \in \mathcal{M}_T,\, 0 < \rho. \quad (2.44)$$

Since each $Q_k^{(\rho)}$ solves the martingale problem corresponding to L^{ρ} the limiting measure solves the problem corresponding to L. □

Denote by $G_k^{(\varepsilon,\rho)}$ the law of $\{K_t^{(\varepsilon,\rho)}(k),\, t \ge 0\}$. To finish the proof of Theorem 2.5 observe that for any $f \in C_b(D[0,+\infty))$ that is $\mathcal{M}_T$ measurable and $0 < \rho$ the laws $G_k^{(\varepsilon)}$ of $\{K_t^{(\varepsilon)}(k),\, t \ge 0\}$ satisfy

$$\limsup_{\varepsilon \downarrow 0} \int f dG_k^{(\varepsilon)} \le \limsup_{\varepsilon \downarrow 0} \int f 1_{[\tau^{\rho} \ge T]} dG_k^{(\varepsilon,\rho)} + \|f\|_\infty \limsup_{\varepsilon \downarrow 0} G_k^{(\varepsilon,\rho)}[\tau^{\rho} \le T] \quad (2.45)$$
$$\le \limsup_{\varepsilon \downarrow 0} \int f dG_k^{(\varepsilon,\rho)} + 2\|f\|_\infty Q_k^{(\rho)}[\tau^{\rho} \le T] = \int f dQ_k^{(\rho)} + 2\|f\|_\infty Q_k^{(\rho)}[\tau^{\rho} \le T].$$

The last equality follows from Proposition 2.7, while the preceding inequality follows from (2.40). Using (2.44) we can further write

$$\limsup_{\varepsilon\to 0+}\int f dG_k^{(\varepsilon)} \le \int f dQ_k + 3\|f\|_\infty Q_k^{(\rho)}[\tau^\rho \le T].$$

Using Lemma 2.9 we obtain that for any $\sigma > 0$ we can adjust $\rho > 0$ in such a way that

$$\limsup_{\varepsilon\to 0+}\int f dG_k^{(\varepsilon)} \le \int f dQ_k + \sigma. \tag{2.46}$$

A similar argument can be used to show that

$$\liminf_{\varepsilon\to 0+}\int f dG_k^{(\varepsilon)} \ge \int f dQ_k - \sigma. \tag{2.47}$$

This finishes the proof of Theorem 2.5. □

We can also consider the continuous trajectory process $\{\tilde{K}_t^{(\varepsilon)}(k),\, t\ge 0\}$ defined by interpolation between the jump points of $\{K_t^{(\varepsilon)}(k),\, t\ge 0\}$. It is easy to show an analogue of Lemma 2.8 for the above processes. We conclude therefore the following.

Corollary 2.11 *The process $\{\tilde{K}_t^{(\varepsilon)}(k),\, t \ge 0\}$ converges in law over $C[0,+\infty)$, as $\varepsilon\downarrow 0$ to the diffusion $\{k_t(k),\, t\ge 0\}$.*

2.5 Application to the linear Boltzmann equation: the end of the proof of Theorem 1.1

Note that

$$W^{(\varepsilon)}(t,x,k) := W_\varepsilon\left(\frac{t}{\varepsilon^3},\frac{x}{\varepsilon^4},\frac{k}{\varepsilon}\right)$$

satisfies

$$\begin{aligned}&\partial_t W^{(\varepsilon)}(t,x,k) + k\cdot\nabla_x W^{(\varepsilon)}(t,x,k) = \mathcal{L}_\varepsilon W^{(\varepsilon)}(t,x,k),\\ &W^{(\varepsilon)}(0,x,k) = \overline{W}_0(x,k),\end{aligned} \tag{2.48}$$

with

$$\mathcal{L}_\varepsilon W(x,k) := \frac{1}{\varepsilon^{3+d}}\int_{\mathbb{R}^d}\frac{2\gamma(|p|/\varepsilon)\hat{R}(|p|/\varepsilon)}{\gamma^2(|p|/\varepsilon)+[p\cdot(p+2k)]^2\varepsilon^{-2}}\Big[W(x,p+k)-W(x,k)\Big]dp$$

for $W(\cdot,\cdot)\in L^2(\mathbb{R}^d\times\mathbb{R}^d)$. Taking the partial Fourier transform with respect to the first variable,

$$\widehat{W}^{(\varepsilon)}(0,p,k) = \int_{\mathbb{R}^d} e^{-ix\cdot p}\overline{W}(x,k)dx,$$

we obtain the equation

$$\begin{aligned}&\partial_t \widehat{W}^{(\varepsilon)}(t,p,k) + ik\cdot p\widehat{W}^{(\varepsilon)}(t,p,k) = \mathcal{L}_\varepsilon \widehat{W}^{(\varepsilon)}(t,p,k),\\ &\widehat{W}^{(\varepsilon)}(0,p,k) = \widehat{W}_0(p,k).\end{aligned} \tag{2.49}$$

The solution of (2.49) can be written as

$$\begin{aligned}\widehat{W}^{(\varepsilon)}(t,p,k) &= \mathbb{E}\left[\widehat{W}_0\left(p,K_t^{(\varepsilon)}(k)\right)\exp\left\{i\int_0^t p\cdot K_s^{(\varepsilon)}(k)\,ds\right\}\right]\\ &\approx \mathbb{E}\left[\widehat{W}_0\left(p,\tilde{K}_t^{(\varepsilon)}(k)\right)\exp\left\{i\int_0^t p\cdot \tilde{K}_s^{(\varepsilon)}(k)\,ds\right\}\right].\end{aligned} \tag{2.50}$$

The last approximate equality follows from the fact that we have replaced the jump process by the linear interpolation process. Taking the limit, as $\varepsilon\downarrow 0$, we obtain from Corollary 2.11 the conclusion of the theorem. □

3 Convergence of the wave-number process on the longer time-scale

We now turn to the proof of Theorem 1.2. As Theorem 1.1 indicates, by the long time scale of Theorem 1.2, the direction of the wave vector becomes equidistributed on the sphere. Hence, here we track only the wave number, the main result being Theorem 3.1. Its proof is different in the case $d \geq 4$ and for $d = 2, 3$. The former case is simpler since we need not worry about the wave number reaching the boundary $\ell = 0$, while in two and three dimensions we need also to ensure that the local time of the wave number at $\ell = 0$ vanishes. The limiting behavior of the spatial component of the process that is also part of Theorem 1.2 is studied in Section 4.

Limit theorem for the wave numbers in the long time

For a given $\ell > 0$ define

$$\mathfrak{K}_t(\ell) := \mathfrak{b}_{2ct}^{1/4}(\ell^4), \quad t \geq 0, \tag{3.1}$$

where $\mathfrak{b}_t(\ell)$ is the square of the Bessel process of the dimension $\delta = d/2$, see (1.16). We also define the process $\mathfrak{K}_t^{(\varepsilon)}(k) := |K_{t/\varepsilon}^{(\varepsilon)}(k/\varepsilon)|$. Our aim is to prove the following result.

Theorem 3.1 $\{\mathfrak{K}_t^{(\varepsilon)}(k),\, t \geq 0\}$ *converge in law over* $D[0, +\infty)$*, as* $\varepsilon \downarrow 0$*, to* $\{\mathfrak{K}_t(t),\, t \geq 0\}$.

Proof of Theorem 3.1 for $d \geq 4$

Since in this case $\delta \geq 2$ the process $\mathfrak{b}_t$ never reaches the boundary point 0, see Chapter XI, p. 442 of [7]. Therefore, $\mathfrak{K}_t$ can be characterized as the solution of the one dimensional SDE on $(0, +\infty)$

$$d\mathfrak{K}_t = \frac{(d-3)cdt}{4{\mathfrak{K}_t}^3} + \frac{c^{1/2}dw_t}{2^{1/2}|\mathfrak{K}_t|}, \quad \mathfrak{K}_0 = \ell, \tag{3.2}$$

and the corresponding martingale problem is well posed.

To show weak convergence, we apply the martingale argument used in Section 2.3 above. For a given $\rho > 0$ we introduce the diffusion $\bar{\mathfrak{K}}_t^{(\rho)}(k)$, starting at k, with the generator

$$M_\rho f(\ell) := \phi^{(\rho)}(\ell) M f(\ell), \quad f \in C_0^\infty(\mathbb{R}_+), \tag{3.3}$$

where the operator M is given by (1.20), and $\phi^{(\rho)}$ is a cut-off function as in (2.34). We will also use the process $\mathfrak{K}_t^{(\varepsilon,\rho)}(k) := |K_{t/\varepsilon}^{(\varepsilon,\rho)}(k)|$. Recall that $K_t^{(\varepsilon,\rho)}(k)$ is the jump process with the regularized generator given by (2.35) that starts at k at $t = 0$. We will prove that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sup_{k \in \mathcal{A}(\rho)} |\mathcal{L}_\varepsilon f(|k|) - M f(|k|)| = 0 \tag{3.4}$$

for any $\rho \in (0, 1)$ and $f \in C_0^\infty(\mathbb{R}_+)$. From that point on we can repeat the argument used in the proof of Proposition 2.7 and conclude that $\{\mathfrak{K}_t^{(\varepsilon,\rho)}(k),\, t \geq 0\}$ converge in law over $D[0, +\infty)$, as $\varepsilon \downarrow 0$, to $\{\bar{\mathfrak{K}}_t^{(\rho)}(k),\, t \geq 0\}$. Since the exit time of the diffusion $\{\bar{\mathfrak{K}}_t^{(\rho)}(k),\, t \geq 0\}$ from $[\rho, \rho^{-1}]$ tends to $+\infty$, as $\rho \to 0+$, a.s., we can remove the truncation corresponding to ρ using the argument made in the course of the proof of Theorem 2.5. The only claim that needs to be shown is, therefore, (3.4), which we now verify with the help of Lemmas 2.3 and 2.4.

For a given function $f \in C_0^\infty(\mathbb{R}_+)$ we have

$$\mathcal{L}_\varepsilon f(k) = \int_{\mathbb{R}^d} [f(|k'|) - f(|k|)] Q^{(\varepsilon)}(k,k')dk', \tag{3.5}$$

with the kernel $Q^{(\varepsilon)}(k,k')$ given by (2.4). Let us denote $g(k) := f(|k|)$ and

$$H_2(k,k') := (k'-k)\cdot\nabla g(k) + \frac{1}{2}(k'-k)^{\otimes 2}\cdot\nabla^2 g(k) + \frac{1}{6}(k'-k)^{\otimes 3}\cdot\nabla^3 g(k). \tag{3.6}$$

where

$$\begin{aligned} \nabla g(k) &= \hat{k} f'(|k|), \qquad \nabla^2 g(k) = \hat{k}^{\otimes 2} f''(|k|) + (I - \hat{k}^{\otimes 2})\frac{f'(|k|)}{|k|}, \\ \nabla^3 g(k) &= \hat{k}^{\otimes 3} f'''(|k|) + S(\hat{k})\frac{f''(|k|)}{|k|} - S(\hat{k})\frac{f'(|k|)}{|k|^2}. \end{aligned} \tag{3.7}$$

Here $S(\hat{k}) = [S_{ijk}(\hat{k})]$ is given by

$$S_{ijm} = \delta_{jm}\hat{k}_i + \delta_{im}\hat{k}_j + \delta_{ij}\hat{k}_m - 3\hat{k}_i\hat{k}_j\hat{k}_m.$$

By Taylor's expansion we conclude that

$$f(|k'|) - f(|k|) - H_2(k,k') = R_\varepsilon(k,k'), \tag{3.8}$$

with the remainder satisfying

$$|R_\varepsilon(k,k')| \le C\|f^{(4)}\|_\infty |k'-k|^4, \quad \forall\, k,k'$$

and some constant C. Using (3.5) we can write

$$\mathcal{L}_\varepsilon f(k) = \int_{\mathbb{R}^d} H_2(k,k')Q^{(\varepsilon)}(k,k')dk' + \int_{\mathbb{R}^d} R_\varepsilon(k,k')Q^{(\varepsilon)}(k,k')dk'. \tag{3.9}$$

For the first term above we have

$$\int_{\mathbb{R}^d} H_2(k,k')Q^{(\varepsilon)}(k,k')dk' = H_{21}f'(|k|) + H_{22}f''(|k|) + H_{23}f'''(|k|),$$

where

$$\begin{aligned} H_{21} &= \int_{\mathbb{R}^d}\left[(k'-k)\cdot\hat{k} + \frac{1}{2|k|}(k'-k)^{\otimes 2}\cdot(I-\hat{k}^{\otimes 2}) - \frac{1}{6|k|^2}(k'-k)^{\otimes 3}\cdot S(k)\right]Q^{(\varepsilon)}(k,k')dk \\ &= b_\varepsilon(k)\cdot\hat{k} + \frac{1}{2|k|}a_\varepsilon(k)\cdot(I-\hat{k}^{\otimes 2}) - \frac{1}{6|k|^2}d_\varepsilon(k)\cdot S(k), \end{aligned}$$

$$\begin{aligned} H_{22}(k) &= \int_{\mathbb{R}^d}\left[\frac{1}{2}(k'-k)^{\otimes 2}\cdot\hat{k}^{\otimes 2} - \frac{1}{6|k|}(k'-k)^{\otimes 3}\cdot S(k)\right]Q^{(\varepsilon)}(k,k')dk \\ &= \frac{1}{2}a_\varepsilon(k)\cdot\hat{k}^{\otimes 2} - \frac{1}{6|k|}d_\varepsilon(k)\cdot S(k), \end{aligned}$$

and, finally,

$$H_{23}(k) = \int_{\mathbb{R}^d}\left[\frac{1}{6}(k'-k)^{\otimes 3}\cdot S(k)\right]Q^{(\varepsilon)}(k,k')dk = \frac{1}{6}d_\varepsilon(k)\cdot S(k).$$

Using Lemma 2.4 we may rewrite the expressions H_{21} and $H_{22}(k)$ for the drift and diffusivity as

$$H_{21}(k) = b(k)\cdot\hat{k} + \frac{1}{2|k|}a(k)\cdot\left(I-\hat{k}^{\otimes 2}\right) + \varepsilon\left\{\bar{b}(|k|) + \frac{d-1}{2|k|}\bar{a}_2(|k|)\right\} + o(\varepsilon) \tag{3.10}$$

and

$$H_{22}(k) = \frac{1}{2}a(k)\cdot\hat{k}^{\otimes 2} + \varepsilon\frac{\bar{a}_1(|k|)}{2} + o(\varepsilon). \tag{3.11}$$

By the same token $H_{23}(k) = o(\varepsilon)$. The above calculations prove that

$$\mathcal{L}_\varepsilon f(k) = Lf(k) + \varepsilon\left\{\left[\bar{b}(|k|) + \frac{d-1}{2|k|}\bar{a}_2(|k|)\right]f'(|k|) + \frac{\bar{a}_1(|k|)}{2}f''(|k|)\right\} + o(\varepsilon)$$

$$= Lf(k) + \varepsilon Mf(|k|) + o(\varepsilon),$$

where M is given by (1.20). Since $Lf(|k|) = 0$ formula (3.4) follows immediately.

