The weak coupling limit for the random Schrödinger equation: The average wave function

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Abstract

We consider the Schrödinger equation with a time-independent weakly random potential of a strength $\varepsilon \ll 1$, with Gaussian statistics. We prove that when the initial condition varies on a scale much larger than the correlation length of the potential, the compensated wave function converges to a deterministic limit on the time scale $t \sim \varepsilon^{-2}$. This is shown under the assumption that the correlation function $R(x)$ of the random potential belongs to the Schoenberg class and decays faster than $1/|x|^2$, which ensures that the effective potential is finite. When $R(x)$ decays slower than $1/|x|^2$, the effective potential is infinite, and we establish an anomalous effective behavior for the averaged wave function on a time scale shorter than ε^{-2}, as long as the initial condition is "sufficiently macroscopic". We also consider the kinetic regime when the initial condition varies on the same scale as the random potential and obtain the limit of the averaged wave function for potentials with the correlation functions decaying faster than $1/|x|^2$. The fact that the correlation belongs to the Schoenberg class allows us to bypass the oscillatory phase estimates.

1 Introduction

We consider the large time behavior of the solutions of the weakly random Schrödinger equation

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\Delta\psi-\varepsilon V(x)\psi=0, \quad t>0, \quad x\in\mathbb{R}^d. \tag{1.1}$$

Here, the random potential $V(x)$ is a mean-zero Gaussian statistically homogeneous random field over a probability space $(\Omega,\mathcal{V},\mathbb{P})$ with the covariance function $R(x)$:

$$R(x)=\mathbb{E}(V(y)V(x+y)). \tag{1.2}$$

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We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. The small parameter $\varepsilon \ll 1$ measures the relative strength of the random fluctuations. The initial condition for (1.1):

$$\psi(0,x) = \psi_0\Big(\frac{x}{l_i}\Big), \tag{1.3}$$

varies on a scale l_i. Scale-wise, it is implicitly assumed in (1.1) that the random potential $V(x)$ varies on a scale $l_c = O(1)$.

The Schrödinger equation (1.1) preserves the total mass

$$M(t) = \int_{\mathbb{R}^d} |\psi(t,x)|^2 dx = M(0),$$

and the total energy

$$E(t) = \int_{\mathbb{R}^d} \Big[\frac{1}{2}|\nabla\psi(t,x)|^2 + \varepsilon V(x)|\psi(t,x)|^2\Big] dx = \int_{\mathbb{R}^d} |\xi|^2|\hat{\psi}(t,\xi)|^2 đ\xi + \varepsilon\int_{\mathbb{R}^d} V(x)|\psi(t,x)|^2 dx.$$

Both here and in what follows we denote $đp := dp/(2\pi)^d$. We also use the notation

$$\hat{\psi}(t,\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\psi(t,x)dx$$

for the Fourier transform of the wave function.

Since the random potential is weak, the preservation of the energy, together with the mass conservation, means, approximately, that the bulk of the energy would remain at the frequency scale $|\xi| \sim l_i^{-1}$ of the initial condition.

The kinetic regime

This problem has been extensively studied in the past when the random potential is rapidly decorrelating and the initial condition varies on the same scale as the random potential. In other words, $l_i = 1$, or, in the dimensional variables, $l_i = l_c$. This is sometimes known as the kinetic regime, and is particularly interesting since it leads to a full interaction between the random fluctuations and the wave function. The first key result here is by Spohn [28] who considered the rescaled wave function

$$\psi_\varepsilon(t,x) = \psi\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon^2}\Big),$$

and its Wigner transform

$$W_\varepsilon(t,x,\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot y}\psi_\varepsilon\Big(t, x - \frac{\varepsilon^2 y}{2}\Big)\psi_\varepsilon^*\Big(t, x + \frac{\varepsilon^2 y}{2}\Big) đy. \tag{1.4}$$

Here, z^* denotes the complex conjugate of $z \in \mathbb{C}$. The main result of [28] is that, if the correlation function $R(x)$ is smooth and sufficiently rapidly decaying, then $\mathbb{E}(W_\varepsilon(t,x,\xi))$ converges, as $\varepsilon \to 0$, in the sense of distributions to the solution $W(t, x, \xi)$ of the radiative transport equation

$$\frac{\partial W(t,x,\xi)}{\partial t} + \xi \cdot \nabla_\xi W(t,x,\xi) = \int_{\mathbb{R}^d} \hat{R}(\xi - p)\delta\left(\frac{|\xi|^2 - |p|^2}{2}\right)(W(t,x,p) - W(t,x,\xi))\frac{dp}{(2\pi)^{d-1}}. \tag{1.5}$$

Here, $\hat{R}(p)$ is the power energy spectrum, the Fourier transform of the correlation function $R(x)$.

It follows that the average density of the solution has the weak limit

$$\lim_{\varepsilon \to 0+} \mathbb{E}|\psi_\varepsilon(t,x)|^2 = \int_{\mathbb{R}^d} W(t,x,\xi)d\xi. \tag{1.6}$$

This result has been established in [28] in dimensions $d \geq 3$, on a finite time interval $0 \leq t \leq T$ with T that is independent of ε (thus, in the original microscopic variables the time interval is T/ε^2) but that does depend on the correlation function $R(x)$. The assumption that the power energy spectrum satisfies $\hat{R} \in L^1 \cap L^\infty$ is essential for the estimates in [28].

The kinetic limit has been further studied by L. Erdös and H. T. Yau in [10]. They have removed the restriction of a finite time interval convergence, and have shown that the kinetic limit holds on any finite time interval $0 \leq t \leq T$. That is, microscopically, it is valid on any time interval of the size $O(\varepsilon^{-2})$. The assumptions on the decay of the correlation function in [7, 10] are more stringent than in [28]. See also [2, 3, 6, 24, 25] for related results. Convergence of the expectation has been strengthened to the L^2-convergence in [7], with some further improvements obtained in [4].

The results of [10] were subsequently extended to the analysis of the diffusive limit in [12, 13]. For random Schrödinger equations coupled to thermal noise, diffusive limits have been studied in [14, 15, 21].

The homogenization regime

Another regime recently investigated by G. Bal and N. Zhang in [31, 32] is $l_i = \varepsilon^{-1} l_c$. That is, the initial condition for (1.1) is of the form $\psi(0, x) = \psi_0(\varepsilon x)$. This is an interesting and convenient regime as in this case the central limit time scale $O(\varepsilon^{-2})$, which comes from the size ε of the random potential, matches the homogeneous Schrödinger time scale $t \sim l_i^2$, which is the time for the linear "Schrödinger phase" $\exp(it|\xi|^2/2)$ to become of the order $O(1)$ for $|\xi| \sim l_i^{-1}$. Accordingly, when $l_i = \varepsilon^{-1}$, a natural object is the rescaled wave function

$$\psi_\varepsilon(t,x) = \psi\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\Big).$$

It has been shown in [32] in $d \geq 3$ that $\psi_\varepsilon(t, x)$ converges in probability, as $\varepsilon \to 0$ to the solution of the deterministic homogenized problem

$$i\frac{\partial \psi_*}{\partial t} + \frac{1}{2}\Delta\psi_* - R_*\psi_* = 0, \tag{1.7}$$

with the initial condition $\psi_*(0,x) = \psi_0(x)$. The effective potential in (1.7) is simply a constant

$$R_* = \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{|p|^2} đp. \tag{1.8}$$

Thus, there is a substantial difference in the behavior of the solutions when $l_i = 1$ and $l_i = \varepsilon^{-1}$. In both cases, the solution is affected in a non-trivial way by the central limit time scale $t \sim \varepsilon^{-2}$. However, in the former case, the solution behaves stochastically on this time scale – this is the kinetic regime, while in the latter it has a deterministic behavior – this is the homogenization regime.

A stochastic (non-deterministic) limit for such problems when the Laplacian is replaced by a higher power $(-\Delta)^m$ with $m > 1$ has been investigated in [31], also for rapidly decorrelating potentials.

The main results

Our main interest in this paper is in the breakdown of the homogenization regime when the effective potential R_* in (1.8) is infinite. However, all of the above results have been established under much more stringent assumptions on the correlation function than just

$$R_* < +\infty. \tag{1.9}$$

Hence, we first carry out the analysis of the behavior of the wave function only assuming (1.9), to reach the threshold of the validity of the homogenization and kinetic regimes. We also obtain some results when $R_* = +\infty$, and these regimes break down.

The correlation function

In order to perform the analysis under weaker assumptions on the decay of the correlation function than previously, we assume that the potential $V(x)$ is isotropic and its correlation function is of the form

$$R(x) := \rho(|x|), \quad x \in \mathbb{R}^d \tag{1.10}$$

where $\rho : [0, +\infty) \to \mathbb{R}$ is of the Schoenberg class, see [27]. That is, it is of the form

$$\rho(y) = \int_0^{+\infty} \exp\left\{-(\lambda y)^2/2\right\} \mu(d\lambda), \quad y \in \mathbb{R}, \tag{1.11}$$

for some finite Borel measure μ on $[0, +\infty)$. It is well known, see Theorem 2 of [27], that $\rho(\cdot)$ belongs to the Schoenberg class if and only if the function $\mathbb{R}^d \ni x \mapsto \rho(|x|)$ is non-negative definite for any dimension $d \geq 1$.

The power energy spectrum of the potential, the Fourier transform of its correlation function, is then of the form $\hat{R}(p) = \mathcal{E}(|p|)$, with

$$\mathcal{E}(q) = \int_0^{+\infty} e^{-(\lambda q)^2/2} \nu(d\lambda), \quad q > 0 \tag{1.12}$$

and $\nu(\cdot)$ is a Borel measure on $(0,+\infty)$ given by

$$\nu(d\lambda) := (2\pi)^{-d/2}\lambda^d \mu S^{-1}(d\lambda). \tag{1.13}$$

Here $S:(0,+\infty)\to(0,+\infty)$ is $S(\lambda):=\lambda^{-1}$. Note that $\mathcal{E}$ need not belong to the Schoenberg class, as the measure $\nu(\cdot)$ might not be finite.

We shall focus our attention on isotropic potentials whose correlation function decays at infinity at an algebraic rate:

$$R(x)\sim\frac{1}{|x|^m},\quad |x|\gg 1, \tag{1.14}$$

for some $m>0$. This leads to the following form of the energy spectrum

$$\mathcal{E}(q)=\int_1^{+\infty} e^{-(\lambda q)^2/2}s(\lambda)\frac{d\lambda}{\lambda^\gamma}, \tag{1.15}$$

with a measurable function $s:[1,+\infty)\to(0,+\infty)$ that satisfies:

$$0<c\le s(\lambda)\le c^{-1},\quad \forall\lambda>1, \tag{1.16}$$

for some $c>0$. After a simple calculation one obtains $m=\gamma+d-1$. To ensure that $m>0$ we assume that $\gamma>1-d$. Note that for $\gamma<1$ the measure $1_{[1,+\infty)}(\lambda)\lambda^{-\gamma}d\lambda$ is not finite and, as a result, the power spectrum has the asymptotics

$$\mathcal{E}(q)\sim q^{\gamma-1},\quad q\ll 1. \tag{1.17}$$

Choosing various $\gamma>1-d$ in (1.15), we may achieve an arbitrarily slow or fast decay of the correlation function in (1.14). A straightforward computation shows that the effective potential in (1.8) is given by

$$R_*=\frac{1}{(2\pi)^{d/2}(d-2)}\int_1^\infty\frac{s(\lambda)d\lambda}{\lambda^{\gamma+d-2}}. \tag{1.18}$$

Thus, the effective potential is finite: $R_*<+\infty$, provided that $\gamma>3-d$, or, in terms of the decay of the correlation function, cf (1.14), we have $m>2$. This assumption does not depend on the spatial dimension.

We should mention that, similarly to the very technical proofs in [10, 28], our strategy relies on the Duhamel expansion of the solutions of the Schrödinger equation, and summation of the Feynman diagram expansions that come up after we evaluate various expectations. Typically, this requires intricate oscillatory phase estimates. One contribution of this paper is a simple observation that for the random potentials whose correlation functions are in the Schoenberg class, we may bypass the oscillatory phase arguments. Instead, one needs to estimate some quite explicit determinants, simplifying substantially the analysis and allowing it to go a bit deeper.

A natural question is how generic the Schoenberg class potentials are. The proofs apply essentially verbatim to isotropic potentials with spectra of the form (1.12), with a signed measure $\nu(\cdot)$ such that $\mathcal{E}(\cdot)$ is non-negative and the signed measure $\mu(d\lambda) := (2\pi)^{d/2}\lambda^{-d}\nu S^{-1}(d\lambda)$ has a finite total variation. As we explain in Section 6, such spectra are dense in L^1, so the respective covariance functions are dense in L^∞. Hence, one expects that the results obtained in this paper hold generically. A more detailed discussion of the examples of random fields whose covariance function is of the form (1.10) and (1.11), and their density is carried out in Section 6.

The microscopic initial conditions

As has been observed in [1], it is convenient to take out the rapidly growing phase $\exp(it|\xi|^2t/2)$ and consider the compensated wave function

$$\hat{\zeta}(t,\xi) := \hat{\psi}(t,\xi)e^{i|\xi|^2t/2}. \tag{1.19}$$

As long as $R_* < +\infty$, we will be interested in the central limit time scales $t \sim \varepsilon^{-2}$, and, accordingly, define

$$\hat{\zeta}_\varepsilon(t,\xi) := \hat{\psi}\left(\frac{t}{\varepsilon^2},\xi\right)e^{i|\xi|^2t/(2\varepsilon^2)}. \tag{1.20}$$

We first consider the initial condition for (1.1) with $l_i = 1$ – this is the kinetic regime considered in [10, 28], or, alternatively, the case of microscopic initial conditions. Let us define

$$r(\xi) = \frac{i}{2(2\pi)^{d/2}}\int_1^\infty \frac{s(\lambda)\kappa(\xi,\lambda)d\lambda}{\lambda^{\gamma+d-2}}, \tag{1.21}$$

with

$$\kappa(\xi,\lambda) := \int_0^{+\infty} \frac{1}{(1+i\tau)^{d/2}}\exp\left\{-\frac{(\lambda|\xi|\tau)^2}{2(1+i\tau)}\right\}d\tau. \tag{1.22}$$

It is straightforward to verify that the real part of $r(\xi)$ is the total scattering cross-section in the radiative transport equation (1.5).

