1 The implicit function theorem

The implicit function theorem deals with the following question: given a function $G : \mathbb{R}^{m+n} \to \mathbb{R}^m$, when is the set

$$S = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{m+n} \mid G(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0\} \subseteq \mathbb{R}^{m+n}$$

locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}^m$? In other words, given $(a_1, \ldots, a_n, b_1, \ldots, b_m) \in \mathbb{R}^{m+n}$ such that $G(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$, when can we find a small neighbourhood $U$ of $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$, a small neighbourhood $V$ of $(b_1, \ldots, b_m)$ in $\mathbb{R}^m$ and a function $f : U \to \mathbb{R}^m$ such that $S \cap (U \times V) = \Gamma(f) \cap (U \times V)$, where $\Gamma(f)$ is the graph of $f$?

**Theorem 1.1** (Implicit function theorem) If $\det\left(\frac{\partial G_i}{\partial y_j}(a_1, \ldots, a_n, b_1, \ldots, b_m)\right)_{i,j} \neq 0$, then the set $G$ is, around the point $(a_1, \ldots, a_n, b_1, \ldots, b_m)$, locally the graph of a function.

This means that there exists $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $G(x_1, \ldots, x_n, f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) = 0$ for all $(x_1, \ldots, x_n) \in U$. We can then use this last equation to find the derivatives of $f$ itself, just by deriving it.

1. First, let’s do a reality check: verify the theorem above for $S$ a circle of radius 1 centered at $(0,0)$ in $\mathbb{R}^2$.

   In other words,

   (a) determine from a picture where $S$ is locally the graph of a function,
   
   (b) find a function $G : \mathbb{R}^2 \to \mathbb{R}$ that defines $S$ (i.e. such that $S = \{(x, y) \in \mathbb{R}^2 \mid G(x, y) = 0\}$), and use the implicit function theorem to confirm your answer to (a),
   
   (c) find an explicit function $f$ so that $S$ is the graph of $f$ near the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and compute its derivative at that point,
   
   (d) check your answer to (c) by pretending that you don’t know $f$ (but you know it exists by point (b)), and by deriving the equation $G(x, f(x)) = 0$ (in which you know $G$).

2. Check that there exists a real-valued one-variable function $y$ defined implicitly in a neighbourhood of 0 by $e^{xy} - 1 = x^2 + y$, and compute

   $$\lim_{x \to 0} \frac{y(x)}{\cos(x) - 1}.$$ 

3. Consider the curve $y^2 + \sin(xy) = 1$.

   (a) Write the equation of the tangent line to this curve at the point $(0,1),$
   
   (b) Give the coordinates of a point where the tangent to this curve is horizontal.
2 Lagrange multipliers

Here, we will consider the problem of minimizing or maximizing a function \( f : \mathbb{R}^n \to \mathbb{R} \), restricted to some subset \( S \) of \( \mathbb{R}^n \). The first question may look childish, but I was taught this stuff using the exact same story, and I still remember it after nine years. So, clearly, it works.

4. A farmer has a cow. He has to milk the cow, and first he must rinse the bucket in the river. But the farmer is in a hurry: he wants to be done as quickly as possible, because he has a date this afternoon with his girlfriend. So, naturally, he decides to rinse the bucket at the point of the river which is closest to his farm. The farm is located at the point \((0,0) \in \mathbb{R}^2\), and the river is given by a curve \( G(x,y) = 0 \) which does not go through the farm. The function \( f : \mathbb{R}^2 \to \mathbb{R} \) that computes the distance of a point \((x,y)\) to the farm is given, of course, by \( f(x,y) = \sqrt{x^2 + y^2} \).

(a) Draw the farm, the river (you can choose what it looks like), and a couple of curves \( f(x,y) = c \) for different values of \( c \).

(b) Find on your picture the smallest value of \( c \) for which there exists a point \((a,b)\) of the river at distance \( c \) of the farm, and draw the corresponding curve \( f(x,y) = c \).

(c) What can you say about the relative position of the river and the curve \( f(x,y) = c \) at the point \((a,b)\)?

(d) What does it mean in terms of the gradients of the function \( f \) and of the function \( g \)?

(e) Rejoice! You have reinvented the following theorem:

**Theorem 2.1 (Lagrange multipliers)** Let \( U \subseteq \mathbb{R}^n \) be an open set, and let \( f, G : U \to \mathbb{R} \) be two \( C^1 \) functions. Let \( S \) be the zero-set of \( G \), i.e. \( S = \{ x \in U \mid G(x) = 0 \} \). Suppose that there exists \( p \in S \) with \( \nabla G(p) \neq 0 \). Then, if the restriction of \( f \) to \( S \) has a critical point at \( p \), there must exist \( \lambda \in \mathbb{R} \) such that \( \nabla f(p) = \lambda \nabla G(p) \).

What this means is that in order to find a critical point of the restriction of \( f \) to a set \( S = G^{-1}(0) \), it suffices to solve the system

\[
\begin{align*}
\nabla f(x) &= \lambda \nabla G(x) \\
G(x) &= 0.
\end{align*}
\]

5. Find the extrema of the function \( f(x,y,z) = x^2y \) on the unit sphere in \( \mathbb{R}^3 \), and compute the values of \( f \) at these points.

6. What is the volume of the largest box that fits inside the following ellipsoid?

\[
E = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}
\]