Proof of Theorem 3.1 for $d = 2$ and $d = 3$

The situation in dimensions two and three differs from the previous one because the dimension of the squared Bessel process $\delta = d/2$, belongs then to $(0, 2)$, so the limiting diffusion is recurrent and enters 0 infinitely many times, see [8] Theorem V.48.6, p. 286. Therefore we modify the former argument truncating the process only for large values. Our presentation deals with the case $d = 3$, as the dimension two can be treated in the same fashion. To simplify the notation, we assume also that $2c = 1$ so that $\mathfrak{b}_t = \mathfrak{K}_t^4$ satisfies the equation

$$\mathfrak{b}_t = \ell^4 + \frac{3t}{2} + 2\int_0^t \sqrt{|\mathfrak{b}_s|}dw_s. \tag{3.12}$$

Its generator $\mathfrak{M}$ is given by (1.14) with $c = 1/2$. Note that the limiting operator $\mathfrak{M}$ does not have a singularity at $b = 0$, and because of that we will work with the process $[\mathfrak{K}_t^{(\varepsilon)}(k)]^4$, where $\mathfrak{K}_t^{(\varepsilon)}(k) := |K_{t/\varepsilon}^{(\varepsilon)}(k)|$, and consider the limit of its generator.

For any $\rho \in (0,1)$ let $\mathcal{B}(\rho) := [k : |k| \le \rho^{-1}]$. Let also $\{K_t^{(\varepsilon,\rho)}(k),\, t \ge 0\}$ be the jump process corresponding to the scattering kernel

$$Q^{(\varepsilon,\rho)}(k,k') := \phi^{(\rho)}(|k|)Q^{(\varepsilon)}(k,k'), \tag{3.13}$$

where $Q^{(\varepsilon)}(k,k')$ is given by (2.4). Here $\phi^{(\rho)} : \mathbb{R} \to [0,1]$ is a C^∞ function that satisfies

$$\phi^{(\rho)}(\ell) \equiv 1 \quad \text{for} \quad \rho^{-1} \ge |\ell| \quad \text{and} \quad \phi^{(\rho)}(\ell) \equiv 0 \quad \text{for} \quad |\ell| \ge 2/\rho. \tag{3.14}$$

Note, once again, that there is no truncation near $\ell = 0$. Let $\{\tilde{\mathfrak{K}}_t^{(\varepsilon)}(k),\, t \ge 0\}$ be the linear interpolation process corresponding to $\mathfrak{K}_t^{(\varepsilon)}(k)$. Define also the respective truncated processes $\mathfrak{K}_t^{(\varepsilon,\rho)}(k)$ and its linear interpolation $\tilde{\mathfrak{K}}_t^{(\varepsilon,\rho)}(k)$. For a function $f : \mathbb{R} \to \mathbb{R}$ belonging to $C^1(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\{0\})$ we introduce the norm

$$\|f\|'_{C^N} := \|f\|_{C^1} + \sup_\ell \sum_{i=1}^{N-1} |\ell^i f^{(i+1)}(\ell)| < +\infty, \quad \forall N \ge 1. \tag{3.15}$$

To prove tightness of $\{\mathfrak{K}_t^{(\varepsilon)}(k),\, t \ge 0\}$, as $\varepsilon \downarrow 0$, and then to identify its limiting law we shall need the following refinement of Proposition 2.6.

Proposition 3.2 *There exists constant $C > 0$ such that*

$$\sup_k |\mathcal{L}_\varepsilon g(k)| \le C\varepsilon \|f\|'_{C^4}, \quad \forall\, \varepsilon \in (0,1] \tag{3.16}$$

for all $f \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$. Here $g(k) := f(|k|^4)$. In addition, for any $\gamma \in (0,1)$ there exists $\delta > 0$ such that

$$\sup_{|k|\ge\varepsilon^\gamma} \left| \frac{1}{\varepsilon}\mathcal{L}_\varepsilon g(k) - \mathfrak{M} f(|k|^4) \right| \le C\varepsilon^\delta \|f\|'_{C^4}, \quad \forall\, \varepsilon \in (0,1] \tag{3.17}$$

for all f as above.

The proof of the lemma is a bit technical and we postpone its presentation till Section 5.2. In the meantime, we apply it to finish the proof of Theorem 3.1.

Tightness in $d = 2, 3$

We use (3.16) to prove tightness of $\mathfrak{K}_t^{(\varepsilon)}(k)$. Let

$$\mathfrak{g}(k) = |k|^4. \tag{3.18}$$

Consider the martingale

$$\mathfrak{m}_{\varepsilon,\rho}(t) := \mathfrak{g}(K_{t/\varepsilon}^{(\varepsilon,\rho)}(k)) - \mathfrak{g}(k) - \frac{1}{\varepsilon}\int_0^t \mathcal{L}_{\varepsilon,\rho}\mathfrak{g}(K_{s/\varepsilon}^{(\varepsilon,\rho)}(k))ds,$$

where $\mathcal{L}_{\varepsilon,\rho}$ is the generator of $\{K_t^{(\varepsilon,\rho)}(k),\, t \ge 0\}$. Recall that $\mathcal{L}_{\varepsilon,\rho} = \phi^{(\rho)}\mathcal{L}_\varepsilon$. The quadratic variation of $\mathfrak{m}_{\varepsilon,\rho}(t)$ equals

$$\langle \mathfrak{m}_{\varepsilon,\rho}\rangle(t) = \frac{1}{\varepsilon}\int_0^t \left(\mathcal{L}_{\varepsilon,\rho}\mathfrak{g}^2 - 2\mathfrak{g}\mathcal{L}_{\varepsilon,\rho}\mathfrak{g}\right)(K_{s/\varepsilon}^{(\varepsilon,\rho)}(k))ds.$$

With the help of (3.16) we conclude that there exists a constant $C > 0$ such that

$$\frac{1}{\varepsilon}\int_{t_1}^{t_2} |\mathcal{L}_{\varepsilon,\rho}\mathfrak{g}(K_{s/\varepsilon}^{(\varepsilon,\rho)}(k))|ds + \langle \mathfrak{m}_{\varepsilon,\rho}\rangle(t_2) - \langle \mathfrak{m}_{\varepsilon,\rho}\rangle(t_1) \le C(t_2 - t_1), \quad \forall\, \varepsilon \in (0,1],\, t_2 > t_1 \ge 0.$$

This according to Theorem VI.5.17, p. 365 of [4] implies tightness of the laws of $\{[\mathfrak{K}_t^{(\varepsilon,\rho)}(k)]^4,\, t \ge 0\}$ over $D[0,+\infty)$ as $\varepsilon \downarrow 0$. We can remove the truncation parameter ρ in the same fashion as it has been done in the course of the proof of Theorem 2.5 by showing that the limiting diffusion does not explode, as in our application of Lemma 2.9.

To prove that the limiting law is indeed supported on $C[0,+\infty)$ we show that for any $\delta, \rho, T > 0$

$$\mathbb{P}\left[\sup_{t\in[0,T]} \left|\tilde{K}_{t/\varepsilon}^{(\varepsilon)}(k) - K_{t/\varepsilon}^{(\varepsilon)}(k)\right| \ge \delta\right] = 0,$$

which can be done in the same fashion as in the proof of Lemma 2.8 (instead of the 5-th moment of the respective chain we should consider 6-th one since the scale is now longer by the factor of ε^{-1}). Summarizing, we have shown so far the following.

Proposition 3.3 *The laws of $\{\mathfrak{K}_t^{(\varepsilon)}(k),\, t \ge 0\}$, and those of $\{\tilde{\mathfrak{K}}_t^{(\varepsilon)}(k),\, t \ge 0\}$, are tight, as $\varepsilon \downarrow 0$, on $D[0,+\infty)$ and $C[0,+\infty)$, respectively. Moreover, the set of the limiting laws for both of these families is identical.*

Limit identification

Let us start with some notation. For any path $\pi \in C(\mathbb{R}_+;\mathbb{R}_+)$ denote by $x(t;\pi) := \pi(t)$. By $(\mathcal{M}_t)$ we denote the canonical filtration on $C(\mathbb{R}_+;\mathbb{R}_+)$. Identification of the limiting law is a simple consequence of the following result.

Proposition 3.4 *Suppose that $f \in C_0^2(\mathbb{R})$ and Q_ℓ is any limiting law of $\{[\mathfrak{K}_t^{(\varepsilon)}]^4(\ell^{1/4}),\, t \ge 0\}$ that starts at ℓ. Then,*

$$f(x(t)) - f(\ell) - \int_0^t \mathfrak{M} f(x(s))ds, \quad t \ge 0,$$

is a Q_ℓ-martingale.

As we have already mentioned in Section 1 the martingale problem corresponding to $\mathfrak{M}$ is well-posed, see Corollary 8.1.6, p. 202 of [9], and is given by the law of the solution of (3.12). Thus establishing the above proposition would conclude the proof of Theorem 3.1.

Proof of Proposition 3.4

We start with the following.

Lemma 3.5 *For any $T > 0$ there exists $0 < \gamma < 1$ such that for $g(k) = f(|k|^4)$, where $f \in C_0^\infty(\mathbb{R})$, we have*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\frac{1}{\varepsilon}\int_0^T \mathcal{L}_\varepsilon g\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right) 1_{[0,\varepsilon^\gamma)}\left(\mathfrak{K}_s^{(\varepsilon)}(k)\right) ds\right]^2 = 0, \quad \forall\, k \ne 0. \tag{3.19}$$

Proof. Suppose that $\gamma' \in (\gamma, 1)$. In light of Proposition 3.2, it suffices to show that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\int_0^T \mathfrak{M} f\left([\mathfrak{K}_s^{(\varepsilon)}(k)]^4\right) 1_{[\varepsilon^{\gamma'},\varepsilon^\gamma)}\left(\mathfrak{K}_s^{(\varepsilon)}(k)\right) ds + \frac{1}{\varepsilon}\int_0^T \mathcal{L}_\varepsilon g\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right) 1_{[0,\varepsilon^{\gamma'})}\left(\mathfrak{K}_s^{(\varepsilon)}(k)\right) ds\right]^2 = 0. \tag{3.20}$$

Taking into account that $\mathfrak{M} f(\ell) = (3/2)f'(0) + O(\ell)$ as $\ell \ll 1$, the above is equivalent to

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\frac{3}{2}f'(0)\int_0^T 1_{[\varepsilon^{\gamma'},\varepsilon^\gamma)}\left(\mathfrak{K}_s^{(\varepsilon)}(k)\right) ds + \frac{1}{\varepsilon}\int_0^T \mathcal{L}_\varepsilon g\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right) 1_{[0,\varepsilon^{\gamma'})}\left(\mathfrak{K}_s^{(\varepsilon)}(k)\right) ds\right]^2 = 0. \tag{3.21}$$

We claim that in fact it suffices to show that, with $\mathfrak{g}(k) = |k|^4$:

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\frac{3}{2}\int_0^T 1_{[\varepsilon^{\gamma'},\varepsilon^\gamma)}\left(\mathfrak{K}_s^{(\varepsilon)}(k)\right) ds + \frac{1}{\varepsilon}\int_0^T \tilde{\mathcal{L}}_\varepsilon \mathfrak{g}\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right) 1_{[0,\varepsilon^{\gamma'})}\left(\mathfrak{K}_s^{(\varepsilon)}(k)\right) ds\right]^2 = 0. \tag{3.22}$$

