Homework # 1.
Math 63CM Homework # 2
Due in class on Friday, January 17.

1. Suppose $A, B$ are similar $n \times n$ matrices, that is, there is an $n \times n$ invertible matrix $C$ with $B = C^{-1}AC$. Prove that $A, B$ have the same eigenvalues. Hint: Show that for all $\lambda \in \mathbb{R}$ we have \(\det(B - \lambda I) \equiv \det(A - \lambda I)\).

2. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \rightarrow \mathbb{R}^n$ be a $C^1(U)$ map. Show that if $\det Df(x) \neq 0$ for each $x \in U$, then $f(U)$ is open.

3. Let $U$ be the annular region $\{x \in \mathbb{R}^2 : 1/2 < \|x\| < 1\}$. Prove that $U$ is open and give an example of a $C^1$ function $f : U \rightarrow U$ such that $\det Df(x) > 0$ for every $x \in U$ and such that $f$ is not one-to-one on $U$.

4. Ellipsoids: Let $A$ be an $n \times n$ invertible matrix (not necessarily symmetric) and let $E = \{x \in \mathbb{R}^n : \|Ax\| \leq 1\}$. Prove that $E$ is an ellipsoid: there exists an orthogonal matrix $Q$ and positive real numbers $\lambda_1, \ldots, \lambda_n$ so that $E$ is the set of all $z \in \mathbb{R}^n$ that can be written as $z = Qy$, with $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ such that
\[
\sum_{j=1}^n \lambda_j y_j^2 \leq 1.
\]
Hint: Start by checking that $\|Ax\| \leq 1$ if and only if $(x \cdot (A^T Ax)) \leq 1$, and think about applying the Spectral Theorem to $A^T A$.

5. Let $x(t)$ be the solution to the initial value problem $x' = x, x(0) = 1$.
You can easily check that its solution is $x(t) = \exp t$ but please do not use this fact answering the questions below.
(i) Show that the function $x(t)$ is actually infinitely differentiable and its Taylor series around any point converges to it.
(ii) Show that the solution satisfies $x(t + s) = x(t)x(s), t, s \in \mathbb{R}$. Hint: For a fixed $s \in \mathbb{R}$, show that both sides solve the same ODE with the same initial condition at $t = 0$.
(iii) Calculate explicitly the iteration sequence given by the mapping
\[
Tx(t) = 1 + \int_0^t x(\tau) \, d\tau,
\]
starting with $x_0(t) = 1$. (This is the iteration if you solve $x' = x$ with $x(0) = 1$ and apply the proof of the contraction mapping theorem with $x_0(t) = 1$ a reasonable first guess as it satisfies the initial condition.)

6. (i) Show that the function $g(x) = e^{-\pi x^2}$ satisfies an ODE
\[
g'(x) + 2\pi x g(x) = 0, \quad g(0) = 1.
\]
(ii) Given a real number $k \in \mathbb{R}$ define
\[
u(k) = \int_{-\infty}^{\infty} \cos(2\pi k x)e^{-\pi x^2} \, dx.
\]
Show that $\nu(k)$ satisfies
\[
u'(k) + 2\pi k \nu(k) = 0, \quad \nu(0) = 1.
\]
Be careful with how you justify differentiation of the integral.

(iii) Show that $u(k) = e^{-\pi k^2}$. 