Theorem 1.1 *Suppose that $d \geq 3$ and $\gamma > 3-d$. Let $\psi(t,x)$ be the solution of (1.1) with the initial condition $\psi(0,x) = \psi_0(x) \in \mathcal{S}(\mathbb{R}^d)$. Then, there exists $t_0 > 0$ such that for all $t \in [0,t_0]$ and $\xi \in \mathbb{R}^d$, we have*

$$\lim_{\varepsilon\to 0+} \mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \hat{\psi}_0(\xi)\exp\{ir(\xi)t\}. \tag{1.23}$$

This result is not surprising – the imaginary part of $r(\xi)$ agrees with the total scattering cross-section for the kinetic equation obtained in [10, 28]. However, in terms of the assumptions on the correlation function $R(x)$, it holds up to the threshold $\gamma = 3-d$, when the effective potential becomes infinite. On the other hand, as $r(0) = -R_*$, the result cannot be true (at least for $\xi = 0$) when $\gamma \leq 3-d$, since in that case $R_* = +\infty$. Let us add that the assumption that ψ_0 is in the Schwartz class can be easily improved but it is not the focus of this paper.

Preliminary computations indicate that the convergence of the expectation in Theorem 1.1 may be bootstrapped to the convergence of the second moment, recovering, in addition, the kinetic limit of [10, 28]. One may also combine the techniques of the present paper with the strategy of [10] to extend the result to all times $t_0 > 0$. However, to keep the paper relatively short, we postpone these directions for a future investigation.

The macroscopic initial conditions: homogenization

Unlike the microscopic initial conditions with the width $l_i = 1$, the macroscopic initial conditions have the initial pulse width $l_i = \varepsilon^{-\beta} \gg l_c = 1$, with some $\beta > 0$. Recall that the special case $\beta = 1$ has been considered in [32] for very rapidly decorrelating potentials. In other words, the initial condition for (1.1) is of the form

$$\psi(0, x) = \varepsilon^{d\beta/2}\psi_0(\varepsilon^\beta x), \tag{1.24}$$

with $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. The pre-factor in (1.24) is introduced simply to keep the L^2 norm of the solution to be of order $O(1)$. Its Fourier transform is

$$\hat{\psi}(0, \xi) = \varepsilon^{-d\beta/2}\hat{\psi}_0(\varepsilon^{-\beta}\xi).$$

To take into account the lower frequencies of the macroscopic initial conditions, and the aforementioned fact that the bulk of the energy is expected to stay at the original frequency, the compensated wave function $\hat{\zeta}_\varepsilon(t, \xi)$ on the time scales $t \sim \varepsilon^{-2}$ is now defined as

$$\hat{\zeta}_\varepsilon(t, \xi) := \varepsilon^{d\beta/2}\hat{\psi}\left(\frac{t}{\varepsilon^2}, \varepsilon^\beta\xi\right) e^{i\varepsilon^{2(\beta-1)}|\xi|^2 t/2}. \tag{1.25}$$

This allows us to track frequencies of the order $O(\varepsilon^\beta)$, present in the initial condition, at the central limit time scale $t \sim O(\varepsilon^{-2})$.

Theorem 1.2 *Suppose that $d \geq 3$ and $\gamma > 3 - d$ and the initial condition for* (1.1) *is of the form* (1.24). *Then, there exists $t_0 > 0$ such that for all $t \in [0, t_0]$ we have*

$$\lim_{\varepsilon \to 0+} \mathbb{E}\|\hat{\zeta}_\varepsilon(t, \cdot) - \bar{\zeta}(t, \cdot)\|_{L^2(\mathbb{R}^d)} = 0, \tag{1.26}$$

with

$$\bar{\zeta}(t, \xi) := \hat{\psi}_0(\xi) \exp\left\{-iR_* t\right\}, \tag{1.27}$$

with the effective potential R_* given by (1.18).

Thus, the homogenization result of [32] holds not just for the somewhat artificial choice $l_i = \varepsilon^{-1}$ but essentially for all $l_i \gg 1$. The threshold to the stochastic behavior is exactly at $l_i = l_c$. The result of Theorem 1.2 is sharp in terms of the assumptions on the correlation function – its conclusion holds for all random potentials such that the effective potential $R_* < +\infty$.

The macroscopic initial data: slowly decorrelating potentials

Next, we consider slowly decorrelating random potentials, with the power spectrum of the form (1.15), and $\gamma < 3 - d$, so that $R(x)$ decays at a rate slower than $1/|x|^2$ as $|x| \to +\infty$ – see (1.14). Then, the effective potential is infinite: $R_* = +\infty$, and the homogenization limit can not hold. We assume that the initial condition is macroscopic: $l_i = \varepsilon^{-\beta}$ with some $\beta > 0$. Typically, in such situations one expects a non-trivial effect of the random fluctuations to be seen on a time scale $t \sim \varepsilon^{-2\alpha}$ with some $\alpha \in (0, 1)$, rather than for $t \sim \varepsilon^{-2}$. Accordingly, the compensated wave function $\hat{\zeta}_\varepsilon(t, \xi)$ is now defined as

$$\hat{\zeta}_\varepsilon(t, \xi) := \varepsilon^{d\beta/2} \hat{\psi}^{(\varepsilon)} \left(\frac{t}{\varepsilon^{2\alpha}}, \varepsilon^\beta \xi \right) e^{i\varepsilon^{2(\beta-\alpha)}|\xi|^2 t/2}. \tag{1.28}$$

We denote the standard Brownian motion by B_t and the expectation with respect to it by $\mathbb{M}$.

Theorem 1.3 *Suppose that $d \geq 3$ and $\beta > \alpha$, where*

$$\alpha := \frac{2}{5 - \gamma - d} \tag{1.29}$$

and $\gamma \in (1 - d, 3 - d)$. Then, there exists $t_0 > 0$ such that

$$\bar{\zeta}(t, \xi) := \lim_{\varepsilon \to 0+} \mathbb{E}\hat{\zeta}_\varepsilon(t, \xi) = \hat{\psi}_0(\xi) \mathbb{M} \exp \left\{ -\frac{\mathfrak{R} t^H}{4} \int_0^1 \int_0^1 |B_s - B_{s'}|^{1-\gamma-d} ds ds' \right\}, \tag{1.30}$$

for all $t \in [0, t_0]$ with $H = 1/\alpha$, and a constant $\mathfrak{R}$ whose real part $\mathrm{Re}\,\mathfrak{R} > 0$.

The assumption $\beta > \alpha$ informally means that the initial condition is "very macroscopic". A short computation, starting from (1.30) shows that there exists $C > 0$ such that

$$|\bar{\zeta}(t, \xi)| \leq C \exp \left\{ -t^{H_*}/C \right\}, \quad (t, \xi) \in [0, +\infty) \times \mathbb{R}^d, \tag{1.31}$$

with

$$H_* = \frac{5 - \gamma - d}{\gamma + d + 1}. \tag{1.32}$$

Note that the randomization time $\varepsilon^{-2\alpha}$ does not depend on β, as long as $\beta > \alpha$. Informally, this means that solutions with all sufficiently slowly varying initial conditions are randomized at the same time scale $\varepsilon^{-2\alpha}$. We expect that solutions with the "less macroscopic" initial conditions varying on a scale $l_i = \varepsilon^{-\beta}$ with $\beta \in (0, \alpha)$ are randomized on time scales that depend on β, but leave this issue for a further investigation.

Let us comment informally on what one may expect in dimension $d = 2$. Here, one can point to two results: first, L. Erdös and H.-T. Yau have established the kinetic limit in [10] for the Wigner transform of the solution also in dimension $d = 2$. On the other hand, the results of A. Debussche and H. Weber in [8], building on the work of M. Hairer and C. Labbé for the parabolic Anderson model in [17], indicate that the phase of the solution needs to be renormalized. This renormalization would not affect the Wigner transform that does not see a constant factor in the phase. We expect a similar need for a renormalization for the phase in our setting.

The paper is organized as follows: in Section 2 we derive a representation of $\mathbb{E}\hat{\zeta}_\varepsilon(t,\xi)$ in terms of the average of the expectation of the exponential of some functional over Brownian paths, see Proposition 2.1. Section 3 is devoted to the presentation of the proofs of Theorems 1.1 and 1.2. We give the proof of Theorem 1.3 in Section 4 , and the short Section 5 contains the proof of a standard auxiliary result. Section 6 discusses the examples of the random potentials of the Schoenberg class, and an explanation why such potential are dense in a certain sense.

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2 A Brownian formula for the averaged wave function

In this section, we consider the Schrödinger equation, without any assumption on the smallness of the random potential

$$
\begin{aligned}
&i\frac{\partial\psi}{\partial t}+\frac{1}{2}\Delta\psi-V(x)\psi=0,\\
&\psi(0,x)=\psi_0(x).
\end{aligned}
\tag{2.1}
$$

We assume that $\psi_0\in\mathcal{S}(\mathbb{R}^d)$ and $V(x)$ is a Gaussian, stationary random field with continuous realizations defined over a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The complex valued spectral measure $\hat{V}(dp)$ corresponding to the field

$$
V(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ip\cdot x}\hat{V}(dp),
$$

has the covariance

$$
\mathbb{E}\left[\hat{V}(dp)\hat{V}^*(dq)\right]=(2\pi)^d\delta(p-q)\hat{R}(p)dpdq,\quad p,q\in\mathbb{R}^d,
$$

with a non-negative function $\hat{R}\in L^1(\mathbb{R}^d)$. As $V(x)$ is real valued and $\hat{R}(p)\geq 0$ for all $p\in\mathbb{R}^d$, we have $\hat{R}(-p)=\hat{R}(p)$, for all $p\in\mathbb{R}^d$. We do not assume in this section that $\hat{R}(p)$ has the Schoenberg form (1.15) – this will be done starting with Section 3 onwards.

The goal of this section is to obtain a convenient representation for the average compensated wave function

$$
\bar{\zeta}(t,\xi)=\mathbb{E}\hat{\zeta}(t,\xi),
\tag{2.2}
$$

with $\hat{\zeta}(t,\xi)$ defined in (1.19). Let us introduce

$$E(t,z,p,\xi) := \exp\left\{z(\sqrt{i}B_t + t\xi)\cdot p\right\}, \quad (t,z,p,\xi) \in [0,+\infty)\times\mathbb{C}\times\mathbb{R}^{2d}, \tag{2.3}$$

where B_t is a d-dimensional standard Brownian motion over a probability space $(\Sigma, \mathcal{A}, \mathbb{Q})$. We denote by $\mathbb{M}$ the expectation w.r.t. $\mathbb{Q}$, and set

$$c_n(t,\xi) := \mathbb{M}\left\{\int_0^t ds \int_0^t ds' \int_{\mathbb{R}^d} E(s,i,p;\xi)E(s',-i,p;\xi)\hat{R}(p)đp\right\}^n. \tag{2.4}$$

The following will be the starting point for our analysis of the asymptotic limits.

Proposition 2.1 *We have*

$$\bar{\zeta}(t,\xi) = \hat{\psi}_0(\xi)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!!}c_n(t,\xi), \quad t\ge 0,\ \xi\in\mathbb{R}^d, \tag{2.5}$$

or equivalently

$$\bar{\zeta}(t,\xi) = \hat{\psi}_0(\xi)\mathbb{M}\mathbb{E}\left\{\exp\left\{\frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{V}(dp)\int_0^t E(s,i,p,\xi)ds\right\}\right\}. \tag{2.6}$$

The rest of this section contains the proof of this proposition.

2.1 The Duhamel expansion

We re-write the Schrödinger equation (2.1) as an integral in time equation

$$\hat{\psi}(t,\xi) = \hat{\psi}_0(\xi)e^{-i|\xi|^2t/2} + \frac{1}{i}\int_0^t\int_{\mathbb{R}^d}\frac{\hat{V}(dp_1)}{(2\pi)^d}\hat{\psi}(s_1,\xi-p_1)e^{-i|\xi|^2(t-s_1)/2}ds_1. \tag{2.7}$$

The compensated wave function $\hat{\zeta}(t,\xi)$ satisfies

$$\hat{\zeta}(t,\xi) = \hat{\psi}_0(\xi) + \frac{1}{i}\int_0^t\int_{\mathbb{R}^d}\frac{\hat{V}(dp_1)}{(2\pi)^d}\hat{\zeta}(s_1,\xi-p_1)\exp\left\{i(|\xi|^2-|\xi-p_1|^2)\frac{s_1}{2}\right\}ds_1. \tag{2.8}$$

Iterating (2.8), we get an infinite series expansion for $\hat{\zeta}(t,\xi)$:

$$\hat{\zeta}(t,\xi) = \sum_{n=0}^{\infty}\hat{\zeta}_n(t,\xi), \tag{2.9}$$

with $\hat{\zeta}_0(t,\xi) = \hat{\psi}_0(\xi)$, and the rest of the individual terms of the form

$$\hat{\zeta}_n(t,\xi) = \left[\frac{1}{i(2\pi)^d}\right]^n\int_{\Delta_n(t)}ds_{1,n}\int_{\mathbb{R}^{dn}}\hat{V}(dp_1)\dots\hat{V}(dp_n)\hat{\psi}_0\left(\xi-\sum_{j=1}^n p_j\right)e^{iG_n}, \tag{2.10}$$

with the phase

$$G_n = G_n(s_{1,n}, p_{1,n}) = \sum_{k=1}^{n} \left(\left| \xi - \sum_{j=1}^{k-1} p_j \right|^2 - \left| \xi - \sum_{j=1}^{k} p_j \right|^2 \right) \frac{s_{n-k}}{2}. \tag{2.11}$$

Here, we denote $p_{1,n} := (p_1, \dots, p_n) \in \mathbb{R}^{nd}$, and $s_{1,n} := (s_1, \dots, s_n) \in \mathbb{R}^n$, so that $ds_{1,n} := ds_1 ds_2 \dots ds_n$. We have also denoted in (2.10) by $\Delta_n(t)$ the time simplex

$$\Delta_n(t) = \{(s_1, s_2, \dots, s_n) : \quad 0 \le s_1 \le s_1 \le \dots \le s_n \le t\}.$$

The next standard proposition shows that we may employ term-wise expectation. Its proof is standard, for the convenience of the reader, we present it in Section 5.