Here

$$\tilde{\mathcal{L}}_\varepsilon g(k) := \int_{|k'| \le \varepsilon^\gamma} [g(k') - g(k)] Q^{(\varepsilon)}(k,k')dk'. \tag{3.23}$$

This can be seen as follows. Due to the rapid decay of $\hat{R}(\cdot)$, see (1.5), from (2.4) we can conclude that $Q^{(\varepsilon)}(k,k') \sim \varepsilon^{10}\hat{R}^{1/2}(|k'-k|)$ when $|k'| \ge \varepsilon^\gamma$ and $|k| \le \varepsilon^{\gamma'}$, since $\gamma < \gamma' < 1$. Thus, for any $m \ge 0$ there exist $C, \delta > 0$ such that

$$\sup_{|k| \le \varepsilon^{\gamma'}} \left|\int_{|k'| \ge \varepsilon^\gamma} |g(k') - g(k)|^m Q^{(\varepsilon)}(k,k')dk'\right| \le C\varepsilon^\delta, \quad \forall\, \varepsilon \in (0,1]. \tag{3.24}$$

Therefore, we can write that for $|k| \le \varepsilon^{\gamma'}$, the generator $(1/\varepsilon)\mathcal{L}_\varepsilon g\,(k)$ is, up to a term of order $o(1)$, equal to

$$\frac{1}{\varepsilon}\int_{|k'|\le\varepsilon^\gamma}[g\,(k')-g\,(k)]Q^{(\varepsilon)}(k,k')dk' = g'(0)\tilde{\mathcal{L}}_\varepsilon\mathfrak{g}(k)+R(k,\varepsilon).$$

The term $R(\varepsilon)$, corresponding to the second order Taylor expansion of function f, in dimension $d=3$, can be estimated as follows

$$\sup_{|k|\le\varepsilon^{\gamma'}}|R(k,\varepsilon)| \le \frac{C}{\varepsilon}\sup_{|k|\le\varepsilon^{\gamma'}}\int_{|k'|\le\varepsilon^\gamma}[\mathfrak{g}\,(k')-\mathfrak{g}\,(k)]^2Q^{(\varepsilon)}(k,k')dk' \le \frac{C\varepsilon^{11\gamma}}{\varepsilon^7}. \tag{3.25}$$

as long as $\gamma > 7/11$. From here we see that (3.21) is indeed equivalent with (3.22).

Before proceeding with the proof of the latter we introduce some notation. Suppose that $\delta > 0$ is a parameter to be adjusted later on, and let

$$f_\delta(\ell) := 2\int_0^\ell \frac{1_{[0,\delta)}(r)}{r^{1/4}}(\ell^{1/4}-r^{1/4})dr.$$

This function satisfies

$$\mathfrak{M}f_\delta = 1_{[0,\delta)}, \tag{3.26}$$

and $f_\delta(0)=0$. In addition, we have

$$f_\delta'(\ell) := \frac{1}{2\ell^{3/4}}\int_0^\ell\frac{1_{[0,\delta)}(r)dr}{r^{1/4}},$$

so $f_\delta'(\ell) = 2/3$ and $f_\delta(\ell) = (2/3)\ell$ for $\ell\in[0,\delta]$. For $\rho < 1/\delta$ let us define $f_{\delta,\rho} := f_\delta\phi^{(\rho)}$. It is elementary to verify that for a fixed ρ we have

$$\lim_{\delta\to0+}\sup_\ell\left(|f_{\delta,\rho}(\ell)|+|\ell|[f_{\delta,\rho}'(\ell)]^2\right)=0, \tag{3.27}$$

and

$$\sup_{\delta\in(0,1]}\|f_{\delta,\rho}\|_{C^N}' < +\infty, \quad \forall\, N\ge 1. \tag{3.28}$$

Let $g_{\delta,\rho}(k) := f_{\delta,\rho}(|k|^4)$, and consider the martingale

$$\mathfrak{m}_\varepsilon(t) := g_{\varepsilon^\gamma,\rho}\left(K_{t/\varepsilon}^{(\varepsilon)}(k)\right) - g_{\varepsilon^\gamma,\rho}(k) - \frac{1}{\varepsilon}\int_0^t\mathcal{L}_\varepsilon g_{\varepsilon^\gamma,\rho}\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right)ds. \tag{3.29}$$

We will show that its second moment vanishes as $\varepsilon\downarrow0$. Combining this with (3.27), we deduce that

$$\lim_{\varepsilon\downarrow0}\mathbb{E}\left[\frac{1}{\varepsilon}\int_0^t\mathcal{L}_\varepsilon g_{\varepsilon^\gamma,\rho}\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right)ds\right]^2 = 0. \tag{3.30}$$

However, (3.24) and the fact that

$$g_{\varepsilon^\gamma,\rho}(k) = \frac{2}{3}\mathfrak{g}(k), \quad |k|\le\varepsilon^\gamma, \tag{3.31}$$

imply that

$$\lim_{\varepsilon\downarrow0}\sup_{|k|\le\varepsilon^{\gamma'}}\left|\mathcal{L}_\varepsilon g_{\varepsilon^\gamma,\rho}(k)-\frac{2}{3}\tilde{\mathcal{L}}_\varepsilon\mathfrak{g}(k)\right| = 0. \tag{3.32}$$

Hence, the integral

$$\frac{1}{\varepsilon}\int_0^t \mathcal{L}_\varepsilon g_{\varepsilon^\gamma,\rho}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right) ds$$

that we know to be small (see (3.30)), can be split into the integrals over the regions where $|K^{(\varepsilon)}_{s/\varepsilon}(k)| \le \varepsilon^{\gamma'}$ and $|K^{(\varepsilon)}_{s/\varepsilon}(k)| \ge \varepsilon^{\gamma'}$. In the first region we can use (3.32) so that it becomes

$$I(t) = \frac{2}{3\varepsilon}\int_0^t \tilde{\mathcal{L}}_\varepsilon \mathfrak{g}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right) 1_{[0,\varepsilon^{\gamma'})}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right) ds,$$

while in the second we can use approximation (3.17), as $|K^{(\varepsilon)}_{s/\varepsilon}(k)| \ge \varepsilon^{\gamma'}$, to show that it equals

$$\begin{aligned} I\!I(t) &= \frac{2}{3}\int_0^t \mathfrak{M} g_{\varepsilon^\gamma,\rho}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right) 1_{[\varepsilon^{\gamma'},+\infty)}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right) ds + o(1)\\ &= \frac{2}{3}\int_0^t 1_{[\varepsilon^{\gamma'},\varepsilon^{\gamma})}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right) ds + o(1). \end{aligned}$$

Adding up the above, we see that the second moment of the sum $I(t)+I\!I(t)$ vanishes as $\varepsilon \downarrow 0$, which is nothing but (3.22).

The only remaining ingredient is to prove that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}[\mathfrak{m}_\varepsilon(t)]^2 = 0. \tag{3.33}$$

We write the quadratic variation of this martingale in the form

$$\langle \mathfrak{m}_\varepsilon\rangle(t) = \langle \mathfrak{m}^{(1)}_\varepsilon\rangle(t) + \langle \mathfrak{m}^{(2)}_\varepsilon\rangle(t),$$

where

$$\begin{aligned} \langle \mathfrak{m}^{(1)}_\varepsilon\rangle(t) &= \frac{1}{\varepsilon}\int_0^t 1_{[0,\varepsilon^{\gamma'})}\left(\mathfrak{K}^{(\varepsilon)}_s(k)\right) ds \int_{\mathbb{R}^3}\left[g_{\varepsilon^\gamma,\rho}(k') - g_{\varepsilon^\gamma,\rho}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right)\right]^2 Q^{(\varepsilon)}\left(K^{(\varepsilon)}_{s/\varepsilon}(k), k'\right) dk',\\ \langle \mathfrak{m}^{(2)}_\varepsilon\rangle(t) &= \int_0^t 1_{[\varepsilon^{\gamma'},+\infty)}\left(\mathfrak{K}^{(\varepsilon)}_s(k)\right) \mathcal{Q}_\varepsilon[g_{\varepsilon^\gamma,\rho}](K^{(\varepsilon)}_{s/\varepsilon}(k))ds, \end{aligned} \tag{3.34}$$

and

$$\mathcal{Q}_\varepsilon[g](k) := \frac{1}{\varepsilon}\left(\mathcal{L}_\varepsilon g^2 - 2g\mathcal{L}_\varepsilon g\right)(k) \tag{3.35}$$

is the carré du champs operator associated to $(1/\varepsilon)\mathcal{L}_\varepsilon$. We will now split the integral over k' in $\langle \mathfrak{m}^{(1)}_\varepsilon\rangle(t)$ as an integral over $\{|k'| \le \varepsilon^\gamma\}$ and its complement. The integral over the complement becomes negligible as $\varepsilon \downarrow 0$ as in (3.24) because k' and $K^{(\varepsilon)}_{s/\varepsilon}(k)$ are well separated. On the other hand, when both $|k'| \le \varepsilon^\gamma$ and $|K^{(\varepsilon)}_{s/\varepsilon}(k)| \le \varepsilon^\gamma$, we may use the fact that

$$g_{\varepsilon^\gamma,\rho}(k) = \frac{2}{3}\mathfrak{g}(k), \quad |k| \le \varepsilon^\gamma, \tag{3.36}$$

leading to

$$\langle \mathfrak{m}^{(1)}_\varepsilon\rangle(t) = \frac{4}{9\varepsilon}\int_0^t 1_{[0,\varepsilon^{\gamma'})}\left(\mathfrak{K}^{(\varepsilon)}_s(k)\right) ds \int_{|k'|\le\varepsilon^\gamma}\left[\mathfrak{g}(k') - \mathfrak{g}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right)\right]^2 Q^{(\varepsilon)}\left(K^{(\varepsilon)}_{s/\varepsilon}(k), k'\right) dk' + o(1). \tag{3.37}$$

Now, (3.25) implies that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\langle \mathfrak{m}_\varepsilon^{(1)} \rangle(t) = 0. \tag{3.38}$$

As for $\langle \mathfrak{m}_\varepsilon^{(2)} \rangle(t)$, since $|K_{s/\varepsilon}^{(\varepsilon)}(k)| \geq \varepsilon^{\gamma'}$, we may use (3.17) to approximate $\mathcal{L}_\varepsilon$ by $\mathfrak{M}$, with the help of the (uniform in ε) bound (3.28), to conclude that there exist $C, \delta > 0$ such that

$$\sup_{|k| \geq \varepsilon^\gamma} \left| \mathcal{Q}_\varepsilon[g_{\varepsilon^\gamma,\rho}](k) - \mathcal{Q}[f_{\varepsilon^\gamma,\rho}](|k|^4) \right| \leq C\varepsilon^\delta, \quad \forall \varepsilon \in (0,1] \tag{3.39}$$

where $\mathcal{Q}$ is the carré du champs operator associated to $\mathfrak{M}$. Hence, $\langle \mathfrak{m}_\varepsilon^{(2)} \rangle(t)$ equals, up to a term of order $o(1)$

$$\int_0^t 1_{[\varepsilon^{\gamma'},+\infty)} \left(\mathfrak{K}_s^{(\varepsilon)}(k) \right) \mathcal{Q}[f_{\varepsilon^\gamma,\rho}] \left(\left[\mathfrak{K}_s^{(\varepsilon)}(k) \right]^4 \right) ds,$$

as $\varepsilon \downarrow 0$. Since $\mathcal{Q}[f](\ell) = 2\ell (f')^2(\ell)$, using (3.27), we conclude that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\langle \mathfrak{m}_\varepsilon^{(2)} \rangle(t) = 0. \tag{3.40}$$

Consequently, combining (3.38) and (3.40) we obtain (3.33). □

To finish the proof of Proposition 3.4 we recall that, given f as in the statement of the proposition and k such that $|k|^4 = \ell$, the process

$$g\left(K_{t/\varepsilon}^{(\varepsilon)}(k)\right) - g(k) - \frac{1}{\varepsilon}\int_0^t \mathcal{L}_\varepsilon g\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right) ds \tag{3.41}$$

is a martingale. Invoking (3.17) together with Lemma 3.5 we conclude that any limiting law Q_ℓ of $\{\mathfrak{K}_t^{(\varepsilon)}(k),\, t \geq 0\}$, as $\varepsilon \downarrow 0$, is the solution of the martingale problem corresponding to $\mathfrak{M}$ that is well posed. This ends the proof of the proposition. □

4 Convergence of the spatial component on the longer time scale

In order to complete the proof of Theorem 1.2 we now consider convergence of the spatial component

$$X_t^\varepsilon = \frac{1}{\varepsilon^{1/2}} \int_0^t K_{s/\varepsilon}^{(\varepsilon)}(k) ds. \tag{4.1}$$

Here, for simplicity of notation, we assume that X_t^ε starts at $x = 0$ at $t = 0$.

