1 On finding a basis of the column space

Assume that $A$ is the $m \times n$ matrix of a linear transformation $T = \mathbb{R}^n \to \mathbb{R}^m$, where we use the standard bases $e_1, \ldots, e_n$ on $\mathbb{R}^n$ and $f_1, \ldots, f_m$ on $\mathbb{R}^m$. After applying Gauss elimination, we get its reduced row echelon form. Let us assume for notational simplicity that the pivot variables are the first $Q$. So, we have a matrix of the form

$$
\text{rref}A = \begin{pmatrix}
1 & 0 & \cdots & 0 & * & \cdots & * \\
0 & 1 & \cdots & 0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & * & \cdots & * \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

Recall that $\text{Ker}(A) = \text{Ker}(\text{rref}A)$ is automatic from the definition of Gauss elimination. In this question, we will prove that the first $Q$ columns of $A$ form a basis for $C(A)$.

**Remark 1.1** It is clear that the first $Q$ columns of $\text{rref}A$ form a basis for $C(\text{rref}A)$, but it is not true in general that $C(A) = C(\text{rref}A)$.

**Remark 1.2** In general, the pivot variables of $\text{rref}A$ do not have to be the first $Q$; they can be any set of $Q$ variables. However, it is still true that the $Q$ columns of $A$ corresponding to these $Q$ variables form a basis of $C(A)$. This can be seen by temporarily renaming the variables, or with a bit of bookkeeping (as on page 26 of Simon’s book).

1. Prove that the first $Q$ columns of $A$ generate $C(A)$. For this,

   (a) Express the $j^{\text{th}}$ column of $\text{rref}A$ in terms of $\text{rref}A$ and the basis vectors $e_1, \ldots, e_n$. Also, express the $j^{\text{th}}$ column of $A$ in terms of $A$ and the basis vectors $e_1, \ldots, e_n$.

   (b) Express the $k^{\text{th}}$ column $\beta_k$ of $\text{rref}A$ (for $k > Q$) in terms of $f_1, \ldots, f_m$.

   (c) Use the fact that $\text{Ker}(A) = \text{Ker}(\text{rref}A)$ to conclude.

2. Prove that the first $Q$ columns of $A$ are linearly independent, again using $\text{Ker}(A) = \text{Ker}(\text{rref}A)$ and the fact that the first $Q$ columns of $\text{rref}A$ are clearly linearly independent (since they are $f_1, \ldots, f_Q$).

3. Quickly prove the rank-nullity theorem.
2 On inhomogeneous systems

Let $A$ be a $m \times n$ matrix, and $b$ a vector in $\mathbb{R}^m$. Let $x$ be an unknown vector in $\mathbb{R}^n$, and consider the system of equations $Ax = b$. Suppose that we know a solution $x_0$ of $Ax = b$.

4. Prove that $x_0 + v$, where $v \in \text{Ker}(A)$, is always a solution of $Ax = b$.

5. Prove that any solution of $Ax = b$ can be written as $x_0 + v$ for some $v \in \text{Ker}(A)$.

These two points prove that the solution set of $Ax = b$ is $x_0 + \text{Ker}(A) = \{ x_0 + v \mid v \in \text{Ker}(A) \}$ (Simon’s book, section 1.10).

6. Write the solution set of the following system in the form $x_0 + W$ for $W$ a subspace of $\mathbb{R}^4$, and find a basis of $W$.

\[
\begin{align*}
  x_1 + 2x_2 - x_4 &= 1 \\
-2x_1 - 3x_2 + 4x_3 + 5x_4 &= 2 \\
  2x_1 + 4x_2 - 2x_4 &= 2
\end{align*}
\]

3 On iterated limits

Recall that for $f : \mathbb{R}^n \to \mathbb{R}^m$ any function (it does not have to be linear), $a \in \mathbb{R}^n$ and $\ell \in \mathbb{R}^m$, we say that $\lim_{x \to a} f(x) = \ell$ if for every $\varepsilon > 0$, there exists $\delta > 0$ so that for all $x \in B(a, \delta)$ we have $f(x) \in B(\ell, \varepsilon)$.

7. Consider the function

\[
f(x, y) = \begin{cases} 
  \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\
  0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]

(a) Prove that $\lim_{x \to 0} (\lim_{y \to 0} f(x, y)) = 0$, and that $\lim_{y \to 0} (\lim_{x \to 0} f(x, y)) = 0$.

(b) Prove that $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist, by considering $f(x, x)$.

One nice way to compute limits in $\mathbb{R}^2$ is to use the polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$, for $r \geq 0$ and $\theta \in [0, 2\pi)$.

8. Use this trick on the function $f$ above, and see what happens. How does it relate to question 7 above?

9. Use this trick to compute

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2 + x^3}.
\]