Proposition 2.2 *(i) The series (2.9) for the function $\hat{\zeta}(t, \xi)$ converges almost surely for any initial data $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$.*
(ii) For each $(t, \xi) \in \mathbb{R}^{1+d}$ fixed, we have

$$\mathbb{E}\hat{\zeta}(t, \xi) = \sum_{n=0}^{\infty} \mathbb{E}\hat{\zeta}_{2n}(t, \xi). \tag{2.12}$$

(iii) Moreover, we have

$$\mathbb{E}\|\hat{\zeta}(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 = \|\hat{\psi}_0\|_{L^2(\mathbb{R}^d)}^2, \quad t \ge 0. \tag{2.13}$$

Note that when we introduce various scaling parameters, this proposition will not guarantee that the convergence will be uniform in those parameters. Thus, while this proposition allows us to interchange the expectation and the summation of the series, it will not allow us to pass to the limit in the individual terms of the series when we will consider appropriate asymptotic limits.

Using Proposition 2.2, and the rule of the expectation for the product of $2n$ mean zero Gaussian random variables, we may re-write the expectation as a series

$$\bar{\zeta}(t, \xi) := \mathbb{E}\zeta(t, \xi) = \left\{ 1 + \sum_{n=1}^{+\infty} \sum_{\mathcal{F} \in \mathfrak{F}_{2n}} I(\mathcal{F})(t) \right\} \hat{\psi}_0(\xi). \tag{2.14}$$

Here, $\mathfrak{F}_{2n}$ is the collection of all pairings $\mathcal{F}$ formed over the set $\mathbb{Z}_{2n} := \{1, \dots, 2n\}$. The individual terms for each pairing have the form

$$I(\mathcal{F}) := (-1)^n \int_{\mathbb{R}^{2dn}} \prod_{(k,\ell) \in \mathcal{F}} \delta(p_k + p_\ell) \hat{R}(p_k) đp_{1,2n} \int_{\Delta_{2n}(t)} e^{iG_{2n}} ds_{1,2n}, \tag{2.15}$$

where $đp_{1,n} = đp_1 \dots đp_n$, and

$$G_{2n}(s_{1,2n}, p_{1,2n}) = \sum_{m=1}^{2n} (s_{2n-m+1} - s_{2n-m}) \Big(\xi \cdot \sum_{j=1}^{m} p_j - \frac{1}{2} \Big| \sum_{j=1}^{m} p_j \Big|^2 \Big), \tag{2.16}$$

where $s_0 := 0$.

2.2 The proof of Proposition 2.1

We now prove Proposition 2.1 via an alternative representation for the series (2.14) for $\bar{\zeta}(t,\xi)$. Identity (2.5) can be restated as

$$\sum_{\mathcal{F}\in\mathfrak{F}_{2n}} I(\mathcal{F})(t) = \frac{(-1)^n c_n(t,\xi)}{(2n)!!}, \quad \forall\, n \geq 1. \tag{2.17}$$

We first find an expression for the left side of (2.17). Given a pairing $\mathcal{F}$, and $m \in \{1,\dots,2n\}$, let us denote by $A_m(\mathcal{F})$ the set of all left vertices $\ell \leq m$ such that the corresponding right vertex r satisfies $r > m$, that is, the edge (ℓr) crosses over m, with the convention $A_0(\mathcal{F}) := \emptyset$. Sometimes, when the pairing is obvious from the context, we simply write A_m. We also denote by $\mathcal{L}(\mathcal{F})$ the set of all left vertices of $\mathcal{F}$. For a given pairing $\mathcal{F}$, we have

$$\sum_{j=1}^{k} p_j = \sum_{j\in A_k} p_j,$$

a.e. in the measure

$$\prod_{(k,\ell)\in\mathcal{F}} \delta(p_k+p_\ell)\hat{R}(p_k)đp_{1,2n}.$$

The phase G_{2n} has, therefore, the form (note that the set A_{2n} is empty)

$$G_{2n}(s_{1,2n},p_{1,2n}) = \sum_{m=1}^{2n-1}(s_{2n-m+1}-s_{2n-m})\Big(\xi\cdot\sum_{j\in A_m}p_j - \frac{1}{2}\Big|\sum_{j\in A_m}p_j\Big|^2\Big). \tag{2.18}$$

This leads to the expression

$$\begin{aligned} I(\mathcal{F})(t) = (-1)^n &\int_{\Delta_{2n}(t)} ds_{1,2n}\int_{\mathbb{R}^{nd}}\Big[\prod_{\ell\in\mathcal{L}(\mathcal{F})}\hat{R}(p_\ell)\Big] \\ \times \exp&\left\{\sum_{m=1}^{2n-1}(s_{2n-m+1}-s_{2n-m})\left(-\frac{i}{2}\Big|\sum_{j\in A_m(\mathcal{F})}p_j\Big|^2 + i\xi\cdot\sum_{j\in A_m(\mathcal{F})}p_j\right)\right\}\prod_{\ell\in\mathcal{L}(\mathcal{F})}đp_\ell. \end{aligned} \tag{2.19}$$

Next, we re-write c_n defined by (2.4), to make it clear that (2.17) holds. To abbreviate somewhat the notation, we will denote $E(s,i,p,\xi)$ by $E(s,p)$ (see (2.3)). We can re-write $c_n(t,\xi)$, see (2.4), as

$$c_n = \mathbb{M}\left\{\int_0^t\dots\int_0^t ds_{1,2n}\int_{\mathbb{R}^{2nd}}đp_{1,2n}\prod_{j=1}^{n}\hat{R}(p_{2j-1})\delta(p_{2j-1}+p_{2j})\prod_{j=1}^{2n}E(s_j,p_j)\right\}. \tag{2.20}$$

For each $(s_1, s_2, \ldots, s_{2n})$, we re-order the times s_j in the increasing order, and re-label the indices j accordingly, so that

$$c_n = \sum_\sigma \int_{\Delta_{2n}(t)} ds_{1,2n} \int_{\mathbb{R}^{2nd}} đp_{1,2n} \prod_{j=1}^{n} \Big[\hat{R}(p_{\sigma(2j-1)})\delta(p_{\sigma(2j-1)} + p_{\sigma(2j)})\Big] \mathbb{M}\Big\{ \prod_{j=1}^{2n} E(s_{\sigma(j)}, p_{\sigma(j)}) \Big\}.$$

Here, the summation extends over all possible permutations $\sigma : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$. The symmetry of the last product above allows us to write

$$c_n = \sum_\sigma \int_{\Delta_{2n}(t)} ds_{1,2n} \int_{\mathbb{R}^{2nd}} đp_{1,2n} \prod_{j=1}^{n} \Big[\hat{R}(p_{\sigma(2j-1)})\delta(p_{\sigma(2j-1)} + p_{\sigma(2j)})\Big] \mathbb{M}\Big\{ \prod_{j=1}^{2n} E(s_j, p_j) \Big\}.$$

Using the independence of the increments of a Brownian motion, and performing the expectation we conclude that (with $s_{2n+1} = 0$)

$$c_n = \sum_\sigma \int_{\Delta_{2n}(t)} ds_{1,2n} \int_{\mathbb{R}^{2nd}} đp_{1,2n} \prod_{j=1}^{n} \Big[\hat{R}(p_{\sigma(2j-1)})\delta(p_{\sigma(2j-1)} + p_{\sigma(2j)})\Big] S_n(s_{1,2n}, p_{1,2n}), \tag{2.21}$$

where

$$S_n(s_{1,2n}, p_{1,2n}) := \prod_{j=1}^{2n} \exp\left\{ (s_{2n-j+1} - s_{2n-j}) \left(-\frac{i}{2}\Big| \sum_{m=1}^{j} p_m \Big|^2 + i\xi \cdot \sum_{m=1}^{j} p_m \right) \right\}.$$

Note that (2.19) and (2.21) are very similar, making (2.17) "plausible", except for the summation taken over all permutations σ in (2.21), as opposed to the summation over all pairings $\mathcal{F}$ in the left side of (2.17). As we will see, the difference in the two summations is responsible for the factor $1/(2n)!!$ in (2.17).

To reconcile the two summations, let $\Pi(2n)$ be the set of all permutations of $\{1, \ldots, 2n\}$, and define the mapping $\mathfrak{f} : \Pi(2n) \to \mathfrak{F}_{2n}$ as follows. Given a permutation σ, we let $\mathfrak{f}(\sigma)$ be the following pairing: a pair (ℓ, r), with $\ell < r$ is in $\mathfrak{f}(\sigma)$ iff there exists j such that $\ell = \sigma(2j-1)$ and $r = \sigma(2j)$, or $\ell = \sigma(2j)$ and $r = \sigma(2j-1)$. In other words, we start with the simple pairing $(1, 2)(3, 4)\ldots(2n-1, 2n)$ and map it by σ to the pairing

$$(\sigma(1), \sigma(2))(\sigma(3), \sigma(4))\ldots(\sigma(2n-1), \sigma(2n)), \tag{2.22}$$

with a slight abuse of notation, as it is possible that $\sigma(2j-1) > \sigma(2j)$. Observe that if $\mathcal{F} = \mathfrak{f}(\sigma)$ then

$$\int_{\mathbb{R}^{2nd}} S_n(s_{1,2n}, p_{1,2n}) \prod_{j=1}^{n} \Big[\hat{R}(p_{\sigma(2j-1)})\delta(p_{\sigma(2j-1)} + p_{\sigma(2j)})\Big] đp_{1,2n} \tag{2.23}$$

$$= \int_{\mathbb{R}^{nd}} S_n(s_{1,2n}, p_{1,2n}) \left[\prod_{\ell \in \mathcal{L}(\mathcal{F})} \hat{R}(p_\ell) đp_\ell \right].$$

On the other hand, given a pairing $\mathcal{F} \in \mathfrak{F}_{2n}$:

$$\mathcal{F} := \{(\ell_1, r_1), \dots, (\ell_n, r_n)\}, \tag{2.24}$$

with $1 = \ell_1 < \ell_2 < \dots < \ell_n$, we may define the corresponding permutation $\mathfrak{g}(\mathcal{F}) \in \Pi(2n)$ as $(\ell_1, r_1, \ell_2, r_2, \dots, \ell_n, r_n)$. This defines the mapping $\mathfrak{g} : \mathfrak{F}_{2n} \to \Pi(2n)$ such that

$$\mathfrak{f}\left(\mathfrak{g}(\mathcal{F})\right) = \mathcal{F}, \quad \forall \mathcal{F} \in \mathfrak{F}_{2n}.$$

Thus, the mapping $\mathfrak{f}$ is onto. Next, suppose that $\sigma = \mathfrak{g}(\mathcal{F})$ and $\mathcal{F}$ is given by (2.24). Note that any permutation σ' obtained from σ by a transposition of ℓ_j and r_j, as well as by permuting in the same fashion $\ell_1, \dots \ell_n$ and $r_1, \dots, r_n$, satisfies $\mathfrak{f}(\sigma') = \mathcal{F}$. For each permutation $\sigma = \mathfrak{g}(\mathcal{F})$ there exist $2^n n! = (2n)!!$ different permutations obtained in that way. Writing $\mathcal{F}_\sigma = \mathfrak{f}(\sigma)$ we obtain from (2.21) and (2.23)

$$\frac{c_n}{(2n)!!} = \frac{1}{(2n)!!} \int_{\Delta_{2n}(t)} ds_{1,2n} \int_{\mathbb{R}^{2nd}} S_n(s_{1,2n}, p_{1,2n}) \prod_{j=1}^{n} \left[\hat{R}(p_{\sigma(2j-1)}) \delta(p_{\sigma(2j-1)} + p_{\sigma(2j)}) \right] đp_{1,2n}$$

$$= \frac{1}{(2n)!!} \sum_{\sigma} \int_{\Delta_{2n}(t)} ds_{1,2n} \int \left[\prod_{\ell \in \mathcal{L}(\mathcal{F}_\sigma)} S_n(s_{1,2n}, p_{1,2n}) \hat{R}(p_\ell) đp_\ell \right]$$

$$= \sum_{\mathcal{F}} \int_{\Delta_{2n}(t)} ds_{1,2n} \int \left[\prod_{\ell \in \mathcal{L}(\mathcal{F})} S_n(s_{1,2n}, p_{1,2n}) \hat{R}(p_\ell) đp_\ell \right].$$

Comparing to (2.19), we conclude that (2.17) holds, finishing the proof of (2.17) and thus also of Proposition 2.1. □

3 The finite effective potential regime

In this section, we present the proofs of Theorems 1.1 and 1.2, both of which hold when the effective potential $R_* < +\infty$, that is, $\gamma > 3 - d$, and the non-trivial behavior takes place at times of the order $t \sim \varepsilon^{-2}$.

