The real numbers

In this section, we will introduce the set of real numbers $\mathbb{R}$ via a collection of several groups of axioms: we will postulate that the real numbers (1) form what is known as a field, (2) they are ordered, and (3) satisfy the completeness axiom. The latter may be less intuitive at the first sight but, as we will soon see, is absolutely crucial for the subject known as ”mathematical analysis”. The intuition behind the axioms is that they should formalize what we are know as ”real numbers” from ”past experience”.

1.1 Axioms of addition and multiplication

First, we need to take care of the arithmetic of real numbers. We will assume that the operations of addition and multiplication are defined for real numbers, and satisfy the following properties:

(A1) The addition operation is associative:

$$(a + b) + c = a + (b + c) \text{ for all } a, b, c \in \mathbb{R},$$

(A2) There exists a zero element denoted by 0, so that

$$x + 0 = 0 + x = x \text{ for all } x \in \mathbb{R}.$$  

(A3) For each $x \in \mathbb{R}$ there exists an element $(−x) \in \mathbb{R}$ such that

$$x + (−x) = 0.$$ 

(A4) The addition operation is commutative:

$$a + b = b + a \text{ for all } a, b \in \mathbb{R}.$$ 

Axioms (A1)-(A3) say that $\mathbb{R}$ is a group with respect to addition, and (A4) says that this group is commutative. We will go over the general notion of a group again in the linear algebra portion of the class.
The next group of axioms concerns multiplication. We will sometimes write the product of two numbers \( a, b \in \mathbb{R} \) as \( ab \) and sometimes as \( a \cdot b \), hoping this causes no confusion.

(M1) The multiplication operation is associative:

\[
(ab)c = a(bc) \quad \text{for all} \quad a, b, c \in \mathbb{R},
\]

(M2) There exists an identity element denoted by 1, so that \( 1 \neq 0 \), and

\[
x \cdot 1 = 1 \cdot x = x \quad \text{for all} \quad x \in \mathbb{R}.
\]

(M3) For each \( x \in \mathbb{R} \) such that \( x \neq 0 \), there exists an element \( x^{-1} \in \mathbb{R} \) such that

\[
x \cdot x^{-1} = 1.
\]

(M4) The multiplication operation is commutative:

\[
ab = ba \quad \text{for all} \quad a, b \in \mathbb{R}.
\]

These axioms mean that the set \( \mathbb{R} \setminus \{0\} \) is a commutative group with respect to multiplication. The last algebraic axiom, when taken together with the previous axioms, says that \( \mathbb{R} \) is a field, and is simply the distributive law with respect to addition and multiplication:

\[
(F1) \quad a(b + c) = ab + ac \quad \text{for all} \quad a, b, c \in \mathbb{R}.
\]

The familiar arithmetic properties of real numbers follow from these axioms relatively easily. For example, to show that there is only one zero in the set of real numbers, let us assume that \( 0_1 \) and \( 0_2 \) are both zeros and write:

\[
0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2.
\]

Make sure that you understand which axioms were used in each identity above! To show that each element \( x \in \mathbb{R} \) has a unique negative element, let us assume that for some \( x \in \mathbb{R} \) there exist \( x_1 \) and \( x_2 \) such that

\[
0 = x + x_1, \quad 0 = x + x_2.
\]

Then we can write:

\[
x_1 = x_1 + 0 = x_1 + (x + x_2) = (x_1 + x) + x_2 = 0 + x_2 = x_2.
\]

Again, make sure that you understand which axioms were used in each identity above. The next exercise lists some of the other elementary arithmetic properties of real numbers.

**Exercise 1.1** Show using the above axioms that the following algebraic properties hold:

1. Given any \( a, b \in \mathbb{R} \), equation

\[
a + x = b
\]

has a unique solution \( x \in \mathbb{R} \).

2. For each \( x \in \mathbb{R} \) there exists a unique \( y \) such that \( xy = 1 \).

3. For each \( a \neq 0 \) and each \( b \in \mathbb{R} \) there exists a unique solution to the equation \( a \cdot x = b \).

4. For any \( x \in \mathbb{R} \) we have \( x \cdot 0 = 0 \).

5. If \( xy = 0 \) then either \( x = 0 \) or \( y = 0 \).

6. For any \( x \in \mathbb{R} \) we have \( -x = (-1) \cdot x \).

7. For any \( x \in \mathbb{R} \) we have \( (-1) \cdot (-x) = x \).

8. For any \( x \in \mathbb{R} \) we have \( (-x) \cdot (-x) = x \cdot x \).
1.2 The order axioms

The next set of axioms has little to do with arithmetic but rather ordering: it says that the real numbers form an ordered set. That is, there is a relation $\leq$ between the real numbers such that

(O1) For each $x \in \mathbb{R}$ we have $x \leq x$.

(O2) If $x \leq y$ and $y \leq x$ then $x = y$.

(O3) If $x \leq y$ and $y \leq z$ then $x \leq z$.

These axioms mean that $\mathbb{R}$ is a (partially) ordered set. We postulate it is totally ordered:

(O4) For each $x, y \in \mathbb{R}$ either $x \leq y$ or $y \leq x$.

To connect the axioms of addition, multiplication to the order axioms, we add the following two axioms.

(OA) If $x \leq y$ and $z \in \mathbb{R}$ then $x + z \leq y + z$.

(OM) If $0 \leq x$ and $0 \leq y$ then $0 \leq xy$.

If $x \leq y$ and $x \neq y$ then we use the notation $x < y$. The relations $\geq$ and $>$ are defined in the obvious way.

Note that we have $0 < 1$. Indeed, the multiplication axioms imply that $0 \neq 1$. If we have $1 < 0$, then adding $(-1)$ to both sides gives $0 < -1$, and (OM) implies that $0 < (-1)(-1) = 1$, which is a contradiction to the assumption $1 < 0$. We used claim (8) of Exercise 1.1 in the last step.

The next exercise shows that ordering is actually a non-trivial structure – not every field can be ordered.

