Writing proofs for MATH 51H
Section 3: Sequences and limits

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Recall

**Definition.** Let \((a_n)\) be a sequence in \(\mathbb{R}\) and \(L \in \mathbb{R}\). We say that \((a_n)\) has limit \(L\) and write \(\lim_{n \to \infty} a_n = L\) if for all \(\varepsilon > 0\) there exists an \(N \in \mathbb{N}\) such that whenever \(n \geq N\), \(|a_n - L| < \varepsilon\). We say that \((a_n)\) is convergent if \((a_n)\) has a limit. If \((a_n)\) has no limit, then we say that it is divergent.

How do you go about proving that a sequence converges to a certain limit? Picking apart the definition, given an arbitrary \(\varepsilon > 0\), you need to produce an \(N = N(\varepsilon)\) and show that every term \(a_n\) after \(a_N\) is within \(\varepsilon\) of \(L\). So one possible structure of the proof is as follows:

Let \(\varepsilon > 0\). Take \(N = [\text{some value depending on } \varepsilon \text{ which makes the proof work}]\). Then for all \(n \geq N\), we have

verification that \(|a_n - L| < \varepsilon\)

Thus, since \(\varepsilon > 0\) was arbitrary, we conclude that \(\lim_{n \to \infty} a_n = L\).

There is one mysterious part of this template: how do you choose \(N = N(\varepsilon)\) that makes the proof work? One way is to get some scratch paper and work out what your \(N\) should be in terms of \(\varepsilon\) so that whenever \(n \geq N\), \(|a_n - L| < \varepsilon\). Then, once you find your \(N\), you can write your proof in the form above, verifying that for all \(n \geq N\), \(|a_n - L| < \varepsilon\).

Alternatively, after taking an arbitrary \(\varepsilon > 0\), you can say something like “We want to show that \(|a_n - L| < \varepsilon\). This is implied by [another inequality]...So we see that if we take \(N = [\text{something in terms of } \varepsilon]\), then \(|a_n - L| < \varepsilon\.” This sort of thing is fine, as long as your proof makes logical sense and all of the implications go in the right direction. Remember that you are trying to prove that \(|a_n - L| < \varepsilon\), so you can’t say something like “Suppose \(|a_n - L| < \varepsilon^*\)” because you can’t assume what you’re trying to prove. This may be faster than the above strategy on an exam, since when you do the verification that \(|a_n - L| < \varepsilon\) in the first strategy, you’re basically redoing the scratch work you did to find \(N\).

Regardless of whether you find your \(N\) first for a given \(\varepsilon > 0\) before writing or include your derivation of what \(N\) should be in your proof, when you’re proving that a sequence has a certain limit from scratch, you should remember to start with some variation of “Let \(\varepsilon > 0\)” because you’re trying to prove a universal statement: that for all \(\varepsilon > 0\) something is true.
1 Some explicit examples

Proposition. Let \((a_n)\) be the constant sequence \(a_n = 0\) for all \(n \in \mathbb{N}\). Then \(\lim_{n \to \infty} a_n = 0\).

Scratch work: Given an \(\varepsilon > 0\), we want to find an \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(|a_n - 0| < \varepsilon\). So let’s start with \(|a_n - 0|\) and see what conditions on \(n\) we need to make this less than \(\varepsilon\). Well, \(a_n = 0\) for all \(n \in \mathbb{N}\), so that \(|a_n - 0| = |0 - 0| = 0\) for all \(n \in \mathbb{N}\), and certainly \(0 < \varepsilon\). Thus, \(|a_n - 0| < \varepsilon\) holds for all \(n \geq 1\). In this case, we can just take \(N = 1\).

Proof. Let \(\varepsilon > 0\). Take \(N = 1\). Then for all \(n \geq N\), we have

\[|a_n - 0| = |0 - 0| = 0 < \varepsilon.\]

Thus, since \(\varepsilon > 0\) was arbitrary, \(\lim_{n \to \infty} a_n = 0\). □

Here is an alternate proof that incorporates the scratch work.

Proof. Let \(\varepsilon > 0\). We want to find an \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(|a_n - 0| < \varepsilon\). Note that \(|a_n - 0| = |0 - 0| = |0| = 0 < \varepsilon\) for all \(n \in \mathbb{N}\). Thus, taking \(N = 1\), we see that for all \(n \geq N\), \(|a_n - 0| < \varepsilon\). So, since \(\varepsilon > 0\) was arbitrary, \(\lim_{n \to \infty} a_n = 0\). □

The same proof shows that the limit of a constant sequence is constant. This is one of the exercises.