Let us add the weak coupling limit to the representation in Proposition 2.1. Recall that, the wave function $\psi^{(\varepsilon)}$ is the solution of

$$\begin{aligned} & i\frac{\partial \psi^{(\varepsilon)}}{\partial t} + \frac{1}{2}\Delta\psi^{(\varepsilon)} - \varepsilon V(x)\psi^{(\varepsilon)} = 0, \\ & \psi^{(\varepsilon)}(0, x) = \varepsilon^{d\beta/2}\psi_0(\varepsilon^\beta x), \end{aligned} \tag{3.1}$$

with $\beta \geq 0$, and the compensated wave function is given by (1.25). We may now apply Proposition 2.1, replacing the random potential $V \to \varepsilon V$, and the time $t \to t/\varepsilon^2$ in (2.4) and (2.6). Using, in addition, the representation (1.15) for the power spectrum $\hat{R}(p)$ turns (2.5), when $\beta = 0$, into

$$\bar{\zeta}_\varepsilon(t,\xi) := \mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \hat{\psi}_0(\xi) \sum_{n=0}^{+\infty} \frac{(-1)^n c_{n,\varepsilon}(t,\xi)}{(2n)!!}, \quad \forall\, (t,\xi) \in \mathbb{R}^{1+d}, \tag{3.2}$$

where

$$c_{n,\varepsilon}(t,\xi) := \varepsilon^{2n} \mathbb{M}\Big\{ \int_1^{+\infty} \frac{s(\lambda) d\lambda}{\lambda^\gamma} \int_0^{t/\varepsilon^2} ds \int_0^{t/\varepsilon^2} ds' \int_{\mathbb{R}^d} E(s,i,p,\xi) E(s',-i,p,\xi) e^{-\lambda^2 |p|^2/2} đp \Big\}^n \tag{3.3}$$

and $E(s,i,p,\xi)$ is given by (2.3). On the other hand, when $\beta > 0$, we have

$$\bar{\zeta}_\varepsilon(t,\xi) = \hat{\psi}_0(\xi) \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!!} c_{n,\varepsilon}(t, \varepsilon^\beta \xi). \tag{3.4}$$

3.1 A uniform bound on $c_{n,\varepsilon}(t,\xi)$

The main step in the proof of Theorems 1.1 and 1.2 is the following uniform bound on $c_{n,\varepsilon}$ that allows us to pass to the limit in representations (3.2) and (3.4).

Proposition 3.1 *Suppose that $d \geq 3$ and $\gamma > 3 - d$. Then, there exists $C > 0$ such that for all $n \geq 0$, $\varepsilon \in (0,1]$, $\xi \in \mathbb{R}^d$ and $t \geq 0$ we have*

$$|c_{n,\varepsilon}(t,\xi)| \leq n!(Ct)^n. \tag{3.5}$$

An alternative representation for $c_{n,\varepsilon}(t,\xi)$

Both in the proof of Proposition 3.1, and in passing to the limit $\varepsilon \to 0$ in $c_{n,\varepsilon}(t,\xi)$, it will be convenient for us to use an expression different from (3.3). Let us first introduce some notation. Given a permutation σ of $\{1,\ldots,2n\}$, we have the corresponding pairing in $\mathfrak{F}_{2n}$ defined by

$$\mathfrak{f}(\sigma) = \{(\ell_1, r_1), \ldots, (\ell_n, r_n)\}, \tag{3.6}$$

with

$$(\ell_k, r_k) := \begin{cases} (\sigma(2k-1), \sigma(2k)), & \text{if } \sigma(2k-1) < \sigma(2k), \\ (\sigma(2k), \sigma(2k-1)), & \text{if } \sigma(2k) < \sigma(2k-1). \end{cases} \tag{3.7}$$

We may then define a $2nd \times 2nd$ symmetric non-negative matrix $A_\sigma(\tau,\lambda)$ corresponding to the quadratic form

$$\Phi_\sigma(y) = (A_\sigma(\tau,\lambda)y, y)_{\mathbb{R}^{2dn}} = \sum_{k=1}^n \frac{1}{\lambda_{r_k}^2} \Big| \sum_{j=\ell_k+1}^{r_k} y_j \tau_j^{1/2} \Big|^2 \tag{3.8}$$

for $y = (y_1, \ldots, y_{2n}) \in \mathbb{R}^{2dn}$ and $\tau_j := s_j - s_{j-1}$, $j = 1, \ldots, 2n$ with $s_0 := 0$. In order to describe the matrix $A_\sigma(\tau, \lambda)$ more explicitly, we introduce some terminology.

For each $j \in \{2, \ldots, 2n\}$ define

$$a_{j,j} := \sum_k{}' \frac{1}{\lambda_{r_k}^2}, \tag{3.9}$$

where the summation extends over those k-s, for which $\ell_k < j \le r_k$. Given $m < j$ we let

$$a_{m,j} = a_{j,m} = \sum_k{}' \frac{1}{\lambda_{r_k}^2}, \tag{3.10}$$

with the summation extending over those k-s, for which $\ell_k < m < j \le r_k$. We also let

$$a_{1,j} = a_{j,1} = 0 \quad \text{for all} \quad j \in \{1, \ldots, 2n\}. \tag{3.11}$$

Then, the matrix A_σ has the form

$$A_\sigma(\tau, \lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0\ldots & 0 \\ 0 & I_d a_{2,2}\tau_2 & I_d a_{23}\tau_2^{1/2}\tau_3^{1/2} & I_d a_{2,4}\tau_2^{1/2}\tau_4^{1/2} & \ldots & I_d a_{2,2n}\tau_2^{1/2}\tau_{2n}^{1/2} \\ 0 & I_d a_{3,2}\tau_3^{1/2}\tau_2^{1/2} & I_d a_{3,3}\tau_3 & I_d a_{34}\tau_3^{1/2}\tau_4^{1/2} & \ldots & I_d a_{3,2n}\tau_3^{1/2}\tau_{2n}^{1/2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & I_d a_{2n,2}\tau_{2n}^{1/2}\tau_2^{1/2} & I_d a_{2n,3}\tau_{2n}^{1/2}\tau_3^{1/2} & I_d a_{2n,4}\tau_{2n}^{1/2}\tau_4^{1/2} & \ldots & I_d a_{2n,2n}\tau_{2n} \end{bmatrix}. \tag{3.12}$$

Here, 0 and I_d are the null and identity $d \times d$ matrices respectively.

We will show the following.

Proposition 3.2 *We have*

$$c_{n,\varepsilon}(t, \xi) = \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n \sum_\sigma \int_1^{+\infty} \frac{s(\lambda_{r_1})d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d}} \cdots \int_1^{+\infty} \frac{s(\lambda_{r_n})d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d}} \int_{\tilde{\Delta}_{2n}(t/\varepsilon^2)} d\tau_{1,2n} \tag{3.13}$$

$$\times \det(I_{2nd} + iA_\sigma(\tau, \lambda))^{-1/2} \exp\left\{-\frac{1}{2}(C_\sigma(\tau, \lambda)\Xi(\tau), \Xi(\tau))_{\mathbb{R}^{2nd}}\right\},$$

with the matrix

$$C_\sigma(\tau, \lambda) = A_\sigma(\tau, \lambda) - i(I_{2nd} + iA_\sigma(\tau, \lambda))^{-1} A_\sigma^2(\tau, \lambda) = (I_{2nd} + iA_\sigma(\tau, \lambda))^{-1} A_\sigma(\tau, \lambda) \tag{3.14}$$

and

$$\Xi^T(\tau) := [\tau_1^{1/2}\xi, \ldots, \tau_{2n}^{1/2}\xi]. \tag{3.15}$$

Proof. To abbreviate, we write $c_{n,\varepsilon} = c_{n,\varepsilon}(t,\xi)$. For simplicity of the notation we assume that $s(\lambda) \equiv 1$. From (3.3) we obtain that

$$c_{n,\varepsilon} = \varepsilon^{2n}\mathbb{M}\left\{\int_1^{+\infty}\frac{d\lambda}{\lambda^\gamma}\int_0^{t/\varepsilon^2}ds\int_0^{t/\varepsilon^2}ds'\int_{\mathbb{R}^d}e^{i(B_s^{(\xi)}-B_{s'}^{(\xi)})\cdot p}e^{-\lambda^2|p|^2/2}\bar{d}p\right\}^n, \tag{3.16}$$

where

$$B_s^{(\xi)} := \sqrt{i}B_s + \xi s. \tag{3.17}$$

Performing the integration over p variable in (3.16), we obtain

$$\begin{aligned}
c_{n,\varepsilon} &= \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n \mathbb{M}\Big[\int_1^{+\infty}\frac{d\lambda}{\lambda^{\gamma+d}}\int_0^{t/\varepsilon^2}ds\int_0^{t/\varepsilon^2}ds'\exp\Big\{-\frac{1}{2\lambda^2}\big(B_s^{(\xi)}-B_{s'}^{(\xi)}\big)^2\Big\}\Big]^n \\
&= \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n\int_1^{+\infty}\frac{d\lambda_1}{\lambda_1^{(\gamma+d)/2}}\cdots\int_1^{+\infty}\frac{d\lambda_{2n}}{\lambda_{2n}^{(\gamma+d)/2}}\int_0^{t/\varepsilon^2}ds_1\ldots\int_0^{t/\varepsilon^2}ds_{2n}\prod_{k=1}^n\delta(\lambda_{2k}-\lambda_{2k-1}) \\
&\times\exp\Big\{-\frac{|\xi|^2}{2}\sum_{k=1}^n\Big(\frac{s_{2k-1}-s_{2k}}{\lambda_{2k}}\Big)^2\Big\} \qquad (3.18) \\
&\times\mathbb{M}\Big\{\exp\Big\{-\frac{i}{2}\sum_{k=1}^n\lambda_{2k}^{-2}|B_{s_{2k}}-B_{s_{2k-1}}|^2-\sqrt{i}\xi\cdot\Big[\sum_{k=1}^n\lambda_{2k}^{-2}(s_{2k}-s_{2k-1})(B_{s_{2k}}-B_{s_{2k-1}})\Big]\Big\}\Big\}.
\end{aligned}$$

Re-arranging again the times s_j in the increasing order, we obtain

$$\begin{aligned}
c_{n,\varepsilon} &= \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n\sum_\sigma\int_1^{+\infty}\frac{d\lambda_1}{\lambda_1^{(\gamma+d)/2}}\cdots\int_1^{+\infty}\frac{d\lambda_{2n}}{\lambda_{2n}^{(\gamma+d)/2}}\prod_{k=1}^n\int_{\Delta_{2n}(t/\varepsilon^2)}d\tau_{1,2n}\delta(\lambda_{\sigma(2k)}-\lambda_{\sigma(2k-1)}) \\
&\times\exp\left\{-\frac{|\xi|^2}{2}\sum_{k=1}^n\left(\frac{s_{\sigma(2k)}-s_{\sigma(2k-1)}}{\lambda_{\sigma(2k)}}\right)^2\right\}\mathbb{M}\left\{\exp\left\{-\frac{i}{2}\sum_{k=1}^n\lambda_{\sigma(2k)}^{-2}|B_{s_{\sigma(2k)}}-B_{s_{\sigma(2k-1)}}|^2\right.\right. \\
&\left.\left.-\sqrt{i}\xi\cdot\left[\sum_{k=1}^n\lambda_{\sigma(2k)}^{-2}(s_{\sigma(2k)}-s_{\sigma(2k-1)})(B_{s_{\sigma(2k)}}-B_{s_{\sigma(2k-1)}})\right]\right\}\right\}.
\end{aligned}$$

Using the formula for the joint probability density of the random vector $(B_{s_{\sigma(1)}},\ldots,B_{s_{\sigma(2n)}})$ leads to

$$\begin{aligned}
c_{n,\varepsilon} &= \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n\sum_\sigma\int_1^{+\infty}\frac{d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d}}\cdots\int_1^{+\infty}\frac{d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d}}\int_{\Delta_{2n}(t/\varepsilon^2)}ds_{1,2n} \qquad (3.19) \\
&\times\exp\left\{-\frac{|\xi|^2}{2}\sum_{k=1}^n\left(\frac{s_{r_k}-s_{\ell_k}}{\lambda_{r_k}}\right)^2\right\}\int_{\mathbb{R}^{2dn}}\exp\left\{-\frac{i}{2}\sum_{k=1}^n\lambda_{r_k}^{-2}\left|\sum_{j=\ell_k+1}^{r_k}y_j(s_j-s_{j-1})^{1/2}\right|^2\right. \\
&\left.-\sqrt{i}\xi\cdot\left[\sum_{k=1}^n\lambda_{r_k}^{-2}(s_{r_k}-s_{\ell_k})\left(\sum_{j=\ell_k+1}^{r_k}y_j(s_j-s_{j-1})^{1/2}\right)\right]\right\}\prod_{\ell=1}^{2n}\left[\frac{1}{(2\pi)^{d/2}}\exp\left\{-\frac{|y_\ell|^2}{2}\right\}\right]dy_{1,2n}.
\end{aligned}$$

Changing variables $\tau_j := s_j - s_{j-1}$ we obtain

$$
\begin{aligned}
c_{n,\varepsilon} = & \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n \sum_\sigma \int_1^{+\infty} \frac{d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d}} \dots \int_1^{+\infty} \frac{d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d}} \int_{\tilde{\Delta}_{2n}(t/\varepsilon^2)} d\tau_{1,2n} \\
& \times \exp\left\{-\frac{1}{2}(A_\sigma(\tau,\lambda)\Xi(\tau)\cdot\Xi(\tau))\right\} \int_{\mathbb{R}^{2dn}} \exp\left\{-\frac{1}{2}\big((I_{2nd} + iA_\sigma(\tau,\lambda))y\cdot y\big)\right\} \\
& \times \exp\left\{-\sqrt{i}(A_\sigma(\tau,\lambda)\Xi(\tau)\cdot y)\right\} \frac{dy_{1,2n}}{(2\pi)^{nd}}.
\end{aligned} \tag{3.20}
$$

Here, $A_\sigma(\tau, \lambda)$ are the $2nd \times 2nd$ block matrices as in (3.32), and $\Xi(\tau)$ is as in (3.15). To perform integration over y variables we shall need the following

Lemma 3.3 *Suppose that $f : \mathbb{C}^N \to \mathbb{C}$ is a holomorphic function such that there exists $C > 0$, for which*

$$
|f(z_1,\dots,z_N)| \le C\exp\left\{C\sum_{j=1}^N |z_j|\right\}, \quad (z_1,\dots,z_N) \in \mathbb{C}^N \tag{3.21}
$$

and A is a symmetric $N \times N$-matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N$. Then,

$$
\begin{aligned}
& \int_{\mathbb{R}^N} \exp\left\{-\frac{1}{2}\left((I_N + zA)x, x\right)_{\mathbb{R}^N}\right\} f(x)dx \\
& \qquad = [\det(I_N + zA)]^{-1/2} \int_{\mathbb{R}^N} \exp\left\{-\frac{|x|^2}{2}\right\} f\left((I_N + zA)^{-1/2}x\right) dx,
\end{aligned} \tag{3.22}
$$

for all $z \in \mathbb{C}$ such that $\mathrm{Re}(1 + z\lambda_j) > 0$, $j = 1,\dots,N$.