This will be done using the familiar techniques of the homogenization theory. The function (called the corrector)

$$\chi_j = \frac{|k|^3}{b(d-1)} k_j, \tag{4.2}$$

is an explicit solution of the cell problem

$$-\nabla_k \cdot \left[\frac{b}{|k|} (I - \hat{k}^{\otimes 2}) \nabla_k \chi_j \right] = k_j. \tag{4.3}$$

Consider the martingales

$$N_j^{(\varepsilon)}(t) := \chi_j(K_{t/\varepsilon}^{(\varepsilon)}(k)) - \chi_j(k) - \frac{1}{\varepsilon} \int_0^t \mathcal{L}_\varepsilon \chi_j(K_{s/\varepsilon}^{(\varepsilon)}(k)) ds, \quad t \geq 0, \tag{4.4}$$

with the generator $\mathcal{L}_\varepsilon$ given by (2.3). The following analogue of Proposition 3.2 holds for χ_j. We present its proof in Section 5.3.

Lemma 4.1 *For any $j = 1, \ldots, d$ we have*

$$\mathcal{L}_\varepsilon \chi_j(k) = L\chi_j(k) + \varepsilon\psi_{j,\varepsilon}(k) \tag{4.5}$$

where L is given by (1.10) *and*

$$\limsup_{\varepsilon\downarrow 0} \sup_{k\in\mathcal{B}(\rho)} |\psi_{j,\varepsilon}(k)| < +\infty, \quad \forall\, \rho \in (0,1), \tag{4.6}$$

and $\mathcal{B}(\rho) := [k : |k| \le \rho^{-1}]$.

Using (4.3) and (4.5) we obtain that

$$\frac{1}{\varepsilon}\int_0^t K^{(\varepsilon)}_{s/\varepsilon}(k)ds = N^{(\varepsilon)}_j(t) + \chi_j(k) - \chi_j(K^{(\varepsilon)}_{t/\varepsilon}(k)) + \int_0^t \psi_{j,\varepsilon}\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right)ds,$$

whence

$$X^\varepsilon_t = \frac{1}{\varepsilon^{1/2}}\int_0^t K^{(\varepsilon)}_{s/\varepsilon}(k)ds = \varepsilon^{1/2}N^{(\varepsilon)}_j(t) + O(\varepsilon^{1/2}), \quad \varepsilon \ll 1, \tag{4.7}$$

in probability. The quadratic co-variation of the martingale part $\{\varepsilon^{1/2}N^{(\varepsilon)}(t),\, t\ge 0\}$ equals

$$\varepsilon\langle N^{(\varepsilon)}_j, N^{(\varepsilon)}_{j'}\rangle(t) = \int_0^t B^{(j,j')}_\varepsilon(K^{(\varepsilon)}_{s/\varepsilon}(k))ds,$$

where

$$\begin{aligned} B^{(j,j')}_\varepsilon(k) &:= \int_{\mathbb{R}^d} [\chi_j(k') - \chi_j(k)][\chi_{j'}(k') - \chi_{j'}(k)]Q^{(\varepsilon)}(k,k')dk' \\ &= \frac{2}{\varepsilon}\int_{\mathbb{R}^d} \frac{(p\otimes p)\cdot\nabla\chi_j(k)\otimes\nabla\chi_{j'}(k)(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|) + (|p|^2 + 2p_1|k|/\varepsilon)^2} + o(1), \end{aligned}$$

where

$$\nabla\chi_j(k) = \frac{1}{b(d-1)}\left\{3k_j|k|k + |k|^3 e_j\right\}.$$

Calculations made in Lemmas 2.1 and 2.4 can be used to conclude that

$$\lim_{\varepsilon\downarrow 0} \sup_{k\in\mathcal{B}(\rho)} |B^{(j,j')}_\varepsilon(k) - B^{(j,j')}(k)| = 0, \quad \forall\, \rho \in (0,1), \tag{4.8}$$

where

$$\begin{aligned} B^{(j,j')}(k) &:= \frac{\pi}{|k|}\int_{\mathbb{R}^{d-1}} \hat{R}(|\bar{p}|)(\bar{p}\otimes\bar{p})\cdot(\nabla\chi_j(k)\otimes\nabla\chi_{j'}(k))d\bar{p} \\ &= \delta_{j,j'}\frac{\pi|k|^5}{(d-1)^2b^2}\int_{\mathbb{R}^{d-1}} \hat{R}(|\bar{p}|)(\bar{p}\cdot e_j)^2 d\bar{p}. \end{aligned}$$

Since $\bar{p}\cdot e_j = \bar{p}\cdot f_j$, where vector $f_j = (I = \hat{k}^{\otimes 2})e_j$ is orthogonal to $\hat{k}$ and $|f_j|^2 = 1 - \hat{k}_j^2$ we obtain

$$B^{(j,j')}(k) = \delta_{j,j'}\frac{|k|^5}{(d-1)b}(1 - \hat{k}_j^2).$$

This, in particular implies that the laws of $\{\varepsilon\sum_{j=1}^d \langle N^{(\varepsilon)}_j, N^{(\varepsilon)}_j\rangle(t),\, t\ge 0\}$ are tight in $D[0,+\infty)$, as $\varepsilon\downarrow 0$, which in turn yields tightness of the laws of $\{\varepsilon^{1/2}N^{(\varepsilon)}(t),\, t\ge 0\}$, see Theorem VI.4.13, p. 358 of [4]. In fact it is clear from (4.7) that the laws are C-tight, i.e. any limiting law is supported on $C[0,+\infty)$. We can also introduce the stopped version of these processes $\{\varepsilon^{1/2}N^{(\varepsilon,\rho)}(t),\, t\ge 0\}$ defined via (4.4) with $K^{(\varepsilon)}_{t/\varepsilon}(k)$ replaced there by $K^{(\varepsilon,\rho)}_{t/\varepsilon}(k)$ for a given $\rho\in(0,1)$.

In the sequel, we shall need the following two results.

Lemma 4.2 *Suppose that $\gamma, \rho \in (0,1)$. Then,*

$$\lim_{\varepsilon \downarrow 0} \sup_{k \in \mathcal{A}(\rho)} \mathbb{E}\left\{\varepsilon^{\gamma} \int_0^{\varepsilon^{-\gamma}} \left[[\hat{K}_{t,j}^{(\varepsilon)}(k)]^2 - \frac{1}{d}\right] dt\right\}^2 = 0, \quad \forall\, j = 1, \ldots, d. \tag{4.9}$$

Proof. Let

$$\mathfrak{v}_j(k) := \frac{1}{2bd}|k|k_j^2, \quad j = 1, \ldots, d.$$

After straightforward calculations we obtain

$$L\mathfrak{v}_j(k) = \frac{1}{d} - \hat{k}_j^2$$

and

$$\mathcal{L}_{\varepsilon}\mathfrak{v}_j(k) = L\mathfrak{v}_j(k) + \varepsilon\theta_{j,\varepsilon}(k) \tag{4.10}$$

where

$$\limsup_{\varepsilon \downarrow 0} \sup_{k \in \mathcal{A}(\rho)} |\theta_{j,\varepsilon}(k)| < +\infty, \quad \forall\, \rho \in (0,1), \tag{4.11}$$

by virtue of Lemma 2.4. Therefore, the integral appearing in (4.9) equals

$$\mathfrak{N}_j^{(\varepsilon)}\left(\frac{1}{\varepsilon^{\gamma}}\right) + \mathfrak{v}_j(k) - \mathfrak{v}_j(K_{\varepsilon^{-\gamma}}^{(\varepsilon)}(k)) + \varepsilon \int_0^{\varepsilon^{-\gamma}} \theta_{j,\varepsilon}\left(K_s^{(\varepsilon)}(k)\right) ds.$$

Here

$$\mathfrak{N}_j^{(\varepsilon)}(t) := \mathfrak{v}_j(K_t^{(\varepsilon)}(k)) - \mathfrak{v}_j(k) - \int_0^t \mathcal{L}_{\varepsilon}\mathfrak{v}_j(K_s^{(\varepsilon)}(k))ds \tag{4.12}$$

is a square integrable martingale with the predictable quadratic variation equal to

$$\langle \mathfrak{N}_j^{(\varepsilon)} \rangle(t) = \int_0^t D_{j,j'}(K_s^{(\varepsilon)}(k))ds, \tag{4.13}$$

where

$$\begin{aligned} D_{\varepsilon}^{(j,j')}(k) &:= \int_{\mathbb{R}^d} [\mathfrak{v}_j(k') - \mathfrak{v}_j(k)][\mathfrak{v}_{j'}(k') - \mathfrak{v}_{j'}(k)]Q^{(\varepsilon)}(k,k')dk' \\ &= \frac{2}{\varepsilon}\int_{\mathbb{R}^d} \frac{(p \otimes p) \cdot \nabla\mathfrak{v}_j(k) \otimes \nabla\mathfrak{v}_{j'}(k)(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|) + (|p|^2 + 2p_1|k|/\varepsilon)^2} + o(1), \end{aligned}$$

and

$$\nabla\mathfrak{v}_j(k) = k_j^2\hat{k} + 2k_j|k|e_j.$$

With the calculation as in Lemmas 2.1 and 2.4 we conclude that

$$\limsup_{\varepsilon \downarrow 0} \sup_{k \in \mathcal{B}(\rho)} D_{\varepsilon}^{(j,j')}(k) < +\infty. \tag{4.14}$$

Let $\tau^{(\varepsilon,\rho)}(k)$ be the exit time of $K_t^{(\varepsilon)}(k)$ from $\mathcal{A}(\rho)$. Thanks to Theorem 3.1, for any $\sigma > 0$ and $\rho \in (0,1)$ we can find $\varepsilon_0 > 0$ such that

$$\sup_{k \in \mathcal{A}(\rho)} \mathbb{P}\left[\tau^{(\varepsilon,\rho/2)}(k) \leq \varepsilon^{-\gamma}\right] < \sigma, \quad \forall\, \varepsilon \in (0,\varepsilon_0]. \tag{4.15}$$

It is clear, in light of (4.11) and Lemma 2.3, that

$$\sup_{k\in\mathcal{A}(\rho)} \mathbb{E}\left\{\varepsilon^{\gamma}\int_0^{\varepsilon^{-\gamma}}\left[[\hat{K}^{(\varepsilon)}_{t,j}(k)]^2-\frac{1}{d}\right]dt,\ \tau^{(\varepsilon,\rho/2)}\geq\varepsilon^{-\gamma}\right\}^2 \tag{4.16}$$
$$=\sup_{k\in\mathcal{A}(\rho)} \mathbb{E}\left[\varepsilon^{2\gamma}\langle\mathfrak{N}^{(\varepsilon)}_j\rangle\left(\frac{1}{\varepsilon^{\gamma}}\right),\ \tau^{(\varepsilon,\rho/2)}\geq\varepsilon^{-\gamma}\right]+o(1)=o(1),\qquad \text{as } \varepsilon\ll 1.$$

The last equality following from (4.14). Thanks to (4.15), the left hand side of (4.9) is as small as we wish, thus the conclusion of the lemma follows. □

The next lemma shows that the momentum and the spatial position become asymptotically independent.

Lemma 4.3 *For any $\rho\in(0,1)$ the quadratic covariation between the martingale $\{\varepsilon^{1/2}N^{(\varepsilon)}(t),\, t\geq 0\}$ and semi-martingale $\{\mathfrak{K}^{(\varepsilon)}_t(k),\, t\geq 0\}$ satisfies*

$$\lim_{\varepsilon\downarrow 0}\sup_{k\in\mathcal{B}(\rho)}\sup_{t\in[0,T]}\left|\langle\varepsilon^{1/2}N^{(\varepsilon)},\mathfrak{K}^{(\varepsilon)}(k)\rangle(t)\right|=0,\quad \forall T>0 \tag{4.17}$$

in probability. The above statement holds also for $\{\varepsilon^{1/2}N^{(\varepsilon,\rho)}(t),\, t\geq 0\}$ and $\{\mathfrak{K}^{(\varepsilon,\rho)}_t(k),\, t\geq 0\}$ for an arbitrary $\rho\in(0,1)$.