Proposition. Let \((a_n)\) be the sequence \(a_n = \frac{n-1}{n+1}\) for all \(n \in \mathbb{N}\). Then \(\lim_{n \to \infty} a_n = 1\).

Scratch work: Given an \(\varepsilon > 0\), we want to find an \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(|a_n - 1| < \varepsilon\). Again, let’s start with \(|a_n - 1|\) and see what conditions on \(n\) we need to make this less than \(\varepsilon\). We compute:

\[\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n-1}{n+1} - \frac{n+1}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1},\]

and so we see that if \(\frac{2}{n+1} < \varepsilon\) holds, then \(\left|\frac{n-1}{n+1} - 1\right| < \varepsilon\). Now, \(\frac{2}{n+1} < \varepsilon\) is equivalent to \(2/\varepsilon < n + 1\), which is equivalent to \(2/\varepsilon - 1 < n\). So we simply take \(N\) to be any integer larger than \(2/\varepsilon - 1\), which we can do by the Archimedean property of \(\mathbb{R}\).

Proof. Let \(\varepsilon > 0\). Take \(N\) to be an integer greater than \(2/\varepsilon - 1\), which we can do by the Archimedean property of \(\mathbb{R}\). Then for all \(n \geq N\), we have, since \(n \geq N\) implies that \(2/\varepsilon - 1 < N \leq n\), which is equivalent to \(\frac{2}{n+1} < \varepsilon\),

\[|a_n - 1| = \left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n-1}{n+1} - \frac{n+1}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} < \varepsilon.\]

Thus, since \(\varepsilon > 0\) was arbitrary, \(\lim_{n \to \infty} a_n = 1\). □
Here is an alternate proof incorporating the scratch work.

**Proof.** Let \( \varepsilon > 0 \). We want to find an \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( |a_n - L| < \varepsilon \). Note that \( |a_n - 1| = |\frac{n-1}{n+1} - 1| = |\frac{n-1}{n+1} - \frac{n+1}{n+1}| = |\frac{2}{n+1}| = \frac{2}{n+1} \). So if \( \frac{2}{n+1} < \varepsilon \), then \( |a_n - 1| < \varepsilon \). The condition \( \frac{2}{n+1} < \varepsilon \) is equivalent to \( \frac{2}{\varepsilon} < n + 1 \), which is equivalent to \( \frac{2}{\varepsilon} - 1 < n \). If \( N > \frac{2}{\varepsilon} - 1 \), then for \( n \geq N \), certainly \( n > \frac{2}{\varepsilon} - 1 \). Take \( N \) to be any integer larger than \( 2/\varepsilon - 1 \), which we can do by the Archimedean property of \( \mathbb{R} \). Then \( n \geq N \) implies that \( n > \frac{2}{\varepsilon} - 1 \), so that \( |a_n - 1| < \varepsilon \) by the argument above. Thus, since \( \varepsilon > 0 \) was arbitrary, \( \lim_{n \to \infty} a_n = 1 \). \( \square \)

## 2 Properties of limits and sequences

Here we’ll prove all of the properties of limits that you saw in book or in lecture.

**Proposition.** Let \( (a_n) \) be a convergent sequence with \( \lim_{n \to \infty} a_n = L \) and \( c \in \mathbb{R} \). Then \( \lim_{n \to \infty} ca_n = cL \).

**Proof.** First of all, if \( c = 0 \), then \( (ca_n) \) is just the constant sequence 0, and we know that the limit of this sequence is 0. Now suppose that \( c \neq 0 \). Let \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} a_n = L \), there exists an \( N' \in \mathbb{N} \) such that for all \( n \geq N' \), \( |a_n - L| < \frac{\varepsilon}{|c|} \). Take \( N = N' \). Then for all \( n \geq N \), we have that \( |a_n - L| < \varepsilon/|c| \), so

\[
|ca_n - cL| = |c||a_n - L| < |c|\frac{\varepsilon}{|c|} = \varepsilon.
\]

Thus, since \( \varepsilon > 0 \) was arbitrary, \( \lim_{n \to \infty} ca_n = cL \). \( \square \)

**Proposition.** Let \( (a_n) \) and \( (b_n) \) be convergent sequences with limits \( L \) and \( M \), respectively. Then \( \lim_{n \to \infty} (a_n + b_n) = L + M \).

