1Determinants

Definition 1.1 Let \(\alpha_i = (\alpha_{i1}, \cdots, \alpha_{in})\) for \(1 \leq i \leq n\) be \(n\) vectors in \(\mathbb{R}^n\). Their determinant is

\[
\mathcal{D}(\alpha_1, \cdots, \alpha_n) = \sum_{\{i_1, \cdots, i_n\} = \{1, \cdots, n\}} (-1)^{N(i_1, \cdots, i_n)} \alpha_{i_11} \alpha_{i_22} \cdots \alpha_{i_n n},
\]

where \(N(i_1, \cdots, i_n)\) is the number of pairs \(k < \ell\) with \(i_k > i_\ell\), or equivalently the number of transpositions needed to achieve the permutation \((i_1, \cdots, i_n)\).

Proposition 1.2 (Theorem 2.1 in Simon) \(\mathcal{D} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}\) is the unique such function which

1. is multilinear, i.e. linear in each entry;
2. is antisymmetric, i.e. it changes sign if we interchange \(\alpha_k\) and \(\alpha_\ell\) for any \(k \neq \ell\);
3. satisfies \(\mathcal{D}(e_1, \cdots, e_n) = 1\).

Definition 1.3 The determinant of a matrix \(A\) is \(\mathcal{D}\) applied to its columns; in this case we write it \(\det A\).

Properties 1.4 The determinant satisfies the following:

P1 \(\det A\) is a linear function of each column of \(A\);

P2 \(\det A\) changes sign if we interchange two different columns of \(A\). It follows that if \(A\) has two columns which are equal, then \(\det A = 0\);

P3 \(\det I = 1\), where \(I\) is the \(n \times n\) identity matrix;

P4 \(\det AB = \det A \det B\);

P5 \(\det A = \det A^T\), hence \(P1\) and \(P2\) also apply to the rows of \(A\);

P6 We can expand along the \(i^{th}\) row or the \(j^{th}\) column:

\[
\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{and} \quad \det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},
\]

where \(A_{ij}\) is obtained from \(A\) by deleting the \(i^{th}\) row and the \(j^{th}\) column of \(A\).

1. Show that if the rows or the columns of \(A\) are linearly dependent, then \(\det A = 0\).

*All the Halloween stuff in this document is there only because I was excited to use it; it has no mathematical significance.*
2. (Do at home:) Show that adding a multiple of a row to another row of a matrix does not change its determinant. Same for columns. Use this to do exercise 2.1 page 69 of Simon’s book.

3. (Do at home:) Using only the properties listed above and the fact that 195, 247 and 403 are divisible by 13 (so, don’t compute it!), show that the following determinant is divisible by 13:

\[
\begin{vmatrix}
1 & 9 & 5 \\
4 & 0 & 3 \\
2 & 4 & 7
\end{vmatrix}
\]

4. Proof of P5

(a) Let \( \sigma \) be a permutation. If \( \sigma = \tau_1 \cdots \tau_k \), find an expression for \( \sigma^{-1} \). Deduce that the sign of a permutation is equal to the sign of its inverse.

(b) Using the definition 1.1 of determinant, express \( \det A \) and \( \det A^T \) in terms of the components of \( A \).

(c) Reorder the terms of \( \det A^T \) to show that \( \det A = \det A^T \).

2 Scary metric spaces 😒

5. Denote by \( d_E \) the Euclidean distance on \( \mathbb{R}^2 \) (i.e. the usual one). The Parisian metric \( d_P \) on \( \mathbb{R}^2 \) is defined as follows: if \( x \) and \( y \) are points in \( \mathbb{R}^2 \) which are on the same line through the origin, then \( d_P(x, y) = d_E(x, y) \). Otherwise, \( d_P(x, y) = d_E(x, 0) + d_E(0, y) \).

(a) (Do at home:) Prove that \( d_P \) is a metric on \( \mathbb{R}^2 \).

(b) Let \( r > 0 \). For \( p \in \mathbb{R}^2 \), describe the ball \( B_D(p, r) = \{ x \in \mathbb{R}^2 \mid d_P(x, p) < r \} \).

6. Let \( (X, d_X) \) be the metric space \( \mathbb{R} \) equipped with the usual metric. Let \( (Y, d_Y) \) be the metric space \( \mathbb{R} \) equipped with the discrete metric, defined by \( d_Y(x, y) = 1 \) if and only if \( x \neq y \) (and of course \( d_Y(x, x) = 0 \) by the definition of metric). Show that 

\[ f : X \to Y : x \mapsto x \]

is not continuous.

7. Let \( E = C([0, 1], \mathbb{R}) \) be the space of \( \mathbb{R} \)-valued continuous functions on the closed interval \([0, 1]\), and let \( F = C((0, 1], \mathbb{R}) \) be the space of \( \mathbb{R} \)-valued continuous functions on the open interval \((0, 1)\). We equip both \( E \) and \( F \) with the norm \( \|f\|_\infty = \sup_x |f(x)| \).

(a) Show that \( A = \{ f \in E \mid f(x) > 0 \quad \forall x \in [0, 1] \} \) is open in \( E \).

(b) Show that \( B = \{ f \in F \mid f(x) > 0 \quad \forall x \in (0, 1) \} \) is not open in \( F \).

8. Let \( E \) be as in the previous question, but now we equip it with the norm \( \|f\|_1 = \int_0^1 |f(x)| \, dx \). Prove that there exists a Cauchy sequence in \( E \) (for this norm) that does not converge. There is a hint at the bottom\(^1\) of the page.

\(^1\text{Hint: consider an appropriately-chosen sequence of functions } f_n \text{ that is 1 on } [0, 1/2] \text{ and 0 on } [1/2 + 1/n, 1].\)