Exercise 1.2 Show that not every set satisfying the addition and multiplication axioms can be ordered. Hint: consider the set $\{0, 1\}$ with $0 + 0 = 0$, $0 + 1 = 1$, $1 + 1 = 0$ and $0 \cdot 0 = 1 \cdot 0 = 0 = 1$, and $1 \cdot 1 = 1$. Show that it satisfies the addition and multiplication axioms but can not be ordered.

1.3 The completeness axiom

As you can readily check, the axioms of addition, multiplication and order do not yet narrow down the intuition we have for the set of real numbers that should, at the very least, include what we know as rational numbers and irrational numbers. Indeed, these axioms are satisfied by the set of rational numbers alone and hence are not yet sufficient. As a side note, strictly speaking, to define the rational numbers we need to know what integers are but let us disregard this for the moment. What is clear is that we need another axiom that would not be satisfied by what we think of as "rational numbers". This is achieved by the completeness axiom.

The completeness axiom. If $X$ and $Y$ are non-empty sets of real numbers such that for any $x \in X$ and $y \in Y$ we have $x \leq y$, then there exists $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.

In a sense, this axiom says that $\mathbb{R}$ has no holes. On other hand, the set $\mathbb{Q}$ of rational numbers does have holes: if we take

$$X = \{r \in \mathbb{Q} : r^2 < 2\}, \quad Y = \{r \in \mathbb{Q} : r^2 > 2 \text{ and } r > 0\},$$

then there is no $c \in \mathbb{Q}$ such that for any $x \in X$ and $y \in Y$ we have $x \leq c \leq y$ – this is because $\sqrt{2}$ is irrational. No rational number squared can equal to 2 and if a rational $c$ separating the sets $X$ and $Y$ were to exist, it is not hard to see that it would have to satisfy $c^2 = 2$ – we will come back to this a little later. Thus, the completeness axiom really distinguishes the reals from the set of rational numbers. To see how this axiom can be used, let us give some definitions.

Definition 1.3 (1) A set $S \subseteq \mathbb{R}$ is bounded from above if there exists a number $K \in \mathbb{R}$ such that any $x \in S$ satisfies $x \leq K$. Such $K$ is called an upper bound for $S$.

(2) A set $S \subseteq \mathbb{R}$ is bounded from below if there exists a number $K \in \mathbb{R}$ such that any $x \in S$
satisfies $K \leq x$. Such $K$ is called a lower bound for $S$.

(3) A set $S \subseteq \mathbb{R}$ is bounded if it is bounded both from above and from below.

**Exercise 1.4** Let us define $|x|$ as $|x| = x$ if $0 \leq x$ and $|x| = -x$ if $0 \leq -x$. Show that a set $S$ is bounded if and only if there exists $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in S$.

**Definition 1.5** We say that $x \in S$ is the maximum of a set $S$ if for any $y \in S$ we have $y \leq x$. Similarly, we say that $x \in S$ is the minimum of a set $S$ if for any $y \in S$ we have $x \leq y$.

Note that the maximum and minimum of a set belong to the set, by definition. However, not every set has a maximum or a minimum.

**Exercise 1.6** Show that the set $S = \{x : 0 < x < 1\}$ has neither a minimum nor a maximum.

**Definition 1.7** (1) A number $K$ is the least upper bound for a set $S \subseteq \mathbb{R}$, denoted as $K = \sup S$, if $K$ is an upper bound for $S$ and any upper bound $M$ for $S$ satisfies $K \leq M$.

(2) A number $K$ is the greatest lower bound for a set $S \subseteq \mathbb{R}$, denoted as $K = \inf S$, if $K$ is a lower bound for $S$ and any lower bound $M$ for $S$ satisfies $M \leq K$.

In other words, $\sup S$ is the minimum of the set of the upper bounds for $S$, and $\inf S$ is the maximum of the set of the lower bounds for $S$. As we have seen, not every set has a maximum or a minimum. The completeness axiom for the real numbers has the following consequence showing that the set of upper bounds for any set has a minimum.

**Lemma 1.8** (The least upper bound principle). Every non-empty set of real numbers that is bounded from above has a unique least upper bound.

**Proof.** Uniqueness of the least upper bound follows from the fact that (if it exists) it is the minimum of the set of all upper bounds, and every set can have have at most one minimum. Thus, we only need to show that a least upper bound exists. Let $Y$ be the set of all upper bounds for a non-empty set $X$. Since $X$ is bounded, the set $Y$ is not empty, and by the definition of an upper bound, we know that for any $x \in X$ and $y \in Y$ we have $x \leq y$. The completeness axiom then implies that there exists $c \in \mathbb{R}$ such that for any $x \in X$ and $y \in Y$ we have $x \leq c \leq y$. The first inequality implies that $c$ is an upper bound for $X$ and the second implies that it is the least upper bound. □

**Exercise 1.9** Show that the conclusion of Lemma 1.8 does not hold for the set $\mathbb{Q}$ of rational numbers. That is, exhibit a set $S$ of rational numbers such that there is no smallest rational upper bound for $S$.

One may wonder if one can construct a version of real numbers explicitly and verify that it satisfies the axioms. A standard way is to start with the rational numbers and use what is known as the Dedekind cuts. We will not do this here for lack of time, and simply note that an interested reader should have no difficulty finding it in a variety of books or on the Internet.

### 1.4 The natural numbers and the principle of mathematical induction

We understand commonly that the natural numbers are $1, 1+1, 1+1+1, \ldots$. In order to make this more formal, we give the following definition.

**Definition 1.10** A set $X \subseteq \mathbb{R}$ is inductive if for each $x \in X$, the set $X$ also contains $x + 1$.

**Definition 1.11** The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets that contain 1.
Exercise 1.12 Show that $\mathbb{N}$ is an inductive set.