**Proof.** First of all, if \( c = 0 \), then \( (ca_n) \) is just the constant sequence 0, and we know that the limit of this sequence is 0. Now suppose that \( c \neq 0 \). Let \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} a_n = L \), there exists an \( N' \in \mathbb{N} \) such that for all \( n \geq N' \), \( |a_n - L| < \frac{\varepsilon}{|c|} \). Take \( N = N' \). Then for all \( n \geq N \), we have that \( |a_n - L| < \varepsilon/|c| \), so

\[
|ca_n - cL| = |c||a_n - L| < |c|\frac{\varepsilon}{|c|} = \varepsilon.
\]

Thus, since \( \varepsilon > 0 \) was arbitrary, \( \lim_{n \to \infty} ca_n = cL \). \( \square \)
certainly suffices. But we know that since \( \lim_{n \to \infty} a_n = L \), there exists an \( N_1 \in \mathbb{N} \) such that whenever \( n \geq N_1 \), \( |a_n - L| < \varepsilon/2 \). Similarly, we know that since \( \lim_{n \to \infty} b_n = M \), there exists an \( N_2 \in \mathbb{N} \) such that whenever \( n \geq N_2 \), \( |b_n - M| < \varepsilon/2 \). Then if we want both \( n \geq N_1 \) and \( n \geq N_2 \) to be true, then it suffices to have \( n \geq \max\{N_1, N_2\} \), the bigger of the two \( N_i \)’s. This is because if \( n \geq N = \max\{N_1, N_2\} \), then \( n \geq N_1 \) and \( n \geq N_2 \), so that both \( |a_n - L| < \varepsilon/2 \) and \( |b_n - M| < \varepsilon/2 \). So let’s take \( N \) to be \( \max\{N_1, N_2\} \).

**Proof.** Let \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} a_n = L \), there exists an \( N_1 \in \mathbb{N} \) such that whenever \( n \geq N_1 \), \( |a_n - L| < \varepsilon/2 \). Similarly, since \( \lim_{n \to \infty} b_n = M \), there exists an \( N_2 \in \mathbb{N} \) such that whenever \( n \geq N_2 \), \( |b_n - M| < \varepsilon/2 \). Take \( N = \max\{N_1, N_2\} \). Then for all \( n \geq N \), we have that

\[
(a_n + b_n) - (L + M) = |(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus, since \( \varepsilon > 0 \) was arbitrary, \( \lim_{n \to \infty} (a_n + b_n) = L + M \).

Recall

**Definition.** A sequence \((a_n)\) is said to be bounded above if there exists a \( B \in \mathbb{R} \) such that for all \( n \in \mathbb{N} \), \( a_n \leq B \). A sequence \((a_n)\) is said to be bounded below if there exists a \( b \in \mathbb{R} \) such that for all \( n \in \mathbb{N} \), \( a_n \geq b \). A sequence which is both bounded above and bounded below is said to be bounded. Equivalently, \((a_n)\) is bounded if there exists a \( B \in \mathbb{R} \) such that for all \( n \in \mathbb{N} \), \( |a_n| \leq B \).

This next example is a different flavor than the previous ones, since we’re not proving that a sequence has a certain limit. The idea is that if a sequence is convergent, then after a finite number of terms, all terms are within say 1 of the limit. So this tail is bounded, and to get a bound on every term of the sequence, we just need to combine the bound for the tail with a bound for the finitely many terms that lie outside of the tail.

**Proposition.** Let \((a_n)\) be a convergent sequence. Then \((a_n)\) is bounded.

**Proof.** Since \((a_n)\) is convergent, it has some limit, say \( L \). Then by the definition of limit, there exists an \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( |a_n - L| < 1 \) (here we took \( \varepsilon = 1 \) in the definition of limit.) So whenever \( n \geq N \), \( |a_n| = |a_n - L + L| \leq |a_n - L| + |L| < |L| + 1 \). Now take \( B = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|\} \). Then for all \( n \in \mathbb{N} \), \( |a_n| \leq B \). This is because either \( n \leq N - 1 \), in which case the absolute value of \( |a_n| \) is included in our maximum, or \( n \geq N \), in which case \( |a_n| < 1 + |L| \) by the computation above. Hence, \((a_n)\) is bounded.  