The formula (3.22) obviously holds for z real. It can be extended to the set in question by the analytic continuation argument. Let us re-write the y-integral using formula (3.22):

$$
\begin{aligned}
& \int_{\mathbb{R}^{2dn}} \exp\big\{-\frac{1}{2}((I_{2nd} + iA_\sigma(\tau,\lambda))y\cdot y)\big\} \exp\big\{-\sqrt{i}(A_\sigma(\tau,\lambda)\Xi(\tau)\cdot y)\big\} \frac{dy_{1,2n}}{(2\pi)^{nd}} \\
& = [\det(I_{2nd} + iA_\sigma(\tau,\lambda))]^{-1/2} \int_{\mathbb{R}^{2nd}} e^{-|x|^2/2} \exp\big\{-\sqrt{i}\Big(A_\sigma(\tau,\lambda)\Xi(\tau)\cdot(I_{2nd} + iA_\sigma(\tau,\lambda))^{-1/2}x\Big)\big\}dx \\
& = [\det(I_{2nd} + iA_\sigma(\tau,\lambda))]^{-1/2} \\
& \times \exp\big\{\frac{i}{2}\Big((I_{2nd} + iA_\sigma(\tau,\lambda))^{-1/2}A_\sigma(\tau,\lambda)\Xi(\tau)\cdot(I_{2nd} + iA_\sigma(\tau,\lambda))^{-1/2}A_\sigma(\tau,\lambda)\Xi(\tau)\Big)\big\}.
\end{aligned} \tag{3.23}
$$

Using this in (3.20) gives (3.13), finishing the proof of Proposition 3.2. □

The proof of Proposition 3.1

We now use representation (3.13) for $c_{n,\varepsilon}$ in order to obtain the bound (3.5) in Proposition 3.1. The main step in the proof is the following lower bound.

Proposition 3.4 *For any permutation σ we have*

$$|\det(I + iA_\sigma(\tau,\lambda))| \geq \frac{1}{2^{dn/2}} \prod_{k=1}^{n} \left(1 + \frac{\tau_{r_k}}{\lambda_{r_k}^2}\right)^d. \tag{3.24}$$

Before proving this proposition, let us show how it implies the required estimate on $c_{n,\varepsilon}$.

The matrix $A_\sigma(\tau,\lambda)$, appearing in (3.13), is symmetric and non-negative, hence $C_\sigma(\tau,\lambda)$, given by (3.14), is diagonalizable with respect to the orthonormal basis of eigenvectors of $A_\sigma(\tau,\lambda)$:

$$A_\sigma(\tau,\lambda) f_j = \gamma_j f_j, \quad \gamma_j \geq 0, \tag{3.25}$$

and

$$C_\sigma(\tau,\lambda) f_j = \mu_j f_j, \quad \mu_j = \frac{\gamma_j}{1 + i\gamma_j},$$

so that

$$(C_\sigma(\tau,\lambda)\Xi, \Xi)_{\mathbb{R}^{2nd}} = \sum_{j=1}^{2nd} \frac{\gamma_j}{1 + i\gamma_j} (\Xi, f_j)^2_{\mathbb{R}^{2nd}}, \tag{3.26}$$

thus

$$\left| \exp\left\{ -\frac{1}{2} (C_\sigma(\tau,\lambda)\Xi, \Xi)_{\mathbb{R}^{2nd}} \right\} \right| = \prod_{j=1}^{2nd} \exp\left\{ -\frac{\gamma_j}{2(1+\gamma_j^2)} (\Xi, f_j)^2_{\mathbb{R}^{2nd}} \right\} \leq 1. \tag{3.27}$$

As a consequence, we have an estimate

$$|c_{n,\varepsilon}(t,\xi)| \leq \left(\frac{\varepsilon^2}{(2\pi)^{d/2}} \right)^n \sum_\sigma \int_1^{+\infty} \frac{d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d}} \cdots \int_1^{+\infty} \frac{d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d}} \int_{\tilde{\Delta}_{2n}(t/\varepsilon^2)} |\det[I_{2nd} + iA_\sigma(\tau,\lambda)]|^{-1/2} \, d\tau_{1,2n} \tag{3.28}$$

Using (3.24) we conclude that that there exists $C > 0$ such that

$$|c_{n,\varepsilon}(t,\xi)| \leq C^n \varepsilon^{2n} \sum_\sigma \int_1^{+\infty} \frac{d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d}} \cdots \int_1^{+\infty} \frac{d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d}} \int_{\sum_{k=1}^n \tau_{\ell_k} \leq t/\varepsilon^2, \tau_{\ell_k} \geq 0} d\tau_{\ell_1} \ldots d\tau_{\ell_n} \prod_{k=1}^{n} \int_0^{+\infty} \frac{d\tau_{r_k}}{(1 + \lambda_{r_k}^{-2} \tau_{r_k})^{d/2}} \tag{3.29}$$

for all $\varepsilon > 0$ and $n \geq 0$. Changing variables $\tau'_{r_k} := \lambda_{r_k}^{-2} \tau_{r_k}$ we obtain

$$|c_{n,\varepsilon}(t,\xi)| \leq C^n \varepsilon^{2n} \sum_\sigma \int_1^{+\infty} \frac{d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d-2}} \cdots \int_1^{+\infty} \frac{d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d-2}} \int_{\sum_{k=1}^n \tau_{\ell_k} \leq t/\varepsilon^2, \tau_{\ell_k} \geq 0} d\tau_{\ell_1} \ldots d\tau_{\ell_n}$$
$$\times \prod_{k=1}^{n} \int_0^{+\infty} \frac{d\tau_{r_k}}{(1 + \tau_{r_k})^{d/2}} \leq (2n)! \frac{(Ct)^n}{n!} \leq n!(Ct)^n, \tag{3.30}$$

provided that $\gamma > 3 - d$, so that (3.5) holds. This proves Proposition 3.1, except for the proof of Proposition 3.4.

The proof of Proposition 3.4

In order to describe the matrix $A_\sigma(\tau, \lambda)$ more explicitly, we make a change of variables:

$$z_{l_k} = y_{l_k} \in \mathbb{R}^d, \text{ and } z_{r_k} = \frac{1}{\lambda_{r_k}} \sum_{j=\ell_k+1}^{r_k} y_j \tau_j^{1/2} \in \mathbb{R}^d, \quad k = 1, \dots, n,$$

so that

$$(A_\sigma(\tau, \lambda)y, y)_{\mathbb{R}^{2dn}} = \sum_{k=1}^{n} |z_{r_k}(\tau, \lambda)|^2 = (P_\sigma z, z) = (P_\sigma L(\tau, \lambda)y, L(\tau, \lambda)y). \tag{3.31}$$

Here P_σ is the projection matrix onto the r_k-components, and $L(\tau, \lambda)$ is the matrix relating z and y: where $z = L(\tau, \lambda)y$. Thus, the matrix A_σ has the form

$$A_\sigma(\tau, \lambda) = L^T(\tau, \lambda) P_\sigma L(\tau, \lambda). \tag{3.32}$$

To get an expression for the change of variables matrix $L(\tau, \lambda)$, set $\rho_{\ell_k, j} := \delta_{\ell_k, j}$, and

$$\rho_{r_k, j} := \begin{cases} 0, & \text{when } j > r_k, \text{ or } 1 \le j \le \ell_k \\ \\ 1, & \text{when } \ell_k < j \le r_k, \end{cases} .$$

With this notation, the lower-triangular matrix $L(\tau, \lambda)$ has the form

$$L(\tau, \lambda) = \begin{bmatrix} \frac{\rho_{11}}{\lambda_1} \tau_1^{1/2} I_d & 0 & 0 & \dots & 0 & 0 \\ \\ \frac{\rho_{21}}{\lambda_2} \tau_1^{1/2} I_d & \frac{\rho_{22}}{\lambda_2} \tau_2^{1/2} I_d & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\rho_{2n-1,1}}{\lambda_{2n-1}} \tau_1^{1/2} I_d & \frac{\rho_{2n-1,2}}{\lambda_{2n-1}} \tau_2^{1/2} I_d & \frac{\rho_{2n-1,3}}{\lambda_{2n-1}} \tau_3^{1/2} I_d & \dots & \frac{\rho_{2n-1,2n-1}}{\lambda_{2n-1}} \tau_{2n-1}^{1/2} I_d & 0 \\ \\ \frac{\rho_{2n,1}}{\lambda_{2n}} \tau_1^{1/2} I_d & \frac{\rho_{2n,2}}{\lambda_{2n}} \tau_2^{1/2} I_d & \frac{\rho_{2n,3}}{\lambda_{2n}} \tau_3^{1/2} I_d & \dots & \frac{\rho_{2n,2n-1}}{\lambda_{2n}} \tau_{2n-1}^{1/2} I_d & \frac{\rho_{2n,2n}}{\lambda_{2n}} \tau_{2n}^{1/2} I_d \end{bmatrix}. \tag{3.33}$$

The matrices A_σ, L and P_σ are all block matrices, with $d \times d$ blocks, which are multiples of the identity matrix I_d. The matrix A_σ is symmetric and non-negative so $\det(I_{2nd} + iA_\sigma)$ is the product $\prod_{j=1}^{2nd}(1 + i\mu_j)$, where μ_j are the eigenvalues of A_σ. It is easy to see that μ_j are the eigenvalues of the matrix $2n \times 2n$ matrix A_σ^r obtained by reducing each $d \times d$ identity block in A_σ to a "1×1" block, except that the corresponding multiplicities are multiplied by d. We conclude that

$$\det(I_{2nd} + iA_\sigma) = [\det(I_{2n} + iA^r_\sigma)]^d. \tag{3.34}$$

The reduced matrix A^r_σ has the form as in (3.32):

$$A^r_\sigma(\tau, \lambda) = L^T_r(\tau, \lambda) P_r(\sigma) L_r(\tau, \lambda),$$

where $P_r(\sigma)$ is the projection on the (now scalar) r_k-components, and $L_r(\tau, \lambda)$ has the same form (3.33) as $L(\tau, \lambda)$ except that each $d \times d$ identity block is contracted to a scalar. Thus, Proposition 3.4 is a consequence of (3.34) and the following lemma.

Lemma 3.5 *For any permutation σ we have*

$$|\det(I_{2n} + iA^r_\sigma(\tau, \lambda))| \geq \frac{1}{2^{n/2}} \prod_{k=1}^{n} \left(1 + \frac{\tau_{r_k}}{\lambda^2_{r_k}}\right) \tag{3.35}$$

for all $n \geq 1$, $(\tau_1, \ldots, \tau_{2n}) \in (0, +\infty)^{2n}$ and $(\lambda_{r_1}, \ldots, \lambda_{r_n}) \in (0, +\infty)^n$.

Proof. The non-negative symmetric $2n \times 2n$ matrix $A^r_\sigma(\tau, \lambda)$ has eigenvalues

$$\gamma_1 \geq \gamma_2 \geq \gamma_n > \gamma_{n+1} = \ldots = \gamma_{2n} = 0.$$

In order to deal with the non-degenerate part, let us denote by $N_r(\tau, \lambda)$ the $n \times n$ matrix obtained from $L_r(\tau, \lambda)$ by removing the rows and columns that correspond to the indices ℓ_k, with $k = 1, \ldots, n$. We will also consider the $n \times n$ matrix

$$a^{(r)}_\sigma(\tau, \lambda) := N^T_r(\tau, \lambda) N_r(\tau, \lambda).$$

Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$ be the eigenvalues of $a^{(r)}_\sigma(\tau, \lambda)$. We claim that

$$\gamma_j \geq \mu_j, \quad j = 1, \ldots, n. \tag{3.36}$$

Consider the quadratic forms $Q(\cdot)$ and $P(\cdot)$ on $\mathbb{R}^{2n}$ and $\mathbb{R}^n$, respectively, that correspond to the matrices $A^r_\sigma(\tau, \lambda)$ and $a^{(r)}_\sigma(\tau, \lambda)$. Let $H_n := \text{span}(e_{r_j}, j = 1, \ldots, n) \subset \mathbb{R}^{2n}$, and $U : \mathbb{R}^n \to H_n$ be given by

$$Uy = \sum_{j=1}^{n} y_j e_{r_j},$$

so that $Q(Uy) = P(y)$. Note that

$$\gamma_1 = \sup_{x \in \mathbb{R}^{2n}, \|x\|=1} Q(x) \geq \sup_{x \in H_n, \|x\|=1} Q(x) = \sup_{y \in \mathbb{R}^n, \|y\|=1} P(y) = \mu_1.$$

Similarly, for $1 < k \leq n$, let $\mathcal{H}_k$ the family of all subspaces of $\mathbb{R}^{2n}$ of dimension k, $\mathcal{H}'_k$ the family of all k-dimensional subspaces of H_n, and $\mathcal{H}''_k$ the family of all k-dimensional subspaces of $\mathbb{R}^n$. Then, by Fisher's principle, see part (i) of Theorem 4, p. 318 of [23], we have

$$\gamma_k = \sup_{H \in \mathcal{H}_k} \inf_{x \in H, \|x\|=1} Q(x) \geq \sup_{H \in \mathcal{H}'_k} \inf_{x \in H, \|x\|=1} Q(x) = \sup_{H \in \mathcal{H}''_n} \inf_{x \in H, \|x\|=1} P(x) = \mu_k.$$

We see that (3.36) holds. This argument allows us to write

$$\begin{aligned} |\det(I + iA^r_\sigma(\tau,\lambda))| &= \prod_{k=1}^{2n} |(1+i\gamma_k)| \geq \prod_{k=1}^{n} |(1+i\gamma_{n+k})| \geq \prod_{k=1}^{n} |(1+i\mu_k)| \geq \prod_{k=1}^{n} \frac{1+\mu_k}{\sqrt{2}} \\ &= \frac{1}{2^{n/2}} \Big(1 + \sum_{k=1}^{n} \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n} \mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}\Big). \end{aligned} \tag{3.37}$$

In order to re-write the summation in the right side, we use an elementary linear algebra result (see p. 88 of [11]). Recall that a $k \times k$ matrix b is a principal minor of rank $k \in \{1, \ldots, n\}$ of an $n \times n$ matrix B, if it is obtained by removing $n - k$ different rows and columns containing the diagonal elements $b_{j_1,j_1}, \ldots, b_{j_{n-k},j_{n-k}}$ for some $j_1 < j_2 < \ldots < j_{n-k}$. Then, we have

$$\sum_{b \in \mathcal{M}_k(B)} \det(b) = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} \prod_{\ell=1}^{k} \eta_{j_\ell}.$$

Here, η_j, $j = 1, \ldots, n$ are the eigenvalues of the matrix B, and $\mathcal{M}_k(B)$ is the collection of all $k \times k$ principal minors of the matrix B. Thus, (3.37) can be written as

$$|\det(I + iA^r_\sigma(\tau,\lambda)) \geq \frac{1}{2^{n/2}} \Big(1 + \sum_{b \in \mathcal{M}_k(B)} \det(b)\Big). \tag{3.38}$$

In order to estimate the right side we will use the following lemma.