**Lemma 1.13** The sum of two natural numbers is a natural number.

**Proof.** Let $S$ be the set of all natural numbers $n$ such that for any $m \in \mathbb{N}$ we have $n + m \in \mathbb{N}$. Then $S$ is not empty because $1 \in S$ (this, in turn, is because $\mathbb{N}$ is an inductive set). Moreover, if $n \in S$ then $n + 1$ is also in $S$ because for any $m \in \mathbb{N}$ we have

$$(n + 1) + m = n + (m + 1) \in \mathbb{N},$$

because $n \in S$ and $m + 1 \in \mathbb{N}$. Thus, $S$ is an inductive set, and, as it is a subset of $\mathbb{N}$, it has to be equal to $\mathbb{N}$. □

Exercise 1.14 Show that the product of two natural numbers is a natural number.

Exercise 1.15 Show that if $y \in \mathbb{N}$ and $y \neq 1$ then $y - 1 \in \mathbb{N}$. Hint: let $E$ be the set of all real numbers of the form $n - 1$ where $n \in \mathbb{N}$ and $n > 1$. Show that $E = \mathbb{N}$.

**Lemma 1.16** If $n \in \mathbb{N}$ then there are no natural numbers $x$ such that $n < x < n + 1$.

**Proof.** We will show that the set $S$ of all natural numbers for which this assertion holds contains 1 and is an inductive set. To show that 1 $\in S$, let $M = \{x \in \mathbb{N} : x = 1 \text{ or } x \geq 2\}$. Then, obviously 1 $\in M$ and if $x \in M$ then either $x = 1$ so that $x + 1 = 2 \in M$ or $x \geq 2$ and then $x + 1 \geq 2$, thus $x + 1 \in M$ also in that case. Thus, $M$ is a subset of $\mathbb{N}$ that is an inductive set containing 1, hence $M = \mathbb{N}$. This means that the assertion of the lemma holds for 1 and 1 $\in S$.

Next, assume that $x \in S$ – we will show that $x + 1 \in S$. Assume that there is $y \in \mathbb{N}$ such that $n + 1 < y < n + 2$. Then $y > 1$, thus $y - 1$ is in $\mathbb{N}$ and $n < y - 1 < n + 1$, meaning that $n \notin S$, which is a contradiction. □

The set $\mathbb{Z}$ of integers consists of numbers $x \in \mathbb{R}$ such that either $x = 0$, or $x \in \mathbb{N}$, or $-x \in \mathbb{N}$. Now, we can define the rational numbers as usual: they have the form $mn^{-1}$ with $m, n \in \mathbb{Z}$, $n \neq 0$. The irrational numbers are those that are not rational. In order to show that irrational numbers exist, we make the following observation.

**Exercise 1.17** (1) Show that if $s > 0$ and $s^2 < 2$ then there exists $\delta > 0$ so that $(s + \delta)^2 < 2$. Hint: you should be able find an explicit $\delta > 0$ in terms of $s$.

(2) Show that if $s > 0$ and $s^2 > 2$ then there exists $\delta > 0$ so that $(s - \delta)^2 < 2$. Hint: again, pick an explicit $\delta > 0$ in terms of $s$.

(3) Let $X = \{x > 0 : x^2 < 2\}$. Show that $X$ is bounded from above and set $\bar{s} = \sup S$. Show that $\bar{s}^2 = 2$ and that $\bar{s}$ is irrational.

### 1.5 The Archimedes principle

The Archimedes principle is a key to the approximation theory, and says the following.

**Lemma 1.18** (The Archimedes principle) For any fixed $h > 0$ and any $x \in \mathbb{R}$ there exists an integer $k \in \mathbb{Z}$ so that

$$(k - 1)h \leq x < kh.$$
Proof. We first note that \( \mathbb{N} \) is not bounded from above. Indeed, if it is bounded, set \( s = \sup \mathbb{N} \). If \( s \not\in \mathbb{N} \), then, by the definition of supremum, there must be \( n \in \mathbb{N} \) such that \( n > s - 1 \), but then \( s < n + 1 \) and thus \( s \) is not an upper bound for \( \mathbb{N} \). However, if \( s \in \mathbb{N} \) then \( s + 1 \in \mathbb{N} \) and \( s + 1 > s \) meaning that \( s \) is again not an upper bound for \( \mathbb{N} \). This shows that \( \mathbb{N} \) is not bounded from above, and thus \( \mathbb{Z} \) is not bounded from above or from below. A similar argument shows that if \( S \) is a set of integers bounded from above, it must contain a maximal element, and if it is a set of integers bounded from below, it must contain a minimal element. Now, turning to the proof of the Archimedes principle, let \( S = \{ n \in \mathbb{Z} : \frac{x}{h} < n \} \). This set is bounded from below and thus contains a minimal element \( k \). Then we have

\[
 k - 1 \leq \frac{x}{h} < k,
\]

and the conclusion of Lemma 1.18 follows. \( \square \)

Here are some important corollaries of the Archimedes principle.

Corollary 1.19  
(i) For any \( \varepsilon > 0 \) there exists \( n \in \mathbb{N} \) such that \( 0 < 1/n < \varepsilon \).  
(ii) For any \( a, b \in \mathbb{R} \) such that \( a < b \) there exists \( r \in \mathbb{Q} \) such that \( a < r < b \).

Exercise 1.20 Prove the assertions of Corollary 1.19.

2 The nested intervals lemma and the Heine-Borel theorem

In this section, we will introduce the first ideas related to the notion of compactness.

2.1 The nested interval lemma

Definition 2.1 An infinite sequence of sets \( X_1, X_2, \ldots, X_n, \ldots \) is nested if for each \( n \in \mathbb{N} \) we have \( X_{n+1} \subseteq X_n \).

A closed interval \([a, b]\) is simply the set \([a, b] = \{x : a \leq x \leq b\}\), and an open interval is the set \((a, b) = \{x : a < x < b\}\).

Lemma 2.2 (The nested intervals lemma). Let \( I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots \) be a nested sequence of closed intervals. Then there exists \( c \in \mathbb{R} \) that belongs to all \( I_k, k \in \mathbb{N} \). If, in addition, for any \( \varepsilon > 0 \) there exists \( k \) such that \( |I_k| < \varepsilon \), then there is exactly one point \( c \) common to all intervals.