One useful trick that shows up a lot in proving things about limits is to add 0 in a clever way. We did this in the proof above when we wrote \( |a_n| \) as \( |a_n + 0| = |a_n + (-L + L)| = |a_n - L + L| \). If you know that you can make \( |a_n - L| \) as small as you want and you’re trying to prove something about limits, you typically want to get \( |a_n - L| \) to appear in your manipulations somewhere so that you can use this fact. Adding 0 cleverly, as we did above, is a good way to get these sort of terms to appear.
Proposition. Let \((a_n)\) and \((b_n)\) be convergent sequences with limits \(L\) and \(M\), respectively. Then \(\lim_{n \to \infty} a_n b_n = LM\).

Scratch work: Given an \(\varepsilon > 0\), we want to find an \(N \in \mathbb{N}\) such that for all \(n \geq N\), 
\[|a_n b_n - LM| < \varepsilon.\] Again, let’s start with \(|a_n b_n - LM|\) and see what conditions on \(n\) we need to make this less than \(\varepsilon\). We compute, adding 0 = \((-Lb_n + Lb_n)\),
\[
|a_n b_n - LM| = |a_n b_n - Lb_n + Lb_n - LM| \\
\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\
= |a_n - L||b_n| + |L||b_n - M|.
\]
So we see that if \(|a_n - L||b_n| + |L||b_n - M| < \varepsilon\), then \(|a_n b_n - LM| < \varepsilon\). By the previous proposition, \((b_n)\) is bounded because it is convergent. Thus, there exists a \(B \in \mathbb{R}\) such that for all \(n \in \mathbb{N}\), \(|b_n| \leq B\). Note that \(B \geq 0\), since \(|b_n| \geq 0\). So it suffices to show that \(B|a_n - L| + |L||b_n - M| < \varepsilon\). What we would like to do here is to take \(N\) large enough so that \(n \geq N\) implies that \(|a_n - L| < \frac{\varepsilon/2}{B+1}\) and \(|b_n - M| < \frac{\varepsilon/2}{|L|+1}\), so then we’d have \(B|a_n - L| + |L||b_n - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon\). But there is a problem with this strategy: \(B\) and/or \(|L|\) could be equal to 0, and we can’t divide by 0. To get around this, you could break the argument up into cases depending on whether \(B = 0\) or \(|L| = 0\) or not. There is an easier way, though. Note that \(0 \leq B < (B+1)\) and \(0 \leq |L| < (|L|+1)\), and hence
\[
B|a_n - L| + |L||b_n - M| \leq (B+1)|a_n - L| + (|L|+1)|b_n - M|.
\]
Thus, it suffices to show that \((B+1)|a_n - L| + (|L|+1)|b_n - M| < \varepsilon\). Here we can divide by \((B+1)\) and \((|L|+1)\) since they are both positive. So to get that \(|a_n b_n - LM| < \varepsilon\), it suffices to have \((B+1)|a_n - L|, (|L|+1)|b_n - M| < \varepsilon/2\). Since \(\lim_{n \to \infty} a_n = L\), there exists an \(N_1 \in \mathbb{N}\) such that for all \(n \geq N_1\), \(|a_n - L| < \frac{\varepsilon/2}{B+1}\). Similarly, there exists an \(N_2 \in \mathbb{N}\) such that for all \(n \geq N_2\), \(|b_n - M| < \frac{\varepsilon/2}{|L|+1}\). So let’s take \(N = \max\{N_1, N_2\}\).

Proof. Let \(\varepsilon > 0\). Since a convergent sequence is also bounded, there exists a \(B \in \mathbb{R}\) such that for all \(n \in \mathbb{N}\), \(|b_n| \leq B\). Also, since \(\lim_{n \to \infty} a_n = L\), there exists an \(N_1 \in \mathbb{N}\) such that for all \(n \geq N_1\), \(|a_n - L| < \frac{\varepsilon/2}{B+1}\). Similarly, there exists an \(N_2 \in \mathbb{N}\) such that for all \(n \geq N_2\), \(|b_n - M| < \frac{\varepsilon/2}{|L|+1}\). Take \(N = \max\{N_1, N_2\}\). Then for \(n \geq N\), we have
\[
|a_n b_n - LM| = |a_n b_n - Lb_n + Lb_n - LM| \\
\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\
= |a_n - L||b_n| + |L||b_n - M| \\
\leq (B+1)|a_n - L| + (|L|+1)|b_n - M| \\
< (B+1)\frac{\varepsilon/2}{B+1} + (|L|+1)\frac{\varepsilon/2}{|L|+1} \\
= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon.
\]
Thus, since $\varepsilon > 0$ was arbitrary, $\lim_{n \to \infty} a_n b_n = LM$. \hfill \Box