Lemma 3.6 *Let $b \in \mathcal{M}_k\big(a^{(r)}_\sigma(\tau,\lambda)\big)$ be the principal minor obtained from $a^{(r)}_\sigma(\tau,\lambda)$ by the removal of the rows and columns that correspond to the indices $1 \leq j_1 < j_2 < \ldots < j_{n-k} \leq n$, then*

$$\det(b) \geq \prod_{j \notin \{j_1,\ldots,j_{n-k}\}} \frac{\tau_{r_j}}{\lambda^2_{r_j}}. \tag{3.39}$$

Proof. Let $\tilde{l}$ be the principal minor obtained from $N_r(\tau,\lambda)$ by the removal of the rows and columns that correspond to the indices $1 \leq j_1 < j_2 < \ldots < j_{n-k} \leq n$, and P be the projection matrix onto $\mathrm{span}\{\mathrm{e}_{r_j},\ j \notin \{j_1, \ldots, j_{n-k}\}\}$, then

$$\det(\tilde{l}) = \det(PN_rP + I - P),$$

and

$$\det(\tilde{\tilde{l}}\tilde{\tilde{l}}^T) = \det(PN_r PPN_r^T P + I - P).$$

It follows that

$$\det(b) = \det(PN_r N_r^T P + I - P) \geq \det(PN_r PPN_r^T P + I - P) = \det(\tilde{\tilde{l}}\tilde{\tilde{l}}^T),$$

as seen by the comparison of the corresponding quadratic forms. We conclude that

$$\det(b) \geq \det(\tilde{l}^t \tilde{l}) = \prod_{j \notin \{j_1, \ldots, j_{n-k}\}} \frac{\tau_{r_j}}{\lambda_{r_j}^2},$$

finishing the proof of Lemma 3.6. □

Using (3.39) in (3.37) we conclude that

$$|\det(I + iA_\sigma^r(\tau, \lambda))| \geq \frac{1}{2^{n/2}} \prod_{k=1}^{n} \left(1 + \frac{\tau_{r_k}}{\lambda_{r_k}^2}\right). \tag{3.40}$$

This finishes the proof of Lemma 3.5, and thus also that of Proposition 3.4. □

Proof of Theorem 1.1

Proposition 3.1 allows us to pass to the limit $\varepsilon \to 0$ termwise in the expression (3.2) for $c_{n,\varepsilon}(t, \xi)$, so that

$$\lim_{\varepsilon \to 0+} \bar{\zeta}_\varepsilon(t, \xi) = \hat{\psi}_0(\xi) \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!!} \lim_{\varepsilon \to 0+} c_{n,\varepsilon}(t, \xi), \tag{3.41}$$

provided that $t \in [0, t_0]$, where t_0 is so small that $Ct_0 < 1$, with C as in (3.5). We will again assume that $s(\lambda) = 1$ to simplify the notation. Let us go back to representation (3.13):

$$c_{n,\varepsilon}(t, \xi) = \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n \sum_\sigma \int_1^{+\infty} \frac{d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d}} \cdots \int_1^{+\infty} \frac{d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d}} \int_{\tilde{\Delta}_{2n}(t/\varepsilon^2)} d\tau_{1,2n} \tag{3.42}$$
$$\times \det(I + iA_\sigma(\tau, \lambda))^{-1/2} \exp\left\{-\frac{1}{2}(C_\sigma(\tau, \lambda)\Xi(\tau), \Xi(\tau))_{\mathbb{R}^{2nd}}\right\}.$$

Each term appearing in the sum in the right side of (3.42) is of the form

$$\frac{1}{(2\pi)^{nd/2}} \int_1^{+\infty} \cdots \int_1^{+\infty} \prod_{j=1}^{n} \lambda_{r_j}^{-\gamma-d} d\lambda_{r_1,r_n} \int_{\tilde{\Delta}_n(t)} \Theta_\varepsilon(\tau_\varepsilon; \sigma) d\tau_{\ell_1,\ell_n}. \tag{3.43}$$

Here $d\lambda_{r_1,r_n} = d\lambda_{r_1} \ldots d\lambda_{r_n}$, $d\tau_{\ell_1,\ell_n} = d\tau_{\ell_1} \ldots d\tau_{\ell_n}$ and the domain of integration is

$$\tilde{\Delta}_n(t; \sigma) := \left[(\tau_{\ell_1}, \ldots, \tau_{\ell_n}) : \sum_{j=1}^{n} \tau_{\ell_j} \leq t, \, \tau_{\ell_j} \geq 0, \, j = 1, \ldots, n\right].$$

The integrand in (3.43) is defined as follows: set $\tau_\varepsilon := (\tau_{1,\varepsilon}, \ldots, \tau_{2n,\varepsilon})$, with $\tau_{\varepsilon,\ell_j} := \varepsilon^{-2}\tau_{\ell_j}$, and $\tau'_{\varepsilon,r_j} := \tau_{r_j}$. Then

$$\Theta_\varepsilon(\tau_\varepsilon;\sigma) := \int_{\widehat{\Delta}_n((t-\tau)/\varepsilon^2)} \det(I + iA_\sigma(\tau_\varepsilon,\lambda))^{-1/2} \exp\left\{-\frac{1}{2}(C_\sigma(\tau_\varepsilon,\lambda)\Xi(\tau_\varepsilon),\Xi(\tau_\varepsilon))_{\mathbb{R}^{2nd}}\right\} d\tau_{r_1,r_n}, \tag{3.44}$$

where $\tau := \sum_{j=1}^n \tau_{\ell_j}$, $d\tau_{r_1,r_n} := d\tau_{r_1}\ldots d\tau_{r_n}$ and

$$\widehat{\Delta}_n(u;\sigma) := \left[(\tau_{r_1},\ldots,\tau_{r_n}) : \sum_{j=1}^n \tau_{r_j} \le u,\ \tau_{r_j} \ge 0,\ j=1,\ldots,n\right].$$

We will distinguish in the computation of the limit between simple and non-simple pairings – note that no such distinction was made in the estimates so far.

Non-simple pairings

Recall that the pairing $\mathfrak{e} := \{(1,2),(3,4),\ldots,(2n-1,2n)\}$ is called *simple*.

Lemma 3.7 *For any $(\lambda_{r_1},\ldots,\lambda_{r_n}) \in (1,+\infty)^n$ and $(\tau_1,\ldots,\tau_{2n}) \in (0,+\infty)^{2n}$, we have*

$$\lim_{\varepsilon\to 0+} |\det(I + iA_\sigma(\tau_\varepsilon,\lambda))| = +\infty, \tag{3.45}$$

provided that σ is a permutation such that $\mathfrak{f}(\sigma) \neq \mathfrak{e}$ (see (2.22) for the definition of the map $\mathfrak{f}$).

Proof. Note that if $a_{\ell,\ell} \neq 0$ for some left vertex ℓ of $\mathfrak{f}(\sigma)$ then choosing $y = (y_1,\ldots,y_{2n})$ with $y_j = 0$ for $j \neq \ell$ and $y_\ell = e$ for some $e \in \mathbb{R}^d$ such that $|e| = 1$, we get, using (3.8) and (3.12):

$$\lim_{\varepsilon\to 0+} (A_\sigma(\tau_\varepsilon,\lambda)y,y)_{\mathbb{R}^{2dn}} = \lim_{\varepsilon\to 0+} a_{\ell,\ell}\frac{\tau_\ell}{\varepsilon^2} = +\infty.$$

It follows that the largest eigenvalue of A_σ satisfies $\gamma_{2n} \to +\infty$, as $\varepsilon \to 0+$, and (3.45) follows. On the other hand, if σ is such that

$$a_{\ell,\ell} = 0 \quad \text{for all left vertices } \ell \text{ of } \mathfrak{f}(\sigma) \tag{3.46}$$

then, according to the defintion (3.9) of $a_{\ell,\ell}$, for any left vertex ℓ of $\mathfrak{f}(\sigma)$ there is no bond (ℓ',r') such that $\ell' < \ell < r'$. This implies that for all bonds we have $\ell = r-1$. Indeed, otherwise we would let ℓ be the smallest left vertex for which $r \neq \ell+1$. Then $\ell+1$ would have to be a left vertex for which $a_{\ell+1,\ell+1} \neq 0$, giving a contradiction to (3.46). This proves that $\mathfrak{f}(\sigma) = \mathfrak{e}$. □

Since, according to (3.24), there exists a constant $C > 0$ such that

$$|\det(I + iA_\sigma(\tau_\varepsilon,\lambda))|^{-1/2} \le C\prod_{k=1}^n \left(1 + \frac{\tau_{r_k}}{\lambda_{r_k}^2}\right)^{-d/2}, \tag{3.47}$$

for all permutations $\sigma \in \Pi(2n)$, $(\tau_1, \dots, \tau_{2n})$, and $(\lambda_{r_1}, \dots, \lambda_{r_n})$, we conclude by the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon \to 0+} \Theta_\varepsilon(\tau_\varepsilon; \sigma) = 0,$$

provided that $\mathfrak{f}(\sigma) \neq \mathfrak{e}$. Using the same theorem once again in (3.43), we conclude that the limit as $\varepsilon \to 0^+$ of the terms in (3.42) corresponding to such permutations, vanishes.

Simple pairings

Observe that for any σ such that $\mathfrak{f}(\sigma) = \mathfrak{e}$, we have $a_{mj} = 0$ if $m \neq j$ and $a_{\ell,\ell} = 0$ if ℓ is a left vertex, in other words, if ℓ is odd. The matrix A_σ has then a particularly simple form

$$A_\sigma(\tau, \lambda) = \begin{bmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & I_d \lambda_2^{-2} \tau_2 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & I_d \lambda_{2n}^{-2} \tau_{2n} \end{bmatrix} \tag{3.48}$$

and the matrix C_σ has the form

$$C_\sigma(\tau, \lambda) = \begin{bmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & I_d (\lambda_2^2 + i\tau_2)^{-1} \tau_2 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & I_d (\lambda_{2n}^2 + i\tau_{2n})^{-1} \tau_{2n} \end{bmatrix}. \tag{3.49}$$

Thus, we obtain for such σ:

$$\lim_{\varepsilon \to 0+} \Theta_\varepsilon(\tau_\varepsilon; \sigma) = \prod_{k=1}^{n} \int_0^{+\infty} \frac{1}{(1 + i\lambda_{2k}^{-2}\tau)^{d/2}} \exp\left\{ -\frac{|\xi|^2 \tau^2}{2(\lambda_{2k}^2 + i\tau)} \right\} d\tau. \tag{3.50}$$

Since, as we already know, there are precisely $(2n)!!$ permutations that yield $\mathfrak{f}(\sigma) \neq \mathfrak{e}$ we conclude from (3.42)-(3.43)and (3.50) that

$$\lim_{\varepsilon \to 0+} c_{n,\varepsilon}(\xi, t) = [-2itr(\xi)]^n \tag{3.51}$$

with

$$r(\xi) = \frac{i}{2} \frac{1}{(2\pi)^{d/2}} \int_1^{+\infty} \frac{\kappa(\lambda, \tau) d\lambda}{\lambda^{\gamma+d-2}}, \quad \kappa(\lambda, \tau) = \int_0^{+\infty} \frac{1}{(1 + i\tau)^{d/2}} \exp\left\{ -\frac{(\lambda|\xi|\tau)^2}{2(1 + i\tau)} \right\} d\tau. \tag{3.52}$$

This leads to the conclusion of the theorem. □

Proof of Theorem 1.2

In the setting of Theorem 1.2, we conclude that there exists $t_0 > 0$ such that

$$\bar{\zeta}_\varepsilon(t,\xi) = \hat{\psi}_0(\xi) \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!!} \lim_{\varepsilon\to 0+} c_{n,\varepsilon}(t, \varepsilon^\beta \xi), \tag{3.53}$$

with $c_{n,\varepsilon}(t,\xi)$ given by (3.42) and $t \in [0, t_0]$. The same computations as in the proof of Theorem 1.1 show that

$$\lim_{\varepsilon\to 0+} c_{n,\varepsilon}(t, \varepsilon^\beta \xi) = [-2itr(0)]^n, \quad n \ge 0,\ t \ge 0.$$

We conclude that for any $t \in [0, t_0]$

$$\lim_{\varepsilon\to 0+} \bar{\zeta}_\varepsilon(t,\xi) = \bar{\zeta}(t,\xi), \tag{3.54}$$

with $\bar{\zeta}(t,\xi)$ given by (1.27), both pointwise in ξ and weakly in $L^2(\mathbb{R}^d)$. In order to show that not only the limit holds for the expectation, but actually the limit is deterministic, observe that, due to (2.13), we have

$$\begin{aligned}
&\mathbb{E}\|\hat{\zeta}_\varepsilon(t,\cdot) - \bar{\zeta}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} = \mathbb{E}\|\hat{\zeta}_\varepsilon(t,\cdot)\|^2 + \|\bar{\zeta}(t,\cdot)\|^2 - 2\mathrm{Re}\left(\bar{\zeta}_\varepsilon(t,\cdot), \bar{\zeta}(t,\cdot)\right)_{L^2(\mathbb{R}^d)} \\
&= 2\|\hat{\psi}_0\|^2 - 2\mathrm{Re}\left(\bar{\zeta}_\varepsilon(t,\cdot), \bar{\zeta}(t,\cdot)\right)_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{3.55}$$

