Math 63CM Homework # 3

Due in class on Friday, January 24.

1. (i) Find the unique solution to the ODE $y'(t) = ay(t)$, with the initial condition $y(0) = y_0$, where $a \in \mathbb{R}$ and $y_0 \in \mathbb{R}$ are given numbers.
(ii) Let $a(t)$ be a given continuous function. Explain why the initial value problem $y'(t) = a(t)y(t)$, with the initial condition $y(0) = y_0$, has a unique solution for all $t \in \mathbb{R}$. Show that if $y_0 > 0$ then $y(t) > 0$ for all $t \in \mathbb{R}$ and if $y_0 < 0$ then $y(t) < 0$ for all $t \in \mathbb{R}$.
(iii) Find this solution. Hint: it may help to look at $z(t) = \log y(t)$ if $y_0 > 0$ and at $z(t) = -\log y(t)$ if $y_0 < 0$.

2. Given an ODE

$$u'(t) = F(u(t)), \quad u(0) = u_0,$$

define

$$T_- := \inf \{T : \text{a solution } u(t) \text{ exists for } t \in (T, 0]\}$$

and

$$T_+ := \sup \{T : \text{a solution } u(t) \text{ exists for } t \in [0, T]\}.$$

(i) Consider the ODE for $u : I \subset \mathbb{R} \to \mathbb{R}$.

$$u'(t) = (u(t))^2, \quad u(0) = u_0.$$

For each $u_0$, find $T_-$ and $T_+$. Conclude that the ODE has a global solution (i.e., a solution $u : \mathbb{R} \to \mathbb{R}$) if and only if $u_0 = 0$.
(ii) Find a smooth function $F : \mathbb{R} \to \mathbb{R}$ such that for some $u_0 \in \mathbb{R}$, the solution to

$$u'(t) = F(u(t)), \quad u(0) = u_0$$

has $T_-, T_+$ both finite.

3. Consider the initial value problem

$$u'(t) = v(t), \quad v'(t) = -4u(t), \quad u(0) = 0, \quad v(0) = 2.$$

(i) Show, by explicit computation, that $u(t) = \sin 2t, v(t) = 2 \cos 2t$ is a solution.
(ii) Consider the Picard’s iteration: $u_0(t) = 0, v_0(t) = 2$ and

$$u_k(t) = \int_0^t v_{k-1}(s) \, ds, \quad v_k(t) = 2 - 4 \int_0^t u_{k-1}(s) \, ds,$$

for $k \geq 1$. Show explicitly that there exists $\varepsilon > 0$ such that for $|t| < \varepsilon$, $u_k(t) \to \sin 2t, v_k(t) \to 2 \cos 2t$ as $k \to \infty$. Hint: Compare $u_k(t)$ with the Taylor’s series of $\sin 2t$ around $t = 0$. You may use any results from 61CM provided they are clearly stated.

4. (i) Let $F_1(x)$ and $F_2(x)$ be two continuously differentiable and uniformly Lipschitz functions on $\mathbb{R}$ such that $F_1(x) < F_2(x)$ for all $x \in \mathbb{R}$. Consider the unique solutions to the ODEs

$$x'_1(t) = F_1(x_1(t)), \quad x'_2(t) = F_2(x_2(t)),$$

with the initial conditions $x_1(0) = a, x_2(0) = b$ and $a < b$. Show that $x_1(t) \leq x_2(t)$ for all $t > 0$.
(ii) Consider the same question as in part (i) but with the weaker assumptions that $F_1(x) \leq F_2(x)$
for all \( x \in \mathbb{R} \) and that \( a \leq b \). Show that we still have \( x_1(t) \leq x_2(t) \) for all \( t > 0 \). Hint: consider solutions to

\[
y'_n(t) = F_2(y_n(t)) + \frac{1}{n}, \quad y_n(0) = b + \frac{1}{n},
\]

show that \( y_n(t) > x_1(t) \) and pass to the limit \( n \to +\infty \). Explain carefully why \( y_n(t) \) converges to \( x_2(t) \) as \( n \to +\infty \), and in which sense this convergence holds.

6. In the statement of the Arzela–Ascoli theorem proven in class, we considered a sequence of functions \( f_n : J \to \mathbb{R}^n \) for some compact interval \( J \subset \mathbb{R} \) and assumed that \( f_n \) satisfy

\[
\forall \epsilon > 0, \exists \delta > 0, \forall n \in \mathbb{N}, \quad |t - s| < \delta, t, s \in J \implies |f_n(t) - f_n(s)| < \epsilon.
\]

Prove that the following (weaker-looking) condition

\[
\forall \epsilon > 0, \forall t \in J, \exists \delta > 0, \forall n \in \mathbb{N}, \quad |t - s| < \delta, s \in J \implies |f_n(t) - f_n(s)| < \epsilon
\]

in fact implies (1).

7. Construct a sequence of continuous functions \( f_n : [0, 1] \to \mathbb{R} \) such that \( f_n \) is uniformly bounded but there is no subsequence of \( f_n \) which converges uniformly to a continuous limit. Show explicitly that the sequence \( f_n \) you constructed is not equicontinuous.

8. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be continuously differentiable. Suppose there is a \( C > 0 \) such that for all \( x \in \mathbb{R}^n \),

\[
\|F(x)\| \leq C\|x\|.
\]

Show that any solution to \( x'(t) = F(x(t)), \quad x(0) = x_0 \) can be extended for all time, i.e. \( J^*_{x_0} = \mathbb{R} \).

9. Consider Newton’s equation \( y''(t) = -\nabla V(y(t)) \) for some smooth \( V : \mathbb{R}^n \to \mathbb{R} \). Suppose \( y(t) \) is a solution with the maximal time interval of existence \((-\infty, T)\), where \( T < \infty \). Show that there exists a sequence \( t_n \to T \) such that \( V(y(t_n)) \to -\infty \).