**Proposition.** Let $(a_n)$ be a convergent sequence with limit $L > 0$. Then for all $n$ sufficiently large, $a_n > \frac{L}{2}$.

**Proof.** Since $\lim_{n \to \infty} a_n = L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \frac{L}{2}$. Then if $n \geq N$, we have for $a_n \geq L$ that $a_n \geq L > L/2$, and if $a_n < L$, then $|a_n - L| = L - a_n$, so that $L - a_n = |a_n - L| < \frac{L}{2}$, i.e. $a_n > L/2$. In either case, $a_n > L/2$. Hence, for all $n$ sufficiently large (for example, all $n \geq N$), $a_n > L/2$. \hfill \Box

Another way to approach the above proof is to use the “reverse triangle inequality”, which is that for all $x, y \in \mathbb{R}$, $||x| - |y|| \leq |x - y|$. This can be derived from the usual triangle inequality, and is part of one of the exercises.

**Proposition.** Let $(a_n)$ be a convergent sequence with limit $L$ where $a_n \neq 0$ for all $n \in \mathbb{N}$ and $L > 0$. Then $\lim_{n \to \infty} \frac{1}{a_n} = \frac{1}{L}$.

**Scratch work:** Given an $\varepsilon > 0$, we want to find an $N \in \mathbb{N}$ such that for all $n \geq N$, $|1/a_n - 1/L| < \varepsilon$. Yet again, let’s start with $|1/a_n - 1/L|$ and see what conditions on $n$ we need to make this less than $\varepsilon$. We compute:

$$|1/a_n - 1/L| = \left| \frac{L - a_n}{a_n L} \right| = \frac{|a_n - L|}{|a_n L|}.$$

So we see that if $\frac{|a_n - L|}{|a_n L|} < \varepsilon$, then $|1/a_n - 1/L| < \varepsilon$. The condition $\frac{|a_n - L|}{|a_n L|} < \varepsilon$ is equivalent to $|a_n - L| < |a_n L|\varepsilon$, and by the previous proposition, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n| > L/2$. This implies that for $n \geq N_1$, $|a_n L|\varepsilon > L/2 \cdot L = L^2/2\varepsilon$. Thus, if $|a_n - L| < \varepsilon L^2/2$, then we have $|1/a_n - 1/L| < \varepsilon$ whenever $n \geq N_1$. Now, because $\lim_{n \to \infty} a_n = L$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - L| < \varepsilon L^2/2$. So let’s take $N = \max\{N_1, N_2\}$, for then $|a_n - L| < \frac{\varepsilon L^2}{2}$ and $|a_n - L| < \varepsilon L^2/2 \Rightarrow |1/a_n - 1/L| < \varepsilon$.

**Proof.** Let $\varepsilon > 0$. By the previous proposition, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n| > L/2$. Also, since $\lim_{n \to \infty} a_n = L$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - L| < \varepsilon L^2/2$. Take $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, since $|a_n| > L/2$ implies that $1/|a_n| < 2/L$, we have

$$|1/a_n - 1/L| = \left| \frac{L - a_n}{a_n L} \right| = \frac{|a_n - L|}{|a_n L|} < \frac{\varepsilon L^2}{2} \cdot \frac{2}{L^2} = \varepsilon.$$

Thus, since $\varepsilon > 0$ was arbitrary, $\lim_{n \to \infty} (1/a_n) = 1/L$. \hfill \Box

The following proposition shows that if $\lim_{n \to \infty} a_n = L$, then we can’t also have $\lim_{n \to \infty} a_n = M$ for any $M$ not equal to $L$.