Proof. Since \( I_k \) is a nested sequence of intervals, we have \( a_k \leq a_m \leq b_m \leq b_k \) for all \( m \geq k \). Thus, the sets \( A = \{a_n : n \in \mathbb{N}\} \) and \( B = \{b_n : n \in \mathbb{N}\} \) satisfy the assumptions of the completeness axiom. Hence, there exists \( c \in \mathbb{R} \) such that \( a_m \leq c \leq b_k \) for all \( m, k \in \mathbb{N} \), and, in particular, we have \( a_n \leq c \leq b_n \) for all \( n \in \mathbb{N} \), thus \( c \in I_n \) for all \( n \in \mathbb{N} \).

If there are two points \( c_1 \neq c_2 \) that belong to all \( I_k \), say, with \( c_1 < c_2 \), then we have

\[
 a_n \leq c_1 < c_2 \leq b_n \quad \text{for all } n,
\]

hence \( |b_n - a_n| > c_2 - c_1 \), and the length of all intervals \( I_n \) is larger than \( c_2 - c_1 \). \( \square \)

Exercise 2.3 Where did we use the fact that the intervals \( I_n \) are closed in the above proof? Give an example of a nested sequence of open intervals \( I_n = (a_n, b_n) \) that has an empty intersection.
2.2 The Heine-Borel lemma

**Definition 2.4** A collection $F$ of sets is said to cover a set $Y$ if for every element $y \in Y$ there exists a set $X \in F$ such that $y \in X$.

A collection of sets $G$ is a subcollection of $F$ if every set $X \in G$ is also in the collection $F$.

**Lemma 2.5** *(The Heine-Borel lemma)* Every collection $F$ of open intervals covering a closed interval $[a,b]$ contains a finite subcollection that also covers $[a,b]$.

**Proof.** Let $F$ be a collection of open intervals covering a closed interval $[a,b]$ and let $S$ be the set of points $y$ in $[a,b]$ with the following property: the interval $[a,y]$ can be covered by finitely many intervals from $F$. As the point $a$ itself is covered by some interval $(\alpha, \beta) \in F$, with $\alpha < a < \beta$, we know that $S$ is not empty: it contains $a$, as well as all points $z \in [a,b]$ such that $a \leq z < \beta$. Moreover, the set $S$ is bounded from above: any element $y \in S$ satisfies $y \leq b$. It is easy to see that if $y \in S$ then any $x$ such that $a \leq x \leq y$ is also in $S$. Let $\bar{s} = \sup S$, and note that $\bar{s} \in [a,b]$. We claim that $\bar{s} \in S$ and $\bar{s} = b$. To see that $\bar{s} \in S$, assume that $\bar{s} \notin S$, and observe that the point $\bar{s}$ is covered by some open interval $(\alpha', \beta') \in F$, with $\alpha' < \bar{s} < \beta'$. Since $\bar{s}$ is the least upper bound for $S$, and $\bar{s} \notin S$, there must be a point $x \in S$ such that $\alpha' < x \leq \bar{s}$. Since $x \in S$, we can cover the interval $[a,x]$ by finitely many intervals from $F$. Adding the interval $(\alpha', \beta') \in F$ to this collection gives us a finite collection covering the interval $[a,\bar{s}]$, contradicting the assumption that $\bar{s} \notin S$. Therefore, we have $\bar{s} \in S$. However, this also shows that unless $\bar{s} = b$, the points in the interval $[\bar{s}, \min(\beta', b)]$ are also in $S$, thus $\bar{s}$ is not an upper bound for $S$, which is a contradiction. Thus, we have $b = \bar{s}$. As we have shown that $\bar{s} \in S$, we know that $b \in S$, and we are done, by the definition of the set $S$. □

**Exercise 2.6** Where did we use the fact that the intervals in the covering collection $F$ are open?

Where did we use the fact that the interval $[a,b]$ is closed? Construct a collection $F$ of closed intervals $[a,b]$ that covers the closed interval $[0,1]$ but no finite sub-collection of $F$ covers $[0,1]$.

**Exercise 2.7** Devise an alternative proof of the Heine-Borel lemma as follows. Assume that there is no finite sub-collection of open intervals from $F$ that covers $I_0 = [a,b]$. Then either there is no finite sub-collection that covers $[a,a_1]$ or there is no finite sub-collection that covers $[a_1,b]$, where $a_1 = (a + b)/2$. Choose the interval $I_2$ out of these two that can not be covered, split it into two sub-intervals, choose a sub-interval $I_3$ that can not be covered by a finite sub-collection and repeat this procedure. This will give you a sequence of nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots$. The Nested Intervals Lemma implies that there is a point $c$ that belongs to all of these intervals. As $c \in [a,b]$, it is covered by some open interval $I$ from the collection $F$. Use this to arrive at a contradiction to the fact that none of $I_k$ can be covered by finitely many intervals from $F$.

2.3 Limit points and the Bolzano-Weierstrass lemma

Let us recall that a neighborhood of a point $x$ is an open interval $(a,b)$ that contains $x$.

**Definition 2.8** A point $z \in \mathbb{R}$ is a limit point of a set $X$ if any neighborhood of $z$ contains infinitely many elements of $X$.

**Exercise 2.9** Show that $z$ is a limit point of a set $X$ if and only if any neighborhood of $z$ contains at least one element of $X$ different from $z$ itself.
As an example, part (i) of the Corollary 1.19 of the Archimedes principle shows that \( z = 0 \) is a limit point of the set \( \{1/n : n \in \mathbb{N}\} \).

**Exercise 2.10** Use the Archimedes principle to show that every \( x \in \mathbb{R} \) is a limit point of the set \( \mathbb{Q} \) of rational numbers.

The following lemma is a cornerstone of many things to come.

