1. Find the solution to
\[
\begin{aligned}
x'(t) &= 1 + (x(t))^2, \\
x(0) &= 0,
\end{aligned}
\]
where $x$ is a real-valued function. In addition, determine its maximal interval of existence.

2. Let $f_n : [0, 1] \to \mathbb{R}$ be an equicontinuous sequence of functions, converging pointwise to a function $f$. Prove that the convergence is in fact uniform.

3. Consider the ODE
\[
\begin{aligned}
x'(t) &= F(x(t)), \\
x(0) &= x_0,
\end{aligned}
\]
where $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and $x_0 \in \mathbb{R}^n$.
Let $H : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $(\nabla H(x) \cdot F(x)) = 0$ for all $x \in \mathbb{R}^n$. Here, $(v \cdot w) = v_1 w_1 + \cdots + v_n w_n$ is the standard inner product in $\mathbb{R}^n$.

(a) Let $x(t)$ be the unique solution to the initial value problem above on a time interval $[-s, s]$, with some $s > 0$. Prove that $H(x(t)) = H(x_0)$ for all $-s \leq t \leq s$.

(b) Suppose that for every $E \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : H(x) = E\} \subset \mathbb{R}^n$ is compact. Prove that the solution exists for all $t \in \mathbb{R}$.

4. Let $F_k : \mathbb{R}^n \to \mathbb{R}^n$ be a sequence of smooth functions such that
\[
\sup_k \sup_{z \in \mathbb{R}^n} \|F_k(z)\| \leq 1.
\]
Suppose $F_k \to F$ uniformly. Let $y_0 \in \mathbb{R}^n$ and $x_k : [0, 1] \to \mathbb{R}^n$ be a sequence of continuous functions such that
\begin{itemize}
  \item $x_k'(t) = F_k(x_k(t))$ for $t \in (0, 1)$,
  \item $x_k(0) = y_0$.
\end{itemize}
Prove that there is a subsequence $\{x_{k_\ell}(t)\}$ and a function $x : [0, 1] \to \mathbb{R}^n$ such that $x_{k_\ell} \to x$ uniformly on $[0, 1]$ as $\ell \to \infty$ and $x(t)$ satisfies
\begin{itemize}
  \item $x'(t) = F(x(t))$ for $t \in (0, 1)$,
  \item $x(0) = y_0$.
\end{itemize}