**Proposition.** Let $(a_n)$ be a convergent sequence. Then the limit of $(a_n)$ is unique.
Proof. Suppose by way of contradiction that \( \lim_{n \to \infty} a_n = L_1 \) and \( \lim_{n \to \infty} a_n = L_2 \) where \( L_1 \neq L_2 \). Since \( L_1 \neq L_2 \), \( d = |L_1 - L_2| > 0 \). Since \( \lim_{n \to \infty} a_n = L_1 \), there exists an \( N_1 \in \mathbb{N} \) such that for all \( n \geq N_1 \), \( |a_n - L_1| < d/2 \). Similarly, since \( \lim_{n \to \infty} a_n = L_2 \), there exists an \( N_2 \in \mathbb{N} \) such that for all \( n \geq N_2 \), \( |a_n - L_2| < d/2 \). Then by the triangle inequality, we have \( |L_1 - L_2| = |(L_1 - a_N) + (a_N - L_2)| \leq |L_1 - a_N| + |L_2 - a_2| < \frac{d}{2} + \frac{d}{2} = d \). So \( |L_1 - L_2| < d \). However, by definition, \( d = |L_1 - L_2| \), which means that \( d < d \), a contradiction. Thus, the limit of \((a_n)\) is unique. \( \square \)

3 Exercises

Exercise 1. Consider the sequence \((a_n)\) where \( a_n = 1/n^2 \) for all \( n \in \mathbb{N} \). Prove that \( \lim_{n \to \infty} a_n = 0 \).

Exercise 2. Fix \( c \in \mathbb{R} \) and let \((a_n)\) be the constant sequence where \( a_n = c \) for all \( n \in \mathbb{N} \). Prove that \( \lim_{n \to \infty} a_n = c \).

Exercise 3. Consider the sequence \((a_n)\) where \( a_n = \frac{9n^2+2}{n^2+2n+1} \) for all \( n \in \mathbb{N} \). Prove that \( \lim_{n \to \infty} a_n = 9 \).

Exercise 4. Show that for any sequence \((a_n)\), \( \lim_{n \to \infty} a_n = 0 \) if and only if \( \lim_{n \to \infty} |a_n| = 0 \).

Exercise 5. Show that for any sequence \((a_n)\), \( \lim_{n \to \infty} a_n = 0 \) if and only if \( \lim_{n \to \infty} a_n^2 = 0 \).

Exercise 6.

1. Prove the “reverse triangle inequality”: for every \( x, y \in \mathbb{R} \), \( ||x| - |y|| \leq |x - y| \).

2. Show that if \( \lim_{n \to \infty} a_n = L \), then \( \lim_{n \to \infty} |a_n| = |L| \).

Exercise 7. Suppose that \((a_n)\) is such that \( a_n > 0 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} a_n = L \) for \( L > 0 \). Show that \( \lim_{n \to \infty} \sqrt{a_n} = \sqrt{L} \).

Exercise 8. Let \((a_n)\) and \((b_n)\) be two convergent sequences such that \( a_n \leq b_n \) for all \( n \in \mathbb{N} \). Show that \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \).

Exercise 9. A sequence \((a_n)\) in \( \mathbb{R} \) is Cauchy if for all \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that whenever \( n, m \geq N \), \( |a_n - a_m| < \varepsilon \).

1. Prove that a convergent sequence is Cauchy.

2. Prove that a Cauchy sequence is bounded.

Exercise 10. Let \((a_n)\) be a sequence in \( \mathbb{R} \). Recall that a sequence \((a_n)\) is a subsequence of \((a_n)\) if all of the \( a_n \)'s are positive integers and \( n_1 < n_2 < n_3 < \ldots \). For example, \( 1, 1/3, 1/5, 1/7, \ldots \) is a subsequence of \( 1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, \ldots \).

1. Suppose that \( \lim_{n \to \infty} a_n = L \). Prove that if \((a_n)\) is any subsequence of \((a_n)\), then \( \lim_{n \to \infty} a_n = L \).

2. Suppose that \((a_n)\) is a Cauchy sequence. Prove that if \((a_n)\) is a subsequence of \((a_n)\) such that \( \lim_{n \to \infty} a_n = L \), then in fact \( \lim_{n \to \infty} a_n = L \) as well. That is, if a subsequence of a Cauchy sequence converges, then the original sequence converges as well (to the same limit.)

3. Prove that any Cauchy sequence in \( \mathbb{R} \) is convergent. (Hint: Use Bolzano-Weierstrass)