**Lemma 2.11** *(The Bolzano-Weierstrass lemma)* Every bounded set of real numbers has at least one limit point.

**Proof.** Let \( X \) be an infinite bounded subset of \( \mathbb{R} \). As \( X \) is bounded, there exists \( M > 0 \) so that \( X \) is contained in the interval \( I = [-M, M] \). Assume that no point in \( I \) is a limit point of \( X \). Then for each \( x \in I \) we can find an open interval \( I_x = (a_x, b_x) \) that contains \( x \) such that there are only finitely many points of \( X \) inside \( I_x \). The open intervals \( I_x \) form an open cover of the closed interval \( I \), and the Heine-Borel lemma implies that there exists a sub-cover of \( I \) by a finite sub-collection of intervals \( I_{x_k} = (a_{x_k}, b_{x_k}) \), \( k = 1, \ldots, N \) with some \( N \in \mathbb{N} \). Each interval \( I_{x_k} \) contains only finitely many elements of \( X \), hence there are only finitely many elements of \( X \) in the union of all these intervals. As no elements of \( X \) can lie outside of this union, there are only finitely many elements in \( X \) which is a contradiction. \( \square \)

Another property we will use often is the following.

**Lemma 2.12** Let \( S \) be a bounded from above set that does not have a maximum. Then \( \sup S \) is a limit point of \( S \).

**Exercise 2.13** Prove Lemma 2.12.

### 3 Limits

#### 3.1 Definition of the limit of a sequence

We first define a limit point of a sequence. One may informally think of them as points where the sequence "bunches up". Here is a way to formalize this.

**Definition 3.1** We say that \( z \in \mathbb{R} \) is a limit point of a sequence \( a_n \) if for any \( \varepsilon > 0 \) there exist infinitely many \( k \in \mathbb{N} \) such that \( |a_k - z| < \varepsilon \).

Note a subtle difference between a limit point of a sequence and a limit point of the set of its values. For instance, for a constant sequence \( 1, 1, 1, 1, \ldots \), that is, \( a_n = 1 \) for all \( n \in \mathbb{N} \), the set of its values is \( \{1\} \) – it consists of one point and has no limit points. However, \( x = 1 \) is a limit point of this sequence. This small difference will be of no "serious" importance for us but one should keep it in mind.

**Exercise 3.2** Show that \( z \in \mathbb{R} \) is a limit point of a sequence \( a_n \) if and only if any open interval \( (c, d) \) that contains \( z \), also contains infinitely many elements of the sequence \( a_n \): there exist infinitely many \( n \) so that \( a_n \in (c, d) \).

A sequence may have more than one limit point. For instance, the sequence \( 1, -1, 1, -1, 1, \ldots \), that is, \( a_n = (-1)^{n+1} \) has exactly two limit points \( z = 1 \) and \( z = -1 \).

**Exercise 3.3** (1) Construct a sequence \( a_n \) that has no limit points. (2) Construct a sequence \( b_k \) that has infinitely many limit points. (3) Construct a sequence \( c_k \) such that any point \( z \in \mathbb{R} \) is a limit point of \( b_k \).
Next, we introduce the notion of the limit of a sequence.

**Definition 3.4** A point $z \in \mathbb{R}$ is the limit of a sequence $a_n$ if for any open interval $(c,d)$ that contains $z$ there exists $N$ so that all $a_k$ with $k \geq N$ lie inside $(c,d)$. We write this as $a_n \to z$ as $n \to +\infty$, or as $\lim_{n \to \infty} a_n = z$.

The following exercise gives a slightly more "practical" definition of the limit.

**Exercise 3.5** Show that a point $z$ is the limit of a sequence $a_n$ if and only if for any $\varepsilon > 0$ there exists $N(\varepsilon)$ so that $|a_k - z| < \varepsilon$ for all $k \geq N(\varepsilon)$.

A small advantage of Definition 3.4 over the one in Exercise 3.5, and the reason to introduce it first, is that it will be easy to generalize to spaces other than $\mathbb{R}$ where the notion of a distance may be not defined but the notion of a neighborhood may be. However, the statement in Exercise 3.5 is much more "practical" and we will mostly use it rather than Definition 3.4 directly.

**Important:** unless otherwise specified, we will usually refer to Exercise 3.5 as the definition of the limit of a sequence. For the first reading, the reader should really think of Exercise 3.5 as the definition of the limit of a sequence.

The next important exercise gives another informal way to think of the limit of a sequence: this is the only point where the sequence "bunches up".

**Exercise 3.6** Let $a_n$ be a bounded sequence. Show that a point $z$ is the limit of $a_n$ if and only if $z$ is the only limit point of $a_n$.

Here are some examples of limits.

**Exercise 3.7** Show that

(a) $\lim_{n \to \infty} \frac{1}{n} = 0$
(b) $\lim_{n \to \infty} \frac{\sin n}{n} = 0$
(c) $\lim_{n \to \infty} \frac{1}{q^n} = 0$ for any $q > 1$.

Hints: for part (a) – use part (i) in Corollary 1.19; for part (c) – write $q = 1 + \delta$ with $\delta > 0$ and use induction to show that $q^n \geq 1 + n\delta$. Then apply part (a).

### 3.2 Basic properties of converging sequences

**Definition 3.8** (1) A sequence $a_n$ is bounded if there exists $M \in \mathbb{R}$ so that $|a_k| \leq M$ for all $k \in \mathbb{N}$.
(2) A sequence $a_n$ is bounded from below if there exists $M \in \mathbb{R}$ so that $a_k \geq M$ for all $k \in \mathbb{N}$.
(3) A sequence $a_n$ is bounded from above if there exists $M \in \mathbb{R}$ so that $a_k \leq M$ for all $k \in \mathbb{N}$.
(4) A sequence $a_n$ is increasing if $a_{k+1} \geq a_k$ for all $k \in \mathbb{N}$.
(5) A sequence $a_n$ is strictly increasing if $a_{k+1} > a_k$ for all $k \in \mathbb{N}$.
(6) A sequence $a_n$ is decreasing if $a_{k+1} \leq a_k$ for all $k \in \mathbb{N}$.
(7) A sequence $a_n$ is strictly decreasing if $a_{k+1} < a_k$ for all $k \in \mathbb{N}$.
(8) A sequence $a_n$ is monotone if it is either decreasing or increasing.
(9) A sequence is converging if it has a limit.