Proof. Denote by $\varepsilon^{1/2}N^{(\varepsilon)}_1$ the components of the martingale in the direction $\hat{k}$, while by $\varepsilon^{1/2}N^{(\varepsilon)}_\perp$ its orthogonal complement. It can easily be checked via a direct calculation that

$$\langle\varepsilon^{1/2}N^{(\varepsilon)},\mathfrak{K}^{(\varepsilon)}(k)\rangle(t)=\int_0^t C_\varepsilon\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right)ds,$$

where $C_\varepsilon(k):=(C_{\varepsilon,1}(k),\ldots,C_{\varepsilon,d}(k))$ and

$$C_{\varepsilon,j}(k):=\frac{1}{\varepsilon^{1/2}}\int_{\mathbb{R}^d}(|k'|-|k|))[\chi_j(k')-\chi_j(k)]Q^{(\varepsilon)}(k,k')dk'.$$

We can write then

$$\langle\varepsilon^{1/2}N^{(\varepsilon)}_1,\mathfrak{K}^{(\varepsilon)}(k)\rangle(t)=\int_0^t C^{(1)}_\varepsilon\left(K^{(\varepsilon)}_{s/\varepsilon}(k)\right)ds, \tag{4.18}$$

where

$$C^{(1)}_\varepsilon(k):=\frac{b}{(d-1)\varepsilon^{1/2}}\int_{\mathbb{R}^d}(|k'|-|k|))[|k'|^3(k'\cdot\hat{k})-|k|^4|]Q^{(\varepsilon)}(k,k')dk'$$

After changing variables $p:=(k'-k)/\varepsilon$ (recall that $p_1:=p\cdot\hat{k}$ and $\bar{p}:=p-(p\cdot\hat{k})\hat{k}$) the expression for $C^{(1)}_\varepsilon(k)$ can be written as $C^{(1)}_{\varepsilon,1}(k)+C^{(1)}_{\varepsilon,2}(k)$ where

$$C^{(1)}_{\varepsilon,1}(k):=\frac{b}{(d-1)\varepsilon^{5/2}}\int_{\mathbb{R}^d}\frac{2(|k+\varepsilon p|-|k|)|k+\varepsilon p|^3 p_1(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|)+(|p|^2+2|k|p_1/\varepsilon)^2},$$
$$C^{(1)}_{\varepsilon,2}(k):=\frac{b|k|}{(d-1)\varepsilon^{7/2}}\int_{\mathbb{R}^d}\frac{2(|k+\varepsilon p|-|k|)(|k+\varepsilon p|^3-|k|^3)(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|)+(|p|^2+2|k|p_1/\varepsilon)^2}. \tag{4.19}$$

Term $C^{(1)}_{\varepsilon,1}(k)$ is of the same order of magnitude, as $\varepsilon\ll 1$, as

$$\frac{2b|k|^3}{(d-1)\varepsilon^{3/2}}\int_{\mathbb{R}^d}\frac{p_1^2(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|)+(|p|^2+2|k|p_1/\varepsilon)^2}.$$

Therefore, by virtue of Lemma 2.4 (see in particular (2.30)), we obtain

$$C_{\varepsilon,1}^{(1)}(k) \le C|k|^{3/2}\left[\left(\frac{\varepsilon}{2|k|}\right)^{1/2} \wedge 1\right]. \tag{4.20}$$

Considering the martingale (3.41) where $g(k) = |k-k_0|^4$ we arrive also at

$$\sup_{t\in[0,T]}\sup_{k_0\in\mathcal{B}(\rho)} \mathbb{E}[\mathfrak{K}_t^{(\varepsilon)}(k_0)]^4 < +\infty, \tag{4.21}$$

implying, in particular, a bound on the 3/2-moment. Therefore, (4.20) implies

$$\lim_{\varepsilon\downarrow 0}\sup_{k\in\mathcal{B}(\rho)}\sup_{t\in[0,T]}\int_0^t C_{\varepsilon,1}^{(1)}\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right)ds = 0.$$

In a similar fashion we obtain also that

$$\lim_{\varepsilon\downarrow 0}\sup_{k\in\mathcal{B}(\rho)}\sup_{t\in[0,T]}\int_0^t C_{\varepsilon,2}^{(1)}\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right)ds = 0.$$

The above argument allows us to conclude that the right side of (4.18) is of order of magnitude $o(1)$, as $\varepsilon \ll 1$. Note also that

$$\langle \varepsilon^{1/2}N_\perp^{(\varepsilon)}, \mathfrak{K}^{(\varepsilon)}\rangle(t) = \int_0^t C_\varepsilon^{(2)}\left(K_{s/\varepsilon}^{(\varepsilon)}(k)\right)ds,$$

where

$$C_\varepsilon^{(2)}(k) := \frac{b}{(d-1)\varepsilon^{5/2}}\int_{\mathbb{R}^d}\frac{2(|k+\varepsilon p|-|k|)|k+\varepsilon p|^3\bar{p}(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|)+(|p|^2+2|k|p_1/\varepsilon)^2},$$

which equals, up to a term of order $o(1)$, to

$$\frac{2|k|^3 b}{(d-1)\varepsilon^{3/2}}\int_{\mathbb{R}^d}\frac{p_1\bar{p}(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|)+(|p|^2+2|k|p_1/\varepsilon)^2} = 0.$$

The conclusion of the lemma then follows. □

Limit identification

Denote by $\mathfrak{Q}_{k,x}^{(\varepsilon)}$, $\mathfrak{Q}_{k,x}^{(\varepsilon,\rho)}$ the laws of $\{(\mathfrak{K}_t^{(\varepsilon)}(k), x+\varepsilon^{1/2}N^{(\varepsilon)}(t)),\ t\ge 0\}$ and the truncated process $\{(\mathfrak{K}_t^{(\varepsilon,\rho)}(k), x+\varepsilon^{1/2}N^{(\varepsilon,\rho)}(t)),\ t\ge 0\}$ on $D[0,+\infty)$, respectively. Recall that the truncation in case of dimension $d\ge 4$ corresponds to the kernel $Q^{(\varepsilon,\rho)}(k,k')$ given by (2.36), while in dimensions $d=2,3$ it corresponds to the kernel defined with the help of the cut-off function as in (3.14). From our previous results it follows that these families are C-tight, as $\varepsilon\downarrow 0$, for any $\rho\in(0,1)$. In addition, let $\mathfrak{Q}_{\ell,x}$ and $\mathfrak{Q}_{\ell,x}^{(\rho)}$ be the respective laws of the diffusion corresponding to the generator $\mathfrak{L}$ defined by (1.19). Generator $\mathfrak{L}_\rho$ is defined analogously, except for the fact that M is replaced in (1.19) by M_ρ given by (3.3).

Let $\pi=(\pi_1,\pi_2)$ be the canonical map, with π_1 and π_2 corresponding to the K and N components. By $(\mathcal{M}_s)$ we denote the canonical filtration and $\mathcal{M}$ the σ-algebra generated by all $\mathcal{M}_s$, Let $f \in C_0^\infty(\mathbb{R}^{1+d})$, $s < t$ and $t_m := m\varepsilon^\gamma$ where $\gamma \in (0,1)$. We partition the interval $[s,t]$ using the points t_m into intervals of size ε^γ. For any $A \in \mathcal{M}_s$ we can write, using Taylor expansion, (3.4), (4.7) and (4.8)

$$E^{(\varepsilon,\rho)}\left[f(\pi(t)) - f(\pi(s)), A\right] = \sum_m E^{(\varepsilon,\rho)}\left[f(\pi(t_{m+1})) - f(\pi(t_m)), A\right] + o(1) \tag{4.22}$$

$$= E^{(\varepsilon,\rho)}\left[\int_s^t M_\rho f(\pi(u))du, A\right] + \frac{1}{2}\sum_m E^{(\varepsilon,\rho)}\left[\int_{t_m}^{t_{m+1}} B^{(j,j)}(\pi(s))\partial^2_{2,jj} f(\pi(u))du, A\right] + o(1),$$

where $E^{(\varepsilon,\rho)}_{x,k}$ is the expectation with respect to $\mathfrak{Q}^{(\varepsilon,\rho)}_{k,x}$ and the summation ranges over m-s such that $t_m \in [s,t]$. Here $\partial^2_{2,jj}$ denotes the second derivative with respect to the j-th component of the variable π_2. We can now make use of Lemma 4.2 and obtain that the utmost right hand side of (4.22) can be approximated by

$$E^{(\varepsilon,\rho)}\left[\int_s^t \mathfrak{L}_\rho f(\pi(u))du, A\right] + o(1).$$

Letting $\varepsilon \to 0$ we conclude that any limiting measure of $\mathfrak{Q}^{(\varepsilon,\rho)}_{k,x}$, as $\varepsilon \downarrow 0$, coincides with the solution of the martingale problem corresponding to the generator $\mathfrak{L}_\rho$. Thus, it equals to $\mathfrak{Q}^{(\rho)}_{\ell,x}$. We can remove the truncation that corresponds to parameter ρ in the same way as in the proof of Theorem 2.5 and conclude that $\mathfrak{Q}^{(\varepsilon)}_{k,x}$ converge weakly, as $\varepsilon \downarrow 0$, towards $\mathfrak{Q}_{\ell,x}$ finishing in this way the proof of Theorem 1.2.

5 Proofs of auxiliary results

In this section we prove the technical estimates of Lemmas 2.4 and 4.1, and Proposition 3.2.

5.1 Proof of Lemma 2.4

Convergence of the drift

We first prove (2.29). After a straightforward computation using (2.25), (2.4) and an elementary change of variables $p := (k' - k)/\varepsilon$ we obtain

$$b_\varepsilon(k) = \frac{\hat{k}}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{2p_1(\gamma\hat{R})(|p|)dp}{\gamma^2(|p|) + (|p|^2 + 2|k|\varepsilon^{-1}p_1)^2}. \tag{5.1}$$

Performing a change of variables $p_1' := 2|k|p_1/\varepsilon$, $\bar{p}' := \bar{p}$ we get

$$b_\varepsilon(k) = \frac{\tilde{b}_{\varepsilon_1}\hat{k}}{2|k|^2} \tag{5.2}$$

with $\varepsilon_1 := \varepsilon/(2|k|)$ and

$$\tilde{b}_\varepsilon := \int_{\mathbb{R}^d} \frac{p_1(\gamma\hat{R})(\{(\varepsilon p_1)^2 + |\bar{p}|^2\}^{1/2})dp}{\gamma^2(\{(\varepsilon p_1)^2 + |\bar{p}|^2\}^{1/2}) + (|\bar{p}|^2 + (\varepsilon p_1)^2 + p_1)^2},$$

Take $\kappa \in (0,1)$ (that will be specified later on), and divide the domain of integration into two regions given by $\{|p_1| > \varepsilon_1^{-1-\kappa}\}$, and $\{\varepsilon_1^{-1-\kappa} \ge |p_1|\}$, respectively. We write, accordingly, $\tilde{b}_{\varepsilon_1} = \tilde{b}^{(1)}_{\varepsilon_1} + \tilde{b}^{(2)}_{\varepsilon_1}$, where $\tilde{b}^{(i)}_{\varepsilon_1}$, $i = 1,2$ are the integrals corresponding to the aforementioned regions of integration. Estimate (2.29) is a consequence of the following.

Lemma 5.1 *There exist constants $C, \delta > 0$ such that*

$$|\tilde{b}_\varepsilon + 2(d-1)b - 2\varepsilon(d-2)c| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1], \tag{5.3}$$

with the constants b and c given by (1.11) *and* (1.15), *respectively.*

Proof. For the sake of abbreviation we introduce the notation

$$g_1(r) := \gamma(r)\hat{R}(r), \quad g_2(r) := \gamma^2(r), \quad r \in [0,+\infty), \tag{5.4}$$

$$G(q,\bar{p},p_1) := g_2\left((q^2+|\bar{p}|^2)^{1/2}\right) + (|\bar{p}|^2 + q^2 + p_1)^2$$

and

$$F(q,p) := \frac{p_1 g_1\left((q^2+|\bar{p}|^2)^{1/2}\right)}{G(q,\bar{p},p_1)}, \quad q \in \mathbb{R}, \quad p \in \mathbb{R}^d. \tag{5.5}$$

Note that $F(q,p)$ is a function of $(d+1)$-variables. Let us define

$$\mathcal{D}(q) = \int_{\mathbb{R}^{d-1}} g_1((q^2+|\bar{p}|^2)^{1/2})d\bar{p}, \tag{5.6}$$

then, as g_1 is rapidly decaying, we have

$$\mathcal{D}(q) \le \frac{C_N}{1+|q|^N}, \tag{5.7}$$

for any $N \in \mathbb{N}$. With this notation, in the first region we have

$$|\tilde{b}_\varepsilon^{(1)}(k)| \le C\int_{\varepsilon^{-1-\kappa}}^{+\infty} |p_1|\mathcal{D}(\varepsilon p_1)dp_1 \le C\varepsilon^N, \quad \forall\, \varepsilon \in (0,1],$$

due to (5.7), with the constant C that depends on $\rho \in (0,1)$ and $N \ge 1$.

To compute the limit of $\tilde{b}_\varepsilon^{(2)}$ we further divide the domain of integration. Let $\kappa_1 \in (0,1)$ and consider the regions, where $|\bar{p}| \le \varepsilon^{-\kappa_1}$, or $|\bar{p}| > \varepsilon^{-\kappa_1}$. The corresponding integrals shall be denoted by $\tilde{b}_\varepsilon^{(2,i)}$, $i = 1,2$.