Letting $\varepsilon \to 0+$ and using (3.54) we conclude that the right hand side of (3.55) tends to

$$2\|\hat{\psi}_0\|^2 - 2\|\bar{\zeta}(t,\cdot)\|_{L^2(\mathbb{R}^d)} = 0,$$

which ends the proof of the theorem. □

4 The infinite effective potential regime

In the present section we give the proof of Theorem 1.3. Adjusting for the scaling, we can write an analogue of (3.28), though this time the simplex $\tilde{\Delta}_{2n}(t/\varepsilon^2)$ is replaced by $\tilde{\Delta}_{2n}(t/\varepsilon^{2\alpha})$. Using Proposition 3.4, we arrive at the estimate:

$$|c_{n,\varepsilon}(t,\xi)| \le C^n \varepsilon^{2n} \sum_\sigma \int_{\tilde{\Delta}_n(t/\varepsilon^{2\alpha})} d\tau_{\ell_1,\ell_n} \prod_{k=1}^n \left[\int_1^{+\infty} \frac{d\lambda_{r_k}}{\lambda_{r_k}^{\gamma+d}} \int_0^{t\varepsilon^{-2\alpha}} \frac{d\tau_{r_k}}{(1+\lambda_{r_k}^{-2}\tau_{r_k})^{d/2}}\right]. \tag{4.1}$$

Changing variables $\tau'_{r_k} := \lambda_{r_k}^{-2}\tau_{r_k}$ we obtain

$$|c_{n,\varepsilon}(t,\xi)| \le C^n \varepsilon^{2n} \sum_\sigma \prod_{k=1}^n \left[\int_1^{+\infty} \frac{d\lambda_{r_k}}{\lambda_{r_k}^{\gamma+d-2}} \phi\left(\frac{t}{\lambda_{r_k}^2 \varepsilon^{2\alpha}}\right)\right] \int_{\tilde{\Delta}_n(t/\varepsilon^{2\alpha})} d\tau_{\ell_1,\ell_n}$$

$$\le (2n)!\frac{(Ct\varepsilon^{2-2\alpha})^n}{n!}\left[\int_1^{+\infty}\phi\left(\frac{t}{\lambda^2\varepsilon^{2\alpha}}\right)\frac{d\lambda}{\lambda^{\gamma+d-2}}\right]^n, \tag{4.2}$$

where

$$\phi(u) := \int_0^u \frac{d\tau}{(1+\tau)^{d/2}}.$$

Note that

$$\lim_{u\to+\infty}\phi(u) = \frac{2}{d-2}, \quad d \ge 3. \tag{4.3}$$

We change variables

$$u := \frac{t}{\lambda^2\varepsilon^{2\alpha}}, \quad k = 1,\ldots,n,$$

in the integral appearing in the right side of (4.2). Taking into account (1.29), the right side of (4.2) can be then rewritten in the form

$$(2n)!\frac{(Ct^H)^n}{n!}\left[\int_0^{t/\varepsilon^\alpha}\phi\left(u\right)u^{(\gamma+d-5)/2}du\right]^n, \tag{4.4}$$

for

$$H := \frac{5-d-\gamma}{2} \tag{4.5}$$

and an appropriate constant $C > 0$. Since $(\gamma + d - 5)/2 < -1$ the integral in (4.4) converges at $+\infty$, due to (4.3). As $\phi(u) \sim u$, for $u \ll 1$ we conclude that it is also convergent close to 0, as $(\gamma + d - 3)/2 > -1$. We conclude therefore that there exists $C > 0$ such that

$$|c_{n,\varepsilon}(t,\xi)| \le (2n)!\frac{(Ct^H)^n}{n!}\left[\int_0^{t/\varepsilon^\alpha}\phi\left(u\right)u^{(\gamma+d-5)/2}du\right]^n, \quad n \ge 0,\ t > 0,\ \varepsilon > 0. \tag{4.6}$$

Computation of the limit

Having the uniform bound (4.6), we now compute the limit. From (3.13), we obtain

$$c_{n,\varepsilon}(t,\xi) = \left(\frac{\varepsilon^2}{(2\pi)^{d/2}}\right)^n \sum_\sigma \int_1^{+\infty}\frac{d\lambda_{r_1}}{\lambda_{r_1}^{\gamma+d}}\cdots\int_1^{+\infty}\frac{d\lambda_{r_n}}{\lambda_{r_n}^{\gamma+d}}\int_{\Delta_{2n}(t/\varepsilon^{2\alpha})}d\tau_{1,2n} \tag{4.7}$$
$$\times \det(I + iA_\sigma^r(\tau,\lambda))^{-d/2}\exp\left\{-\frac{\varepsilon^{2\beta}|\xi|^2}{2}(C_\sigma^r(\tau,\lambda)\top^{1/2}\mathbf{1}\cdot\top^{1/2}\mathbf{1})\right\},$$

Here, we denote $\top := \mathrm{diag}[\tau_1,\ldots,\tau_{2n}]$, $\mathbf{1}^T := \underbrace{[1,\ldots,1]}_{2n}$ and

$$C_\sigma^r(\tau,\lambda) := A_\sigma^r(\tau,\lambda)[I_{2n} + iA_\sigma^r(\tau,\lambda)]^{-1}.$$

We change variables $\tau'_k := \varepsilon^{2\alpha}\tau_k/t$, $k = 1, \dots, 2n$, $u_k := t\lambda_{r_k}^{-2}\varepsilon^{-2\alpha}$, $k = 1, \dots, n$, and recall that

$$H = \frac{1}{\alpha} = \frac{5-\gamma-d}{2}.$$

We can write then

$$c_{n,\varepsilon}(t,\xi) = \left(\frac{t^H}{2(2\pi)^{d/2}}\right)^n \sum_\sigma \int_0^{t/\varepsilon^{2\alpha}} u_1^{(\gamma+d-3)/2} du_1 \dots \int_0^{t/\varepsilon^{2\alpha}} u_n^{(\gamma+d-3)/2} du_n \int_{\tilde{\Delta}_{2n}(1)} d\tau_{1,2n}$$
$$\times \det(I + i\tilde{A}^r_\sigma(\tau,u))^{-d/2} \exp\left\{-\frac{t\varepsilon^{2(\beta-\alpha)}|\xi|^2}{2}(\tilde{C}^r_\sigma(\tau,u)\top^{1/2}\mathbf{1}, \top^{1/2}\mathbf{1})_{\mathbb{R}^{2nd}}\right\}, \quad (4.8)$$

where $\tilde{A}^r_\sigma(\tau,u) = [\tilde{a}_{j,m}]$ is an $2n \times 2n$ matrix satisfying, as in (3.9)–(3.11):

$$\tilde{a}_{1,j} = \tilde{a}_{j,1} = 0 \quad \text{for all } j \in \{1, \dots, 2n\},$$

and

$$\tilde{a}_{j,j} := \tau_j {\sum_k}' u_k, \quad \text{for each } j \in \{2, \dots, 2n\}. \quad (4.9)$$

The summation above extends over those k-s, for which $\ell_k < j \le r_k$. Given $m < j$ we let

$$\tilde{a}_{m,j} = \tilde{a}_{j,m} = (\tau_m\tau_j)^{1/2} {\sum_k}' u_k$$

and the summation extends over those k-s, for which $\ell_k < m < j \le r_k$. We also let

$$\tilde{C}^r_\sigma(\tau,u) := \tilde{A}^r_\sigma(\tau,u)(I_{2n} + i\tilde{A}^r_\sigma(\tau,u))^{-1}.$$

In the limit we get

$$\bar{c}_n(t,\xi) := \lim_{\varepsilon\to 0+} c_{n,\varepsilon}(t,\xi) = \left(\frac{t^H}{2(2\pi)^{d/2}}\right)^n \sum_\sigma \int_0^{+\infty} u_1^{(\gamma+d-3)/2} du_1 \dots \int_0^{+\infty} u_n^{(\gamma+d-3)/2} du_n \quad (4.10)$$
$$\times \int_{\tilde{\Delta}_{2n}(1)} d\tau_{1,2n} \det(I + i\tilde{A}^r_\sigma(\tau,u))^{-d/2}.$$

Repeating the derivation of (3.13), see the calculation from (3.16) to (3.23), this time for $\xi = 0$ and in the reverse order, we obtain

$$\bar{c}_n(t,\xi) = \mathbb{M}\left\{\frac{t^H}{2(2\pi)^{d/2}} \int_0^{+\infty} u^{(\gamma+d-3)/2} du \int_0^1 ds \int_0^1 ds' \exp\left\{-\frac{iu}{2}|B_s - B_{s'}|^2\right\}\right\}^n, \quad (4.11)$$

where B_t is the d-dimensional standard Brownian motion and $\mathbb{M}$ is the corresponding expectation. Using the bound (4.6) we conclude that there exists $t_0 > 0$ such that it is possible to interchange the limit as $\varepsilon \to 0+$ with the summation over n, as in (3.41) above. Performing the summation $\sum_{n=0}^{+\infty} \bar{c}_n(t,\xi)$ we arrive at (1.30).

5 Proof of Proposition 2.2

The conclusion of part (i) of the proposition and formula (2.12) follow, provided we can show that

$$\sum_{n=0}^{\infty}[\mathbb{E}|\hat{\zeta}_n(t,\xi)|^2]^{1/2} < +\infty.$$

We have

$$\mathbb{E}|\hat{\zeta}_n(t,\xi)|^2 = \frac{1}{(2\pi)^{2nd}} \int\limits_{\Delta_n(t)} ds_{1,n} \int\limits_{\Delta_n(t)} ds_{n+1,2n} \int_{\mathbb{R}^{2dn}} \hat{\psi}_0(\xi - \sum_{j=1}^{n} p_j)\hat{\psi}_0^*(\xi - \sum_{j=1}^{n} p_{n+j})$$
$$\times e^{iG_n} e^{-i\tilde{G}_n} \mu(dp_1, \dots, dp_{2n}), \tag{5.1}$$

where the measure

$$\mu(dp_1, \dots, dp_{2n}) := \sum_{\mathcal{F}\in\mathfrak{F}_{2n}} \prod_{(k,\ell)\in\mathcal{F}} \delta\left(p_k + (-1)^{s(k,\ell)} p_\ell\right) \hat{R}(p_k) \text{đ}p_{1,2n}$$

and

$$s(k,\ell) := \begin{cases} 0, & \text{if either } k,\ell \in \{1,\dots,n\} \text{ or } k,\ell \in \{n+1,\dots,2n\}, \\ 1, & \text{if otherwise.} \end{cases}$$

Here $ds_{m,m'} = ds_m \dots ds_{m'}$ for $1 \le m < m' \le 2n$ and $\tilde{G}_n$ is given by (2.11), with the variables $s_1, \dots s_n$, $p_1, \dots, p_n$ replaced by $s_{n+1}, \dots s_{2n}$, $p_{n+1}, \dots, p_{2n}$. One can easily obtain

$$\mu(\mathbb{R}^{2dn}) = (2n-1)!! \left(\int_{\mathbb{R}^d} \hat{R}(p)dp\right)^n, \tag{5.2}$$

therefore

$$\mathbb{E}|\hat{\zeta}_n(t,\xi)|^2 \le \frac{C^n t^{2n}(2n-1)!!}{n!^2} \left(\int_{\mathbb{R}^d} \hat{R}(p)dp\right)^n \|\hat{\psi}_0\|_\infty^2 \le \frac{C^n}{n!}, \tag{5.3}$$

with a constant $C > 0$ independent of n, and part (i) of the proposition follows. Part (ii) is a simple consequence of the fact that we can interchange the summation and expectation in (2.9), by virtue of the estimate obtained in the proof of part (i). This, combined with the fact that the odd moments vanish, yields (2.12).

Concerning part (iii), let us take a smooth radially symmetric non-negative function θ such that $\theta(x) = 1$ for $|x| \le 1$ and $\theta(x) = 0$ for $|x| > 2$. To prove (2.13), we consider the regularization of the potential: $V_M(x) := \theta_M(x)V(x)$, with $\theta_M(x) := \theta(x/M)$, which is still Gaussian but not stationary. Let $\hat{\zeta}^{(M)}(t,\xi)$ and $\hat{\zeta}_n^{(M)}(t,\xi)$ be the random fields given by modifications of formulas (2.9) and (2.10), with the spectral measure $\hat{V}(dp)$ replaced by

$$\hat{V}_M(p) = \int_{\mathbb{R}^d} \hat{\theta}_M(p-q)\hat{V}(dq),$$

where $\hat\theta_M(p) := M^d\hat\theta(Mp)$ and $\hat\theta$ – the Fourier transform of θ. Note that since θ is radially symmetric, smooth and compactly supported its Fourier transform $\hat\theta$ is real valued and of Schwartz class. We also have $\int_{\mathbb{R}^d}\hat\theta_M(p)dp = (2\pi)^d\theta(0)$. The covariance of $\hat V_M$ equals

$$\Gamma_M(p,p') := \mathbb{E}\left[\hat V_M(p)\hat V_M^*(p')\right] = (2\pi)^d\int_{\mathbb{R}^d}\hat\theta_M(p-q)\hat\theta_M(p'-q)\hat R(q)dq, \quad p,p'\in\mathbb{R}^d.$$

The function

$$\hat\psi^{(M)}(t,\xi) := e^{-it|\xi|^2/2}\hat\zeta^{(M)}(t,\xi), \quad t\ge 0,$$

is the solution of the Schrödinger equation with the potential $V_M(x)$. The L^2-norm conservation for the solutions of the Schrödinger equaiton with a decaying potential means that

$$\|\hat\psi^{(M)}(t)\|_{L^2(\mathbb{R}^d)} = \|\hat\psi_0\|_{L^2(\mathbb{R}^d)},$$

thus also

$$\|\hat\zeta^{(M)}(t)\|_{L^2(\mathbb{R}^d)} = \|\hat\psi_0\|_{L^2(\mathbb{R}^d)} \text{ for all } t, M>0 \text{ a.s.}$$