**Theorem 3.9** A convergent sequence is bounded.
Proof. Let \(a_n\) be a convergent sequence, with \(\lim_{n \to \infty} a_n = \ell\). Take \(\varepsilon = 1\) in the definition of the limit of a sequence (again, in Exercise 3.5!) – this implies existence of \(N\) so that \(|a_k - \ell| < 1\) for all \(k \geq N\). It follows that all \(a_k\) with \(k \geq N\) satisfy \(\ell - 1 \leq a_k \leq \ell + 1\). There are only finitely many elements \(a_j\) with \(j < N\). Thus, if we set
\[
m = \min(a_1, \ldots, a_{N-1}, \ell - 1), \quad M = \max(a_1, \ldots, a_{N-1}, \ell + 1),
\]
then all \(a_k\) satisfy \(m \leq a_k \leq M\), hence the sequence \(a_n\) is bounded. □

An immediate consequence is that any unbounded sequence can not have a limit.

Exercise 3.10 Not all bounded sequences converge: give an example of a bounded sequence that has no limit.

The next theorem gives a condition for convergence that is actually very useful in many applications as monotone sequences arise very often.

Theorem 3.11 If \(a_n\) is monotone and bounded, then it converges. Moreover, if \(S = \{a_1, a_2, \ldots, a_k, \ldots\}\), then
1. If \(a_n\) is increasing and bounded from above then \(\lim_{n \to \infty} a_n = \sup S\).
2. If \(a_n\) is decreasing and bounded from below, the \(\lim_{n \to \infty} a_n = \inf S\).

Proof. We will only prove (1). Let us assume that \(a_n\) is increasing and bounded from above. Then the set \(S\) is bounded from above, thus \(\ell = \sup S\) exists. Let \(\varepsilon > 0\), then, as \(\ell - \varepsilon\) is not an upper bound for \(S\) (this follows from the definition of \(\sup S\)), there exists \(N(\varepsilon)\) so that \(a_{N(\varepsilon)} > \ell - \varepsilon\). As the sequence \(a_n\) is monotonically increasing, it follows that for all \(k \geq N(\varepsilon)\) we have \(a_k \geq \ell - \varepsilon\) as well. However, as \(\ell\) is an upper bound for the sequence \(a_n\), we also know that \(a_k \leq \ell\). Thus, we have \(|a_k - \ell| < \varepsilon\) for all \(k \geq N(\varepsilon)\) and we are done. □

Theorem 3.12 Let \(a_n\) and \(b_n\) be convergent sequences with \(\lim_{n \to \infty} a_n = A\) and \(\lim_{n \to \infty} b_n = B\).
1. The sequence \(a_n + b_n\) also converges and \(\lim_{n \to \infty} (a_n + b_n) = A + B\).
2. The sequence \(a_n b_n\) also converges and \(\lim_{n \to \infty} (a_n b_n) = AB\).
3. If, in addition, \(B \neq 0\), then the sequence \(a_n/b_n\) also converges and \(\lim_{n \to \infty} (a_n/b_n) = A/B\).

Proof. We will only prove (2). Let us write
\[
a_n b_n - AB = a_n b_n - a_n B + a_n B - AB = a_n(b_n - B) + (a_n - A)B.
\]
The triangle inequality implies that
\[
|a_n b_n - AB| \leq |a_n| |b_n - B| + |a_n - A||B|.
\]
Since \(a_n\) is a convergent sequence, there exists \(M\) so that \(|a_n| \leq M\) for all \(n \in \mathbb{N}\), which leads to
\[
|a_n b_n - AB| \leq M |b_n - B| + |a_n - A||B|.
\]
Now, given \(\varepsilon > 0\) find \(N_1\) so that \(|a_n - A| \leq \varepsilon/(2|B|)\) for all \(n \geq N_1\), and \(N_2\) so that \(|a_n - A| \leq \varepsilon/(2M)\) for all \(n \geq N_1\). Then, for all \(n \geq N = \max(N_1, N_2)\), we have
\[
|a_n b_n - AB| \leq M |b_n - B| + |a_n - A||B| \leq M \frac{\varepsilon}{2M} + \frac{\varepsilon}{2|B|} |B| = \varepsilon.
\]
Hence, the sequence \(a_n b_n\) converges to \(AB\). □
Exercise 3.13 (1) Prove assertion (1) in Theorem 3.12.
(2) Prove assertion (3) in Theorem 3.12. Hint: it is easier to first show that the sequence $1/b_n$ converges to $1/B$.
(3) Let $a_n = 1/n$, find a sequence $b_n$ such that the limit of $a_n/b_n$ exists, and a sequence $d_n$ such that the limit of $a_n/b_n$ does not exist.
(4) Assume that $\lim_{n\to\infty} a_n = A > 0$, and $\lim_{n\to\infty} b_n = 0$. Show that the sequence $a_n/b_n$ does not converge.

Theorem 3.14 Assume that $a_n$ and $b_n$ are two convergent sequences with
\[
\lim_{n\to\infty} a_n = A \text{ and } \lim_{n\to\infty} b_n = B.
\]
If $A < B$ then there exists $N$ so that $a_n < b_n$ for all $n \geq N$.

Exercise 3.15 Prove this theorem.

3.3 The number $e$

Theorem 3.16 The sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ converges. Its limit is denoted as $e$:
\[
e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n
\]  
(3.1)