To estimate $\tilde{b}_\varepsilon^{(2,2)}$ note that the rapid decay of the function $g_1(p)$ implies that

$$|\tilde{b}_\varepsilon^{(2,2)}| \le C\varepsilon^{-2-2\kappa} \sup_{|q|\le\varepsilon^{-\kappa}} \int_{|\bar{p}|\ge\varepsilon^{-\kappa_1}} g_1(\{q^2+|\bar{p}|^2\}^{1/2})d\bar{p} \le C\varepsilon^{1+\delta}$$

for some $C, \delta > 0$. Finally, we deal with $\tilde{b}_\varepsilon^{(2,1)}$ that can be written as

$$\tilde{b}_\varepsilon^{(2,1)} = \int_{|p_1|\le\varepsilon^{-1-\kappa}} \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} F(\varepsilon p_1, p)dp = \beta_\varepsilon + r_\varepsilon, \tag{5.8}$$

where

$$\beta_\varepsilon := \int_{|p_1|\le\varepsilon^{-1-\kappa}} \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} F(0,p)dp$$

and (since $F(q,p)$ is even in q)

$$r_\varepsilon := \int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} [F(\varepsilon p_1, p) - F(0,p) + F(\varepsilon p_1, -p_1, \bar{p}) - F(0,-p_1,\bar{p})]d\bar{p}.$$

A simple calculation shows that

$$\beta_\varepsilon = -\int_{|\bar{p}|\le \varepsilon^{-\kappa_1}} |\bar{p}|^2 \hat{R}(|\bar{p}|)\left[\arctan\left(\frac{\varepsilon^{-1-\kappa}+|\bar{p}|^2}{\gamma(|\bar{p}|)}\right) - \arctan\left(\frac{-\varepsilon^{-1-\kappa}+|\bar{p}|^2}{\gamma(\bar{p}|)}\right)\right] d\bar{p}$$
$$+\frac{1}{2}\int_{|\bar{p}|\le \varepsilon^{-\kappa_1}} g_1(\bar{p}) \log\left[\frac{G(0,\bar{p},\varepsilon^{-1-\kappa})}{G(0,\bar{p},-\varepsilon^{-1-\kappa})}\right] d\bar{p}.$$

Hence, there exist $C,\delta>0$ such that

$$\sup_{k\in\mathcal{A}(\rho)} |\beta_\varepsilon + 2(d-1)b| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon\in(0,1].$$

On the other hand

$$r_\varepsilon = \varepsilon\int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}|\le \varepsilon^{-\kappa_1}} d\bar{p}\int_0^{p_1}\{\partial_q F(\varepsilon q, p_1,\bar{p}) + \partial_q F(\varepsilon q,-p_1,\bar{p})\}dq. \tag{5.9}$$

Note that

$$\partial_q F(q,p) = \frac{p_1 g_1'(\{|\bar{p}|^2+q^2\}^{1/2})q}{\{|\bar{p}|^2+q^2\}^{1/2}G(q,\bar{p},p_1)} - \frac{p_1 g_3(\{|\bar{p}|^2+q^2\}^{1/2})q}{\{|\bar{p}|^2+q^2\}^{1/2}G^2(q,\bar{p},p_1)} \tag{5.10}$$
$$-4(|\bar{p}|^2+q^2+p_1)g_1(\{|\bar{p}|^2+q^2\}^{1/2})\frac{p_1 q}{G^2(q,\bar{p},p_1)},$$

where $g_3(r)=g_1(r)g_2'(r)$. Substituting (5.10) into (5.9) we can write the above formula as the sum

$$r_\varepsilon = \sum_{i=1}^3 r_\varepsilon^{(i)},$$

with terms $r_\varepsilon^{(i)}$ $i=1,2,3$ corresponding to the respective terms in the right hand side of (5.10).

We can write that (in order to symmetrize $r_\varepsilon^{(1)}$)

$$r_\varepsilon^{(1)} = \sum_{i=1}^3 r_\varepsilon^{(1,i)},$$

where

$$r_\varepsilon^{(1,1)} := \sum_{\sigma=\pm1}\sigma\varepsilon^2\int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} d\bar{p}\int_0^{p_1}\frac{[p_1+\sigma(|\bar{p}|^2+(\varepsilon q)^2)]g_1'(\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2})q\,dq}{\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2}G^2(\varepsilon q,\bar{p},\sigma p_1)} \tag{5.11}$$

and

$$r_\varepsilon^{(1,2)} := -\sum_{\sigma=\pm1}\varepsilon^4\int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} d\bar{p}\int_0^{p_1}\frac{q^3 g_1'(\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2})dq}{\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2}G^2(\varepsilon q,\bar{p},\sigma p_1)} \tag{5.12}$$

while

$$r_\varepsilon^{(1,3)} := -\sum_{\sigma=\pm1}\varepsilon^2\int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} d\bar{p}\int_0^{p_1}\frac{|\bar{p}|^2 q g_1'(\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2})dq}{\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2}G^2(\varepsilon q,\bar{p},\sigma p_1)}. \tag{5.13}$$

Interchanging the integration in p_1 and q, and integrating out p_1 variable in (5.11) we get

$$r_\varepsilon^{(1,1)} = \frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-1-\kappa}} dq \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} \frac{g_1'(\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2})q}{\{|\bar{p}|^2+(\varepsilon q)^2\}^{1/2}}\left\{\mathcal{L}(\varepsilon q,\varepsilon^{-1-\kappa},\bar{p})-\mathcal{L}(\varepsilon q,q,\bar{p})\right\}d\bar{p}, \tag{5.14}$$

where

$$\mathcal{L}(q,r,\bar{p}) := \log\left[\frac{G(q,\bar{p},r)}{G(q,\bar{p},-r)}\right].$$

After changing the variables $q' := \varepsilon q$ we conclude that the utmost right hand side of (5.11) equals

$$r_\varepsilon^{(1,1)} = \frac{\varepsilon}{2}\int_0^{\varepsilon^{-\kappa}} dq \int_{|\bar{p}|\le\varepsilon^{-\kappa_1}} \frac{g_1'(\{|\bar{p}|^2+q^2\}^{1/2})q}{\{|\bar{p}|^2+q^2\}^{1/2}}\cdot\frac{\mathcal{L}(q,\varepsilon^{-1-\kappa},\bar{p})-\mathcal{L}(q,q/\varepsilon,\bar{p})}{\varepsilon}d\bar{p}. \tag{5.15}$$

We use an elementary estimate. For any $\kappa,\kappa_1\in(0,1/16)$ there exist $C,\delta>0$ such that

$$\sup_{\gamma_0^2\le A\le\gamma_0^{-2}}\ \sup_{\varepsilon^\delta\le y\le\varepsilon^{-\kappa}}\ \sup_{|a|\le\varepsilon^{-\kappa_1}}\left|\frac{1}{\varepsilon}\log\left[\frac{(y/\varepsilon+a)^2+A}{(y/\varepsilon-a)^2+A}\right]-\frac{4a}{y}\right|\le C\varepsilon^\delta,\quad\forall\,\varepsilon\in(0,1].$$

With this estimate in hand we conclude that for some $C,\delta>0$ we have

$$\left|r_\varepsilon^{(1,1)}+\varepsilon\int_{\mathbb{R}^d} g_1'(|p|)|p|dp\right|\le C\varepsilon^{1+\delta},\quad\forall\,\varepsilon\in(0,1],$$

or equivalently

$$\left|r_\varepsilon^{(1,1)}-\varepsilon d\int_{\mathbb{R}^d} g_1(|p|)dp\right|\le C\varepsilon^{1+\delta},\quad\forall\,\varepsilon\in(0,1]. \tag{5.16}$$

On the other hand, changing order of integration between q and p_1 and variables according to $q' := \varepsilon q$ we get

$$r_\varepsilon^{(1,2)} = -\sum_{\sigma=\pm1}\varepsilon\int_0^{\varepsilon^{-\kappa}} dq\int_{|\bar{p}|\le\varepsilon^{-\kappa_1}}\frac{q^3 g_1'(\{|\bar{p}|^2+q^2\}^{1/2})d\bar{p}}{\{|\bar{p}|^2+q^2\}^{1/2}}\left\{\frac{1}{\varepsilon}\int_{q/\varepsilon}^{\varepsilon^{-1-\kappa}}\frac{dp_1}{G(q,\bar{p},\sigma p_1)}\right\}. \tag{5.17}$$

One can verify the following elementary property of integrals. For any $\kappa,\kappa_1\in(0,1/16)$ there exist $C,\delta>0$ such that

$$\sup_{\gamma_0^2\le A\le\gamma_0^{-2}}\ \sup_{\varepsilon^\delta\le y\le\varepsilon^{-\kappa}}\ \sup_{|a|\le\varepsilon^{-\kappa_1}}\left|\frac{1}{\varepsilon}\int_{y/\varepsilon}^{\varepsilon^{-1-\kappa}}\frac{dx}{A+(x+a)^2}-\frac{1}{y}\right|\le C\varepsilon^\delta,\quad\forall\,\varepsilon\in(0,1]. \tag{5.18}$$

Hence, there exist $C,\delta>0$ such that

$$\left|r_\varepsilon^{(1,2)}+\frac{\varepsilon}{d}\int_{\mathbb{R}^d} g_1'(|p|)|p|dp\right|\le C\varepsilon^{1+\delta},\quad\forall\,\varepsilon\in(0,1]. \tag{5.19}$$

Finally, in the same fashion we obtain

$$\left|r_\varepsilon^{(1,3)}+\varepsilon\left(1-\frac{1}{d}\right)\int_{\mathbb{R}^d} g_1'(|p|)|p|dp\right|\le C\varepsilon^{1+\delta},\quad\forall\,\varepsilon\in(0,1]. \tag{5.20}$$

This together with (5.19) yields

$$\left|r_\varepsilon^{(1,2)}+r_\varepsilon^{(1,3)}-\varepsilon d\int_{\mathbb{R}^d} g_1(|p|)dp\right|\le C\varepsilon^{1+\delta},\quad\forall\,\varepsilon\in(0,1] \tag{5.21}$$

for some $C, \delta > 0$. Summarizing, from (5.16) and (5.21) it follows that

$$\left| r_\varepsilon^{(1)} - 2\varepsilon d \int_{\mathbb{R}^d} g_1(|p|) dp \right| \le C\varepsilon^{1+\delta}, \quad \forall \varepsilon \in (0,1] \tag{5.22}$$

for some $C, \delta > 0$.

Concerning $r_\varepsilon^{(2)}$ we can write it as $r_\varepsilon^{(2)} = r_\varepsilon^{(2,1)} + r_\varepsilon^{(2,2)}$, where

$$r_\varepsilon^{(2,1)} = \sum_{\sigma=\pm 1} \frac{\varepsilon^2}{2} \int_0^{\varepsilon^{-1-\kappa}} q dq \int_{|\bar{p}| \le \varepsilon^{-\kappa_1}} \frac{g_3(\{|\bar{p}|^2 + (\varepsilon q)^2\}^{1/2})}{\{|\bar{p}|^2 + (\varepsilon q)^2\}^{1/2}} \left\{ \frac{1}{G(\varepsilon q, \bar{p}, \sigma \varepsilon^{-1-\kappa})} - \frac{1}{G(\varepsilon q, \bar{p}, \sigma q)} \right\} d\bar{p}, \tag{5.23}$$

and

$$r_\varepsilon^{(2,2)} = \sum_{\sigma=\pm 1} \varepsilon^2 \int_0^{\varepsilon^{-1-\kappa}} q dq \int_{|\bar{p}| \le \varepsilon^{-\kappa_1}} (|\bar{p}|^2 + (\varepsilon q)^2)^{1/2} g_3(\{|\bar{p}|^2 + (\varepsilon q)^2\}^{1/2}) d\bar{p} \int_q^{\varepsilon^{-1-\kappa}} \frac{dp_1}{G^2(\varepsilon q, \bar{p}, \sigma p_1)} \tag{5.24}$$

Changing variables $q' := \varepsilon q$ in both formulas expressing $J_{2,1}^{(\varepsilon)}$ and $J_{2,2}^{(\varepsilon)}$ and using an elementary estimate

$$\sup_{\gamma_0^2 \le A \le \gamma_0^{-2}} \sup_{\varepsilon^\delta \le y \le \varepsilon^{-\kappa}} \sup_{|a| \le \varepsilon^{-\kappa_1}} \left| \frac{1}{\varepsilon} \int_{y/\varepsilon}^{\varepsilon^{-1-\kappa}} \frac{dx}{[A + (x+a)^2]^2} - \frac{1}{3y^3} \right| \le C\varepsilon^\delta, \quad \forall \varepsilon \in (0,1].$$

to bound $r_\varepsilon^{(2,2)}$ we conclude that

$$\left| r_\varepsilon^{(2)} \right| \le C\varepsilon^{1+\delta}, \quad \forall \varepsilon \in (0,1]$$

for some $C, \delta > 0$.