Note that

$$\|\hat\psi_0\|^2_{L^2(\mathbb{R}^d)} = \mathbb{E}\|\hat\zeta^{(M)}(t)\|^2_{L^2(\mathbb{R}^d)} = \sum_{m,n\ge 0}\mathbb{E}\left[\left(\hat\zeta_n^{(M)}(t),\hat\zeta_m^{(M)}(t)\right)_{L^2(\mathbb{R}^d)}\right].$$

By the Cauchy-Schwarz inequality, the absolute value of the term of the series on the right hand side is bounded from above by $(a_{n,M}a_{m,M})^{1/2}$, where

$$a_{n,M} := \mathbb{E}\int_{\mathbb{R}^d}|\hat\zeta_n^{(M)}(t,\xi)|^2 d\xi.$$

Using an analogue of (5.1) for $\hat V_M(p)$, this time with the signed measure

$$\mu_M(dp_1,\ldots,dp_{2n}) := \sum_{\mathcal{F}\in\mathfrak{F}_{2n}}\prod_{(k,\ell)\in\mathcal{F}}\Gamma_M\left(p_k,(-1)^{s(k,\ell)}p_\ell\right)đp_{1,2n}$$

we conclude that

$$a_{n,M} \le \frac{C^n t^{2n}(2n-1)!!}{n!^2}\left[\left(\int_{\mathbb{R}^d}|\hat\theta(p)|dp\right)^2\int_{\mathbb{R}^d}\hat R(q)dq\right]^n\|\hat\psi_0\|^2_{L^2(\mathbb{R}^d)},$$

with the constant $C>0$ independent of $M>0$ and $n\ge 0$. It follows that that

$$\|\hat\psi_0\|^2_{L^2(\mathbb{R}^d)} = \lim_{M\to+\infty}\mathbb{E}\|\hat\zeta^{(M)}(t)\|^2_{L^2(\mathbb{R}^d)} = \sum_{m,n\ge 0}\lim_{M\to+\infty}\mathbb{E}\left[\left(\hat\zeta_n^{(M)}(t),\hat\zeta_m^{(M)}(t)\right)_{L^2(\mathbb{R}^d)}\right]. \tag{5.4}$$

It is an elementary calculation to verify that

$$\lim_{M\to+\infty}\mathbb{E}\left[\left(\hat\zeta_n^{(M)}(t),\hat\zeta_m^{(M)}(t)\right)_{L^2(\mathbb{R}^d)}\right] = \mathbb{E}\left[\left(\hat\zeta_n(t),\hat\zeta_m(t)\right)_{L^2(\mathbb{R}^d)}\right]$$

for each $n,m\ge 0$. Therefore, the right hand side of (5.4) equals $\mathbb{E}\|\hat\zeta(t)\|^2_{L^2(\mathbb{R}^d)}$, and (2.13) follows.

6 A discussion of the potentials of the Schoenberg class

In the present section we give several examples of the covariance functions that belong to the Schoenberg class and introduce an *extended Schoenberg class*. The main results of the paper hold also for the covariances of the latter class, with essentially identical proofs. We also show that the extended Schoenbeerg class correlation functions are dense in L^1 and L^∞.

6.1 Examples of the covariance functions of the Schoenberg class

Isotropic fields with a completely monotone energy per shell density

According to Theorem 1, p. 816 of [27] any isotropic random field in dimension $d \geq 2$ has the covariance function of the form $R(x) = \rho(|x|)$, $x \in \mathbb{R}^d$, where

$$\rho(y) = \int_0^{+\infty} \Omega_d(qy)\tilde{\mathcal{E}}(q)dq, \quad y \in \mathbb{R}. \tag{6.1}$$

The function $\tilde{\mathcal{E}}(q)$, that we call the energy per shell density, has the form

$$\tilde{\mathcal{E}}(q) := q^{d-1}\mathcal{E}(q),$$

where $\mathcal{E}(\cdot)$ is the power energy spectrum of the field, and

$$\Omega_d(y) = \omega_d \int_{-1}^{1} e^{iyu}(1-u^2)^{(d-3)/2}du, \quad y \in \mathbb{R}. \tag{6.2}$$

Let us assume that $\tilde{\mathcal{E}}(\cdot)$ is completely monotone: it is C^∞ and $(-1)^n\tilde{\mathcal{E}}^{(n)}(\cdot) \geq 0$ for each $n \geq 1$. According to the Bernstein theorem, then $\tilde{\mathcal{E}}$ is a Laplace transform of a Borel measure $\mu(d\lambda)$ on $[0, +\infty)$ (not necessarily finite):

$$\tilde{\mathcal{E}}(z) = \mathcal{L}[\mu](z) := \int_0^{+\infty} e^{-\lambda z}\mu(d\lambda), \tag{6.3}$$

for $z \in \mathbb{C}_+$ – the open right half-plane. Since $\rho(0) < +\infty$ we must have

$$\int_0^{+\infty} \frac{\mu(d\lambda)}{\lambda} < +\infty. \tag{6.4}$$

Using the above, we can write

$$\begin{aligned} \rho(y) &= 2\omega_d \int_0^{+\infty} \left\{ \int_{-1}^{1} \int_0^{+\infty} \exp\left\{-q(\lambda - iyu)\right\} dq \right\} (1-u^2)^{(d-3)/2}\mu(d\lambda)du \\ &= 2\omega_d \int_0^{+\infty} \mu(d\lambda) \int_0^{1} (1-u^2)^{(d-3)/2} \frac{\lambda du}{\lambda^2 + (yu)^2} \end{aligned} \tag{6.5}$$

$$= 2\omega_d \int_0^{+\infty} \frac{\mu(d\lambda)}{\lambda} \int_0^1 (1-u^2)^{(d-3)/2} du \int_0^{+\infty} \exp\left\{-v[1+(yu/\lambda)^2]\right\} dv.$$

Finally, upon the change of variables $v \mapsto \theta = (u/\lambda)\sqrt{v}$ we conclude that $\rho(\cdot)$ belongs to the Schoenberg class as it can be written in the form

$$\rho(y) = \int_0^{+\infty} h_*(\theta) e^{-(\theta y)^2} d\theta, \tag{6.6}$$

where

$$h_*(\theta) := 4\theta\omega_d \int_0^{+\infty} \int_0^1 \frac{(1-u^2)^{(d-3)/2}}{u^2} \exp\left\{-(\theta\lambda/u)^2\right\} \lambda\mu(d\lambda) du. \tag{6.7}$$

It is straightforward to verify that (6.4) ensures that $h_* \in L^1([0,+\infty))$, so that ρ is, indeed, of the Schoenberg class.

Fields whose covariance is completely monotone

Next, assume that the function $\rho(y)$, $y \in [0,+\infty)$, itself is a completely monotone function. Then, there exists a finite Borel measure μ on $[0,+\infty)$ such that

$$\rho(y) = \int_0^{+\infty} e^{-\lambda y} \mu(d\lambda), \quad y \geq 0. \tag{6.8}$$

Note that $\rho(\cdot)$ is non-negative definite, as its Fourier transform is non-negative:

$$e^{-\lambda y} = \frac{\lambda}{\pi} \int_{\mathbb{R}} \frac{e^{i\xi y} d\xi}{\lambda^2 + \xi^2} = \frac{1}{\pi\lambda} \int_{\mathbb{R}} e^{i\xi y} d\xi \left\{ \int_0^{+\infty} e^{-v(1+(\xi/\lambda)^2)} dv \right\}, \quad \lambda,\, y \geq 0.$$

Substituting into the right hand side of (6.8) and integrating out the ξ variable, we conclude that

$$\rho(y) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mu(d\lambda) \int_0^{+\infty} e^{-(\lambda y)^2/(4v)} \frac{e^{-v} dv}{\sqrt{v}}.$$

Making a change of variable$v \mapsto \theta := \lambda/(2\sqrt{v})$, we conclude that the function $\rho(\cdot)$ is in the Schoenberg class, as:

$$\rho(y) = \int_0^{+\infty} e^{-(\theta y)^2} F(\theta) d\theta, \quad y \geq 0, \tag{6.9}$$

with

$$F(\theta) := \frac{1}{\sqrt{\pi}\theta^2} \int_0^{+\infty} \lambda e^{-\lambda^2/(4\theta^2)} \mu(d\lambda).$$

Once again, $F \in L^1([0,+\infty))$ because of (6.4).

Fields with the energy spectrum of the exponential type

A slightly longer computation shows that covariances with the energy spectra of the form

$$\mathcal{E}(q) = \int_1^{+\infty} e^{-(\lambda q)^\alpha} s(\lambda) \frac{d\lambda}{\lambda^\gamma}, \quad q > 0, \tag{6.10}$$

are of the Schoenberg class not only for $\alpha = 2$ but for all $\alpha \in (0, 2)$ and $\gamma > 1 - d$.

6.2 The extended Schoenberg class

Our arguments can be easily generalized to fields whose covariance function belongs to the *extended Schoenberg class* – functions of the form (1.11) with $\mu(\cdot)$ a signed measure of a finite total variation. Since $\rho(\cdot)$ is a covariance, we obviously require additionally that it is non-negative definite. For example Theorems 1.1-1.3 hold if we assume that the function $s(\cdot)$ in (1.15) satisfies $|s(\lambda)| \ge c > 0$. The power energy spectrum corresponding to a covariance function in the extended Schoenberg class can be extended to a complex domain as

$$\mathcal{E}(z) = \mathcal{L}[\nu](z^2), \quad z^2 \in \mathbb{C}_+ := [z \in \mathbb{C} : \operatorname{Re} z > 0], \tag{6.11}$$

with the signed measure $\nu(\cdot)$ given by (1.13). Conversely, $\mathcal{E}(\cdot)$ is a power energy spectrum of a covariance function from the extended Schoenberg class if $\mathcal{E}(z) = \mathcal{L}[\nu](z^2)$ for a signed measure $\nu(d\lambda) = \lambda^d \mu(d\lambda)$, with μ a signed measure of a finite total variation, such that $\mathcal{L}[\nu](q) \ge 0$ for $q \ge 0$.

For example, let $\mu(d\lambda) := g(\lambda)d\lambda$ with a non-negative definite, real valued function g in $L^1(\mathbb{R})$. We claim that then $\mathcal{L}[\mu](q) \ge 0$ for $q \ge 0$. Indeed, we have the representation

$$g(\lambda) = \int_{\mathbb{R}} e^{i\lambda y} \nu(dy),$$

with a finite Borel measure $\nu(\cdot)$. Substituting for $\mu(d\lambda)$ into (6.11) we see that for $q \ge 0$

$$\mathcal{L}[\mu](q) = \int_0^{+\infty} e^{-\lambda q} g(\lambda) d\lambda = \int_0^{+\infty} d\lambda \left\{ \int_{\mathbb{R}} e^{\lambda(iy-q)} \nu(dy) \right\} = q \int_{\mathbb{R}} \frac{\nu(dy)}{y^2 + q^2} \ge 0. \tag{6.12}$$

In the last equality we have used the fact that $\nu(\cdot)$ is symmetric, since g is real valued.

6.3 Density of the covariances of the extended Schoenberg class

Let $\mathcal{E}(q)$ be the power spectrum of a covariance $R(x) = \rho(|x|)$, such that $\tilde{\mathcal{E}}(q) := q^{d-1}\mathcal{E}(q)$, is in $L^1[0, +\infty)$. We claim that for any $\sigma > 0$ there exists $\mathcal{E}_0(\cdot)$ that is non-negative, such that

$$\|\rho - \rho_0\|_{L^\infty} \le \int_0^{+\infty} q^{d-1} |\mathcal{E}(q) - \mathcal{E}_0(q)| dq < \sigma, \tag{6.13}$$

and the corresponding covariance function $\rho_0(\cdot)$ given by (6.1) is in the extended Schoenberg class. Indeed, let $\Psi(v) := \tilde{\mathcal{E}}(-\log v)$, $v \in (0,1]$. Then Ψ is non-negative on $[0,1]$ and satisfies

$$\int_0^1 \Theta(v)dv < +\infty,$$

with $\Theta(v) := \Psi(v)/v$. Let Θ_1 be a continuous, non-negative, compactly supported in $(0,1]$ and such that $\|\Theta - \Theta_1\|_{L^1[0,1]} < \sigma/2$. Note that then $\Psi_1(v) := \Theta_1(v)v$, $v \in [0,1]$ is non-negative and belongs to $C[0,1]$. Thanks to the Weierstrass approximation theorem, we can find a real coefficient polynomial

$$\Theta_0(v) = \sum_{k=0}^{n} a_k v^k, \quad v \in [0,1],$$

that is non-negative and satisfies $\|\Theta_1 - \Theta_0\|_\infty < \sigma/2$. Let $\Psi_0(v) := \Theta_0(v)v$ and note that

$$\int_0^1 |\Psi(v) - \Psi_0(v)|\frac{dv}{v} \le \|\Theta - \Theta_1\|_{L^1[0,1]} + \|\Theta_1 - \Theta_0\|_\infty < \sigma.$$

We set $\tilde{\mathcal{E}}_0(q) := \Psi_0(e^{-q})$ and $\mathcal{E}_0(q) := \tilde{\mathcal{E}}_0(q)q^{1-d}$, $q > 0$. We have a representation

$$\tilde{\mathcal{E}}_0 = \mathcal{L}[\mu], \qquad \mu := \sum_{k=1}^{n} a_{k-1}\delta_k.$$

As in the first example of this section, we conclude that the covariance function ρ_0, corresponding to $\tilde{\mathcal{E}}_0$ via (6.1), belongs to the extended Schoenberg class. In addition, we have the approximation

$$\|\rho - \rho_0\|_{L^\infty} \le \int_0^{+\infty} q^{d-1}|\mathcal{E}(q) - \mathcal{E}_0(q)|dq = \|\tilde{\mathcal{E}} - \tilde{\mathcal{E}}_0\|_{L^1[0,+\infty)} = \int_0^1 |\Psi(v) - \Psi_0(v)|\frac{dv}{v} < \sigma.$$

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