Proof. We will look instead at the sequence
\[
y_n = \left(1 + \frac{1}{n}\right)^{n+1}
\]
and show that it is decreasing. For that, we need the inequality
\[
(1 + \alpha)^n \geq 1 + n\alpha
\]  
(3.2)
that holds for all $\alpha \geq 0$ and all $n \in \mathbb{N}$. Recall that we have seen this inequality in Exercise 3.7. Now, we write
\[
y_{n-1}/y_n = \frac{(1 + \frac{1}{n-1})^n}{(1 + \frac{1}{n})^{n+1}} = \frac{n^n}{(n-1)^n(n+1)^{n+1}} = \frac{n^{n+1}}{(n^2-1)^n(n+1)} = \left(1 + \frac{1}{{n^2-1}}\right)^n \frac{n}{n+1}.
\]
Now, we use (3.2) with $\alpha = 1/(n^2-1)$, to get
\[
\frac{y_{n-1}}{y_n} \geq \left(1 + \frac{n}{n^2-1}\right) \frac{n}{n+1} \geq \left(1 + \frac{1}{n}\right) \frac{n}{n+1} = 1.
\]
Thus, the sequence $y_n$ is a positive decreasing sequence, and Theorem 3.11 implies that $\lim_{n\to\infty} y_n$ exists. Thus, the sequence $x_n$ can be written as a product of two converging sequences:
\[
x_n = \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^{n+1} \frac{1}{1+1/n},
\]
hence $x_n$ is itself converging. $\square$
Let us explain informally why (3.1) agrees with what we may know from a standard calculus course. It follows from (3.1) that we can write
\[ e = \left(1 + \frac{1}{n}\right)^n + \alpha_n, \]  
with a sequence \(\alpha_n\) that goes to zero as \(n \to +\infty\). Taking the logarithm with respect to base \(e\), that we will denote simply by \(\log\), gives
\[ 1 = n \log \left(1 + \frac{1}{n}\right) + \beta_n, \]  
with
\[ \beta_n = \log \left[1 + \alpha_n \left(1 + \frac{1}{n}\right)^{-n}\right]. \]

**Exercise 3.17** Show that \(\lim_{n \to \infty} \beta_n = 0\). Hint: use the fact that \(\alpha_n \to 0\).

Using the result of Exercise 3.17 in (3.4) gives
\[ \log \left(1 + \frac{1}{n}\right) = \frac{1}{n} + \frac{\beta_n}{n}. \]  

**Exercise 3.18** Recalling the calculus definition of the derivative (that we officially do not know yet), show that (3.5) implies that if the derivative of \(f(x) = \log x\) exists at \(x = 1\), then \(f'(x) = 1\). Show that \(e\) is the unique number \(a\) such that
\[ \left. \frac{d}{dx} (\log_a x) \right|_{x=1} = 1. \]

### 3.4 The Cauchy criterion

**Definition 3.19** We say that \(a_n\) is a Cauchy sequence if for any \(\varepsilon > 0\) there exists \(m \in \mathbb{N}\) so that \(|a_n - a_m| < \varepsilon\) for all \(n, m \geq N\).

**Theorem 3.20** A sequence \(a_n\) converges if and only if \(a_n\) is a Cauchy sequence.

**Proof.** One direction is easy. Assume that \(a_n\) converges and \(A = \lim a_n\). Given \(\varepsilon > 0\) we can find \(N \in \mathbb{N}\) so that \(|a_n - A| < \varepsilon/2\) for all \(n \geq N\). Then, for all \(n, m \geq N\) we have
\[ |a_n - a_m| \leq |a_n - A| + |A - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \]
hence \(a_n\) is a Cauchy sequence.

Next, assume that \(a_n\) is a Cauchy sequence. First, we claim that \(a_n\) is bounded. Indeed, taking \(\varepsilon = 1\), we can find \(N \in \mathbb{N}\) such that \(|x_n - x_m| < 1\) for all \(n, m \geq N\). In particular, we have \(x_N - 1 < x_m < x_N + 1\) for all \(m \geq N\). In addition, there are only finitely many elements of the sequence \(a_n\) with \(n < N\), so the set \(\{a_1, a_2, \ldots, a_{N-1}\}\) is a bounded set. Hence, \(\{a_n\}\) is a union of two bounded sets, hence it is also bounded, and the sequence \(a_n\) is bounded as well. Thus, for each \(n \in \mathbb{N}\) we can define
\[ x_n = \inf_{k \geq n} a_k, \quad y_n = \sup_{k \geq n} a_k. \]

It is clear from the definition that \(x_n \leq x_{n+1} \leq y_{n+1} \leq y_n\) for all \(n \in \mathbb{N}\), so that the sequence \(x_n\) is increasing and the sequence \(y_n\) is decreasing. By the nested intervals theorem there exists a point \(A\) common to all intervals \([x_n, y_n]\):
\[ x_n \leq A \leq y_n \text{ for all } n \in \mathbb{N}. \]
In addition, we have
\[ x_n \leq a_n \leq y_n \text{ for all } n \in \mathbb{N}, \]
and it follows that
\[ |A - a_n| \leq |x_n - y_n|. \tag{3.6} \]
However, given any \( \varepsilon > 0 \) we can find \( N \) so that for all \( m, N \geq N \) we have
\[ |a_n - a_m| < \frac{\varepsilon}{10}, \]
and in particular,
\[ |a_n - a_N| < \frac{\varepsilon}{10}. \]
Now, it follows from the definition of \( x_n \) and \( y_n \) that for all \( n \geq N \) we have
\[ |x_n - a_N| \leq \frac{\varepsilon}{10}, \quad |y_n - a_N| \leq \frac{\varepsilon}{10}, \]
hence
\[ |x_n - y_n| \leq \frac{2\varepsilon}{10} < \varepsilon. \]
We conclude from this and (3.6) that \( |A - a_n| < \varepsilon \) for all \( n \geq N \), thus \( a_n \) converges to \( A \) as \( n \to +\infty \). □

**Exercise 3.21** Use the Cauchy criterion to show that the sequence
\[ a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \]
does not converge. Hint: show that \( a_{2n} - a_n \geq 1/2 \) for all \( n \).

**Exercise 3.22** (i) Show that the claim of Theorem 3.20 does not hold for the set \( \mathbb{Q} \) of rational numbers. In other words, show that a Cauchy sequence of rational numbers does not necessarily converge to a rational number. This is another crucial difference between \( \mathbb{Q} \) and \( \mathbb{R} \). (ii) However, show, without using the results about real numbers, that if a sequence \( a_k \) of rational numbers is Cauchy and a subsequence \( a_{n_k} \) converges to a rational number \( \ell \in \mathbb{Q} \), then the sequence \( a_n \) converges to \( \ell \). In other words, a Cauchy sequence that has a converging subsequence must converge.