To estimate $r_\varepsilon^{(3)}$ observe that it can be rewritten as $r_\varepsilon^{(3)} = \sum_{i=1}^3 r_\varepsilon^{(3,i)}$, where

$$\begin{aligned} r_\varepsilon^{(3,1)} &:= -4 \sum_{\sigma=\pm 1} \varepsilon^2 \int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}| \le \varepsilon^{-\kappa_1}} d\bar{p} \int_0^{p_1} \frac{g_1(\{|\bar{p}|^2 + (\varepsilon q)^2\}^{1/2}) q dq}{G(\varepsilon q, \bar{p}, \sigma p_1)}, \\ r_\varepsilon^{(3,2)} &:= 4 \sum_{\sigma=\pm 1} \sigma \varepsilon^2 \int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}| \le \varepsilon^{-\kappa_1}} d\bar{p} \int_0^{p_1} [p_1 + \sigma(|\bar{p}|^2 + (\varepsilon q)^2)] \\ &\quad \times g_1(\{|\bar{p}|^2 + (\varepsilon q)^2\}^{1/2}) (|\bar{p}|^2 + (\varepsilon q)^2) \frac{q dq}{G^2(\varepsilon q, \bar{p}, \sigma p_1)}, \end{aligned}$$

and

$$r_\varepsilon^{(3,3)} := 4 \sum_{\sigma=\pm 1} \varepsilon^2 \int_0^{\varepsilon^{-1-\kappa}} dp_1 \int_{|\bar{p}| \le \varepsilon^{-\kappa_1}} d\bar{p} \int_0^{p_1} dq \frac{g_4(\{|\bar{p}|^2 + (\varepsilon q)^2\}^{1/2}) q}{G^2(\varepsilon q, \bar{p}, \sigma p_1)},$$

with $g_4(\cdot) := g_1(\cdot) g_2(\cdot)$. Arguing for $r_\varepsilon^{(3,2)}$ as in case of $r_\varepsilon^{(2,1)}$ and for $r_\varepsilon^{(3,3)}$ as in case of $r_\varepsilon^{(2,2)}$ we conclude that

$$\left| r_\varepsilon^{(3,2)} \right| + \left| r_\varepsilon^{(3,3)} \right| \le C\varepsilon^{1+\delta}, \quad \forall \varepsilon \in (0,1]$$

for some $C, \delta > 0$. The case of $r_\varepsilon^{(3,1)}$ can be argued as $r_\varepsilon^{(1,2)}$ and we conclude that

$$\left| r_\varepsilon^{(3)} + 4\varepsilon \int_{\mathbb{R}^d} g_1(|p|) dp \right| \le C\varepsilon^{1+\delta}, \quad \forall \varepsilon \in (0,1] \tag{5.25}$$

for some $C, \delta > 0$.

Summarizing, we have shown that

$$|r_\varepsilon - 2(d-2)c\varepsilon| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1], \tag{5.26}$$

which in turn, upon the substitution of ε_1 for ε, implies (2.29). □

Convergence of the covariance: the radial component

Formula for $a_\varepsilon(k)$ can be rewritten in the form

$$a_\varepsilon(k) = \frac{\hat{a}_{\varepsilon_1}(\hat{k})}{|k|}, \tag{5.27}$$

where

$$\hat{a}_\varepsilon(\hat{k}) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \frac{(p\otimes p)\gamma(|p|)\hat{R}(|p|)dp}{\gamma^2(|p|) + (|p|^2 + \varepsilon^{-1}p_1)^2}.$$

We set

$$\tilde{a}_\varepsilon(\hat{k}) := \hat{a}_\varepsilon[\hat{k},\hat{k}] = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \frac{p_1^2\gamma(|p|)\hat{R}(|p|)dp}{\gamma^2(|p|) + (|p|^2 + \varepsilon^{-1}p_1)^2}. \tag{5.28}$$

Lemma 5.2 *There exist $C, \delta > 0$*

$$\sup_{\hat{k}} \left|\tilde{a}_\varepsilon(\hat{k}) - c\varepsilon\right| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1], \tag{5.29}$$

with c given by (1.15).

Proof. From (5.28) we obtain that for a certain constant $C > 0$

$$\left|\tilde{a}_\varepsilon(\hat{k}) - c\varepsilon\right| \le \varepsilon \int_{\mathbb{R}^d} \left|\frac{(p_1/\varepsilon)^2}{\gamma^2(|p|) + (|\bar{p}|^2 + p_1^2 + (p_1/\varepsilon))^2} - 1\right| \gamma(|p|)\hat{R}(|p|)dp \le C\varepsilon D(\varepsilon) \tag{5.30}$$

with

$$D(\varepsilon) := \int_{\mathbb{R}^d} \frac{(|p_1/\varepsilon| + |p|^2)|p|^2\gamma(|p|)\hat{R}(|p|)dp}{\gamma_0^2 + (|p|^2 - |p_1/\varepsilon|)^2}. \tag{5.31}$$

To estimate of $D(\varepsilon)$ note that

$$\frac{(|p_1/\varepsilon| + |p|^2)|p|^2\gamma(|p|)\hat{R}(|p|)}{\gamma_0^2 + (|p|^2 - |p_1/\varepsilon|)^2} \le \frac{C\hat{R}^{1/2}(|p|)}{\{\gamma_0^2 + (|p|^2 - |p_1/\varepsilon|)^2\}^{1/2}}, \tag{5.32}$$

where

$$C := \sup_{u,v>0} \frac{\gamma_0^{-1}(u+v^2)v^2\hat{R}^{1/2}(v)}{\{\gamma_0^2 + (v^2-u)^2\}^{1/2}} < +\infty, \tag{5.33}$$

due to the rapid decay of function $\hat{R}(\cdot)$. We can write $D(\varepsilon) = \sum_{i=1}^2 D_i(\varepsilon)$, where the terms $D_1(\varepsilon)$, $D_2(\varepsilon)$ correspond to the integration over the regions $|p_1| \ge 2\varepsilon^\gamma$ and $|p_1| < 2\varepsilon^\gamma$. In the first case, (since $|p| < \varepsilon^{\gamma-1}$) we can easily conclude from (5.32) that

$$D_1(\varepsilon) \le C\varepsilon^\delta, \quad \forall\, \varepsilon \in (0,1],$$

for some $\delta > 0$, while in the second, again from (5.32), we get

$$D_2(\varepsilon) \le \frac{C}{\gamma_0}\int_{|p_1|\le 2\varepsilon^\gamma} dp_1 \sup_{p_1}\int_{\mathbb{R}^{d-1}} \hat{R}^{1/2}(\{p_1^2 + |\bar{p}|^2\}^{1/2})d\bar{p} \le C_1\varepsilon^\gamma,$$

for some $C_1 > 0$.□

Convergence of the covariance: the angular component

We now compute the covariance in the directions orthogonal to k. The main result of this section can be stated in the following form.

Lemma 5.3 *There exist constants $C, \delta > 0$ such that*

$$\sup_{\hat{k}} \left| \hat{a}_\varepsilon(\hat{k}) \cdot (I - \hat{k}^{\otimes 2}) - 2(d-1)b - \varepsilon c \right| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1], \tag{5.34}$$

with b and c given by (1.11) *and* (1.15)*, respectively.*

Proof. Suppose that $k^\perp$ is orthogonal to k and $|k^\perp| = 1$. We have

$$\tilde{a}_\varepsilon(k^\perp) - \tilde{a}(k^\perp) = \varepsilon \int_{\mathbb{R}^d} dp \int_0^{p_1} \partial_q F_1(\varepsilon q, p) dq,$$

where

$$F_1(q,p) := \frac{(\bar{p} \cdot k^\perp)^2 g_1(\{|\bar{p}|^2 + q^2\}^{1/2})}{G(q, \bar{p}, p_1)},$$

and

$$\tilde{a}(k^\perp) := \int_{\mathbb{R}^d} F_1(0,p)dp = \pi \int_{\mathbb{R}^{d-1}} (\bar{p} \cdot k^\perp)^2 \hat{R}(|\bar{p}|) d\bar{p},$$
$$\tilde{a}_\varepsilon(k^\perp) := \hat{a}_\varepsilon(k)[k^\perp, k^\perp] = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \frac{(\bar{p} \cdot k^\perp)^2 g_1(|p|) dp}{\gamma^2(|p|) + (|p|^2 + \varepsilon^{-1} p_1)^2}.$$

The functions $g_1(\cdot)$, $g_2(\cdot)$ are as in (5.4), while $G(q, \bar{p}, p_1)$ is defined in (5.4). Using the truncation argument we conclude that

$$\left| \tilde{a}_\varepsilon(k^\perp) - \tilde{a}(k^\perp) - \tilde{r}_\varepsilon \right| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1],$$

where $\tilde{r}_\varepsilon$ is defined as in (5.9), with $F_1(q,p)$ replacing $F(q,p)$ and $C, \delta > 0$ are some constants. In analogy to the calculations done for r_ε we can write

$$\tilde{r}_\varepsilon = \sum_{i=1}^{3} \tilde{r}_\varepsilon^{(i)},$$

with terms $\tilde{r}_\varepsilon^{(i)}$ $i = 1, 2, 3$ corresponding to the terms of the expression

$$\partial_q F_1(q,p) = \frac{(\bar{p} \cdot k^\perp)^2 g_1'(\{|\bar{p}|^2 + q^2\}^{1/2}) q}{\{|\bar{p}|^2 + q^2\}^{1/2} G(q, \bar{p}, p_1)} - \frac{(\bar{p} \cdot k^\perp)^2 g_3(\{|\bar{p}|^2 + q^2\}^{1/2}) q}{\{|\bar{p}|^2 + q^2\}^{1/2} G^2(q, \bar{p}, p_1)}$$
$$- \frac{4(\bar{p} \cdot k^\perp)^2 q}{G^2(q, \bar{p}, p_1)} (|\bar{p}|^2 + q^2 + p_1) g_1(\{|\bar{p}|^2 + q^2\}^{1/2}). \tag{5.35}$$

We have

$$\tilde{r}_\varepsilon^{(1)} = \sum_{\sigma = \pm 1} \int_0^{\varepsilon^{-\kappa}} q dq \int_{|\bar{p}| \le \varepsilon^{-\kappa_1}} d\bar{p} \int_{q/\varepsilon}^{\varepsilon^{-1-\kappa}} \frac{(\bar{p} \cdot k^\perp)^2 g_1'(\{|\bar{p}|^2 + q^2\}^{1/2}) dp_1}{\{|\bar{p}|^2 + q^2\}^{1/2} G(q, \bar{p}, \sigma p_1)}, \tag{5.36}$$

which, upon an application of (5.18), yields

$$\left| \tilde{r}_\varepsilon^{(1)} - \varepsilon \int_{\mathbb{R}^d} (\bar{p} \cdot k^\perp)^2 g_1'(|p|) \frac{dp}{|p|} \right| \le C\varepsilon^{1+\delta}.$$

We conclude from the symmetry considerations (recall that $|k^\perp| = 1$) that the integral above equals

$$\frac{1}{d}\int_{\mathbb{R}^d} |p| g_1'(|p|)dp = -c.$$

Repeating the calculations concerning $r_\varepsilon^{(2)}$, $r_\varepsilon^{(3)}$ we obtain that

$$\left|\tilde{r}_\varepsilon^{(i)}\right| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1],\, i = 2,3.$$

Finally, for the mixed component of the diffusivity matrix, let $k^\perp \in \mathbb{R}^d$ be a vector satisfying $|k^\perp| = 1$ and $k^\perp \perp \hat{k}$. Integrating out $\bar{p}$ first and using odd parity of the expression in this variable we conclude that $\tilde{a}_\varepsilon(\hat{k}, k^\perp) = 0$, where

$$\tilde{a}_\varepsilon(\hat{k}, k^\perp) := \hat{a}_\varepsilon(k)[\hat{k}, k^\perp] = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \frac{p_1(\bar{p}\cdot k^\perp)\gamma(|p|)\hat{R}(|p|)dp}{\gamma^2(|p|) + (|p|^2 + \varepsilon^{-1}p_1)^2}.$$

From the above we conclude (5.29).□

Lemmas 5.2 and 5.3 together imply (2.30).

Estimates of the third moment

Tensor $d_\varepsilon(k) := \hat{d}_{\varepsilon_1}(\hat{k})$, where

$$\hat{d}_\varepsilon(\hat{k}) = \int_{\mathbb{R}^d} \frac{2p^{\otimes 3}\gamma(|p|)\hat{R}(|p|)dp}{\gamma^2(|p|) + (|p|^2 + p_1/\varepsilon)^2}, \tag{5.37}$$

and $\varepsilon_1 = \varepsilon/(2|k|)$. Formula (2.31) is a consequence of the following.

Lemma 5.4 *There exist constants $C, \delta > 0$ such that*

$$\sup_{\hat{k}} |\hat{d}_\varepsilon(\hat{k})| \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1]. \tag{5.38}$$

Proof. Observe that tensor vanishes in a direction $(k^\perp)^{\otimes 3}$, due to the odd parity of the third moment in the $\bar{p}$ variable. We consider only the value of the tensor in the direction $\hat{k}^{\otimes 3}$. The proof is reduced to showing that

$$|\tilde{d}(\varepsilon)| \le C\varepsilon^\delta, \quad \forall\, \varepsilon \in (0,1], \tag{5.39}$$

where

$$\tilde{d}(\varepsilon) := \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \frac{2p_1^3\gamma(|p|)\hat{R}(|p|)dp}{\gamma^2(|p|) + (|p|^2 + p_1/\varepsilon)^2}.$$

Choose $\gamma \in (0,1)$. We can write $\tilde{d}(\varepsilon) = \sum_{i=1}^2 \tilde{d}_i(\varepsilon)$, where $\tilde{d}_1(\varepsilon)$, $\tilde{d}_2(\varepsilon)$ correspond to the integration over the regions $|p| \ge \varepsilon^{\gamma-1}$ and $|p| < \varepsilon^{\gamma-1}$. Due to the rapid decay of function $\hat{R}(\cdot)$ we can easily estimate

$$|\tilde{d}_1(\varepsilon)| \le C\varepsilon^\delta, \quad \forall\, \varepsilon \in (0,1]. \tag{5.40}$$

To find an estimate of $\tilde{d}_2(\varepsilon)$ we use the following upper bound

$$\frac{1}{\varepsilon}\frac{2|p_1|^3\gamma(|p|)\hat{R}(|p|)}{\gamma^2(|p|) + (|p|^2 + p_1/\varepsilon)^2} \le \frac{2|p_1/\varepsilon||p|^2\gamma(|p|)\hat{R}(|p|)}{\gamma_0^2 + (|p|^2 - |p_1/\varepsilon|)^2} \le \frac{C\hat{R}^{1/2}(|p|)}{\{\gamma_0^2 + (|p|^2 - |p_1/\varepsilon|)^2\}^{1/2}}, \tag{5.41}$$

where

$$C := \sup_{u,v>0} \frac{2\gamma_0 uv^2 \hat{R}^{1/2}(v)}{\{\gamma_0^2 + (v^2 - u)^2\}^{1/2}} < +\infty. \tag{5.42}$$

due to the rapid decay of function $\hat{R}(\cdot)$. From this point on the estimates are identical with those done in the course of the proof of Lemma 5.2. We can write $\tilde{d}_2(\varepsilon) = \sum_{i=1}^2 \tilde{d}_{2i}(\varepsilon)$, where the terms $\tilde{d}_{21}(\varepsilon)$, $\tilde{d}_{22}(\varepsilon)$ correspond to the integration over the regions $|p_1| \geq 2\varepsilon^\gamma$ and $|p_1| < 2\varepsilon^\gamma$. In the first case, (since $|p| < \varepsilon^{\gamma-1}$) we can easily conclude from (5.41) that

$$|\tilde{d}_{21}(\varepsilon)| \leq C\varepsilon^\delta, \quad \forall\, \varepsilon \in (0,1],$$

while in the second, again from (5.41), we get

$$|\tilde{d}_{22}(\varepsilon)| \leq \frac{C}{\gamma_0} \int_{|p_1|\leq 2\varepsilon^\gamma} dp_1 \sup_{p_1} \int_{\mathbb{R}^{d-1}} \hat{R}^{1/2}(\{p_1^2 + |\bar{p}|^2\}^{1/2}) d\bar{p} \leq C_1 \varepsilon^\gamma,$$

for some $C_1 > 0$, and (5.38) follows in the direction $\hat{k}^{\otimes 3}$. It is clear that the same argument can be applied in any other "mixed" directions formed over $\hat{k}$ and $k^\perp$. $\square$

5.2 Proof of Proposition 3.2

We first prove (3.16). Let $\ell := |k|$, $\hat{k} := k/\ell$. Recall that $g(k) := f(|k|^4)$, where f satisfies the assumptions of the proposition. With this notation we can write formulas for $\nabla g(k)$, $\nabla^2 g(k)$ and $\nabla^3 g(k)$ using (3.7). In addition

$$\begin{aligned}\nabla^4 g(k) \quad &:= \quad \hat{k}^{\otimes 4} \frac{d^4}{d\ell^4} f(\ell^4) + 2S^{(1)}(\hat{k}) \frac{1}{\ell} \frac{d^3}{d\ell^3} f(\ell^4) + (S^{(2)}(\hat{k}) - 3S^{(1)}(\hat{k})) \frac{1}{\ell^2} \frac{d^2}{d\ell^2} f(\ell^4) \\ &+ \quad (3S^{(1)}(\hat{k}) - S^{(2)}(\hat{k})) \frac{1}{\ell^3} \frac{d}{d\ell} f(\ell^4).\end{aligned} \tag{5.43}$$

Here $S^{(1)}(\hat{k}) = [S^{(1)}_{i_1 i_2 i_3 i_4}(\hat{k})]$, $S^{(2)}(\hat{k}) = [S^{(2)}_{i_1 i_2 i_3 i_4}(\hat{k})]$ are given by

$$\begin{aligned} S^{(1)}_{i_1 i_2 i_3 i_4}(\hat{k}) &= \sum_{\{j,l\},\{m,n\}} \left(\delta_{i_m i_n} - \hat{k}_{i_m} \hat{k}_{i_n}\right) \hat{k}_{i_j} \hat{k}_{i_l}, \\ S^{(2)}_{i_1 i_2 i_3 i_4}(\hat{k}) &= \sum_{\{j,l\},\{m,n\}} \left(\delta_{i_m i_n} - \hat{k}_{i_m} \hat{k}_{i_n}\right) \delta_{i_j, i_l}. \end{aligned}$$

The summation extends over all partitions of $\{1,2,3,4\}$ into two element subsets. In addition,

$$\begin{aligned} &\frac{d}{d\ell} f(\ell^4) = 4\ell^3 f'(\ell^4), \qquad (5.44) \\ &\frac{d^2}{d\ell^2} f(\ell^4) = 12\ell^2 f'(\ell^4) + 16\ell^6 f''(\ell^4), \\ &\frac{d^3}{d\ell^3} f(\ell^4) = 24\ell f'(\ell^4) + 144\ell^5 f''(\ell^4) + 64\ell^9 f'''(\ell^4), \\ &\frac{d^4}{d\ell^4} f(\ell^4) = 24 f'(\ell^4) + 816\ell^4 f''(\ell^4) + 1152\ell^8 f'''(\ell^4) + 256\ell^{12} f^{(4)}(\ell^4). \end{aligned}$$

Recall that $H_2(k,k')$ is given by (3.6). Then, $R_\varepsilon(k,k') := g(k') - g(k) - H_2(k,k')$ can be estimated by (see (3.15))

$$|R_\varepsilon(k,k')| \leq C\|f\|'_{C^4} |k' - k|^4.$$

The estimate (3.16) will follow from the bound

$$\sup_k \left| \frac{1}{\varepsilon}\mathcal{L}_\varepsilon g(k) - \sum_{i=1}^2 J_i(k;\varepsilon) \right| \le \frac{C}{\varepsilon}\|f\|'_{C^4} \sup_k \hat{q}_\varepsilon^{(4)}(k), \tag{5.45}$$

where

$$\begin{aligned} J_1(k;\varepsilon) &:= \frac{1}{\varepsilon} b_\varepsilon(k)\cdot\nabla g(k) + \frac{1}{2\varepsilon} a_\varepsilon(k)\cdot\nabla^2 g(k), \\ J_2(k;\varepsilon) &:= \frac{1}{6\varepsilon} d_\varepsilon(k)\cdot\nabla^3 g(k). \end{aligned} \tag{5.46}$$

Recall, see (2.16), that the right hand side of (5.45) stays bounded, as $\varepsilon \in (0,1]$. Hence, in order to establish (3.16) it suffices to obtain bounds on J_1 and J_2. To this end, we show that there exist constants $C, \delta > 0$ such that

$$\sup_k |J_1(k;\varepsilon) - \mathfrak{M} f(|k|^4)| \le C\varepsilon^\delta \|f\|'_{C^2} \tag{5.47}$$

and

$$\sup_k |J_2(k;\varepsilon)| \le C\varepsilon^\delta \|f\|'_{C^3}, \quad \forall\, \varepsilon \in (0,1]. \tag{5.48}$$

The combination of (5.45), (5.47) and (5.48) implies (3.16). In order to prove (5.47), using (5.2) and (5.27) we write (recall that $\varepsilon_1 = \varepsilon/(2|k|)$)

$$J_1(k;\varepsilon) = \frac{\tilde{b}_{\varepsilon_1}}{4\varepsilon_1|k|^3}\hat{k}\cdot\nabla g(k) + \frac{1}{4\varepsilon_1|k|^2}\hat{a}_{\varepsilon_1}(\hat{k})\cdot\nabla^2 g(k). \tag{5.49}$$

Using formulas (3.7) and (5.44) we can further rewrite the right hand side of (5.49) in the form

$$\left[\frac{\tilde{b}_{\varepsilon_1}}{\varepsilon_1} + \frac{\hat{a}_{\varepsilon_1}(\hat{k})}{\varepsilon_1}\cdot(I - \hat{k}^{\otimes 2})\right] f'(|k|^4) + \frac{\hat{a}_{\varepsilon_1}(\hat{k})}{\varepsilon_1}\cdot\hat{k}^{\otimes 2}\left[3f'(|k|^4) + 4|k|^4 f''(|k|^4)\right]$$

Estimate (5.47) follows easily from Lemmas 5.1, 5.2 and 5.3. Estimate (5.48) can be derived in a similar fashion from Lemma 5.4 using formula for $\nabla^3 g(k)$, cf. (3.7).

To prove (3.17) it suffices only to show that for any $\gamma \in (0,1)$ there exist $C, \delta > 0$ such that

$$\sup_{|k|\ge\varepsilon^\gamma} \hat{q}_\varepsilon^{(4)}(k) \le C\varepsilon^{1+\delta}, \quad \forall\, \varepsilon \in (0,1]. \tag{5.50}$$

Note that

$$\frac{1}{\varepsilon}\sup_{|k|\ge\varepsilon^\gamma} \hat{q}_\varepsilon^{(4)}(k) \le C \sup_{|k|\ge\varepsilon^\gamma} \int_{\mathbb{R}^d} \frac{|p|^4(\gamma\hat{R})(|p|)dp}{\gamma_0^2 + (|p|^2 + 2kp_1/\varepsilon)^2} \le C \sup_{0<\varepsilon_1\le\varepsilon^{1-\gamma}} \int_{\mathbb{R}^d} \frac{|p|^4(\gamma\hat{R})(|p|)dp}{\gamma_0^2 + (|p|^2 - |p_1|/\varepsilon_1)^2}.$$

The right hand side can be estimated by $C\varepsilon^\delta$ for some $C, \delta > 0$ similarly as the expression in the right hand side of (5.32). □

5.3 Proof of Lemma 4.1

We proceed as in the proof of Proposition 3.2. Observe that

$$\nabla^n \chi_j(k) = \nabla^n \mathfrak{h}(k) k_j + n\nabla^{n-1}\mathfrak{h}(k) \otimes e_j,$$

where $\mathfrak{h}(k) = |k|^3$. Using formulas (3.7) and (5.43), with the function $f(\ell^4)$ replaced there by ℓ^3, we conclude that

$$\sup_{k\in\mathcal{B}(\rho)} \left| \frac{1}{\varepsilon}\mathcal{L}_\varepsilon \chi_j(k) - \sum_{i=1}^{2} \tilde{J}_i(k;\varepsilon) \right| \le \frac{C}{\varepsilon} \sup_{k\in\mathcal{B}(\rho)} |\nabla^4 \chi_j(k)| \hat{q}_\varepsilon^{(4)}(k) \le \frac{C_1}{\varepsilon} \sup_{k\in\mathcal{B}(\rho)} \hat{q}_\varepsilon^{(4)}(k). \tag{5.51}$$

Here

$$\tilde{J}_1(k;\varepsilon) := \frac{1}{\varepsilon} b_\varepsilon(k) \cdot \nabla\chi_j(k) + \frac{1}{2\varepsilon} a_\varepsilon(k) \cdot \nabla^2 \chi_j(k),$$
$$\tilde{J}_2(k;\varepsilon) := \frac{1}{6\varepsilon} d_\varepsilon(k) \cdot \nabla^3 \chi_j(k).$$

We deal with $\tilde{J}_i(k;\varepsilon)$, $i = 1,2$ as with $J_i(k;\varepsilon)$, $i = 1,2$ defined in (5.46). As in that argument, we can prove that there exist constants $C, \delta > 0$ such that

$$\sup_{k\in\mathcal{B}(\rho)} |\tilde{J}_1(k;\varepsilon) - \frac{1}{\varepsilon} L\chi_j(k)| \le C\varepsilon^\delta \tag{5.52}$$

and

$$\sup_{k\in\mathcal{B}(\rho)} |\tilde{J}_2(k;\varepsilon)| \le C\varepsilon^\delta, \quad \forall\, \varepsilon \in (0,1]. \tag{5.53}$$

This ends the proof of Lemma 4.1.

References

[1] G. Bal, T. Komorowski, L. Ryzhik, *Kinetic limits for waves in a random medium*, Kinetic and Related Models **3**, 2010, 529 – 644.

[2] Billingsley, P., *Convergence of probability measures. Second edition. Wiley Series in Probability and Statistics: Probability and Statistics.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999

[3] A. Fannjiang, Self-averaging scaling limits for random parabolic waves, Arch. Rat. Mech. Anal., **175**, 2005, 343–387.

[4] Jacod, J., Shiryaev, A., *Limit theorems for stochastic processes.* Second edition. Grundlehren der Mathematischen Wissenschaften, 288. Springer-Verlag, Berlin, 2003.

[5] Karatzas, I., Shreeve, S., *Brownian Motion and Stochastic Calculus*, Springer 2-nd edition, 1991.

[6] Oksendal, B., *Stochastic differential equations. An introduction with applications.* Sixth edition. Universitext. Springer-Verlag, Berlin, 2003.

[7] Revuz, D; Yor, M, *Continuous martingales and Brownian motion.* Third edition. Grundlehren der Mathematischen Wissenschaften ser., 293. Springer-Verlag, Berlin, 1999.

[8] Rogers, L. C. G.; Williams, D. *Diffusions, Markov processes, and martingales. Vol. 2. It calculus.* Reprint of the second (1994) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000.

[9] Stroock, D. W.; Varadhan, S. R. S. *Multidimensional diffusion processes.* Grundlehren der Mathematischen Wissenschaften, 233. Springer-Verlag, Berlin-New York, 1979.