### 3.5 The Bolzano-Weierstrass theorem, \( \lim \sup \) and \( \lim \inf \)

**Definition 3.23** If \( x_n \) is a sequence, and \( n_k \in \mathbb{N} \) is a strictly increasing sequence: \( n_1 < n_2 < n_3 \cdots < n_k < \ldots \), then the sequence \( y_k = x_{n_k} \) is called a subsequence of \( x_n \).

**Theorem 3.24** Every bounded sequence of real number contains a convergent subsequence.

**Proof.** Let \( E \) be the set of values of a bounded sequence \( x_n \). If \( E \) is finite, then there exists \( a \in \mathbb{R} \) so that \( x_n = a \) for infinitely many \( n \). It is then an easy exercise to see that there exists a subsequence \( x_{n_k} \) of \( x_n \) such that \( x_{n_k} = a \) for all \( k \), hence \( \lim_{k \to \infty} x_{n_k} = a \) and we are done. If the set \( E \) is infinite, then, as \( x_n \) is a bounded sequence, \( E \) is an infinite bounded set, hence by the Bolzano-Weierstrass Lemma 2.11, it has a limit point \( A \). Then one can choose \( n_1 \) so that \( |x_{n_1} - A| < 1 \). Next, note that, since \( A \) is a limit point of \( x_n \), the interval \((A - 1/2, A + 1/2)\) contains infinitely many elements of the sequence, hence one can choose \( n_2 > n_1 \) so that \( |x_{n_2} - A| < 1/2 \), and so on, at each step choosing \( n_{k+1} > n_k \) so that \( |x_{n_{k+1}} - A| < 1/(k+1) \). As \( \lim_{k \to \infty} 1/(k+1) = 0 \), the sequence \( x_{n_k} \) converges to \( A \), and we are done. □
**Definition 3.25** If there exists a subsequence $x_{n_k}$ such that $\ell = \lim_{k \to \infty} x_{n_k}$, then we say $\ell$ is a limit of $x_n$ along a subsequence.

We now define $\limsup$ and $\liminf$ of a sequence $x_n$. As we have done in the proof of the Cauchy criterion for convergence, let us set

$$a_n = \inf_{k \geq n} x_k, \quad b_n = \sup_{k \geq n} x_k.$$ 

As we have observed in that proof, if $x_n$ is bounded from below, then $a_n$ is well-defined, and the sequence $a_n$ is increasing, while if $x_n$ is bounded from, then $b_n$ are well-defined, and the sequence $b_n$ is decreasing. We denote the corresponding limits by

$$\limsup x_n = \lim_{n \to \infty} (\sup_{k \geq n} x_k), \quad \liminf x_n = \lim_{n \to \infty} (\inf_{k \geq n} x_k).$$

**Proposition 3.26** Let $x_n$ be a bounded sequence, then $\liminf x_n$ and $\limsup x_n$ are the largest and the smallest limits of $x_n$ along a subsequence.


## 4 Continuity and limits of a function

### 4.1 Limit of a function at a point

Let $f(x)$ be a function $f : [c, d] \to \mathbb{R}$ and $a \in [c, d]$.

**Definition 4.1** We say that

$$\lim_{x \to a} f(x) = A,$$

in the sense of Cauchy, if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for any $x \neq a$ such that $|x - a| < \delta$ and $x \in [c, d]$ we have $|f(x) - A| < \varepsilon$.

Remark that we exclude $x = a$ in the above definition. An alternative definition is as follows.

**Definition 4.2** We say that $A$ is a sequential limit of $f(x)$ as $x \to a$ if for any sequence $x_n \to a$, with $x_n \neq a$ and $x_n \in [c, d]$, we have

$$\lim_{n \to \infty} f(x_n) = A.$$

**Theorem 4.3** The two definitions of the limit of $f(x)$ at a point $a$ are equivalent.

**Proof.** Let us first assume that

$$\lim_{x \to a} f(x) = A,$$

in the sense of Cauchy, and let $x_n$ be a sequence such that $x_n \to a$. Given $\varepsilon > 0$ there exists $\delta > 0$ so that for any $x \neq a$ such that $|x - a| < \delta$ we have $|f(x) - A| < \varepsilon$. As $x_n \to a$, given this $\delta > 0$, we can find $N$ so that $|x_n - a| < \delta$ for all $n \geq N$, and then $|f(x_n) - A| < \varepsilon$ by the above, hence $f(x_n) \to A$.

Next, assume that $f(x)$ converges in the sequential sense to $A$ as $x \to a$ but that $A$ is not the limit of $f(x)$ in the Cauchy sense as $x \to a$. This means that there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exists $x \neq a$ such that $|x - a| < \delta$ and $|f(x) - A| > \varepsilon_0$. Let us take $\delta_n = 1/n$—this will generate the corresponding sequence $x_n$. It is easy to see that $x_n \to a$ but $f(x_n)$ does not converge to $A$, which is a contradiction. □

**Exercise 4.4** Verify that the usual arithmetic and inequality properties holds for the limits of a function at a point.
4.2 Continuous functions and their basic properties

Definition 4.5 (1) A function \( f(x) : [c, d] \rightarrow \mathbb{R} \) is continuous at a point \( a \in [c, d] \) if

\[
\lim_{x \to a} f(x) = f(a).
\]

(2) A function \( f : [c, d] \rightarrow \mathbb{R} \) is continuous on the interval \([c, d]\) if it is continuous at every point of \([c, d]\).

We summarize some basic properties of continuous functions in the following proposition.

Proposition 4.6 (i) Let \( f(x) \) defined on an interval \([c, d]\) be continuous at a point \( a \in [c, d] \) and \( f(a) \neq 0 \), then there is a neighborhood \( U \) of \( a \) such that \( f(x) \) has the same sign as \( f(a) \) for all \( x \in U \) such that \( x \in [c, d] \).

(ii) If the functions \( f \) and \( g \) defined on \([c,d]\) are continuous at \( a \in [c,d] \), then so are the functions \( f + g \) and \( fg \). Also, the function \( fd/g \) is continuous at a provided that \( g(a) \neq 0 \).

(iii) Let \( f \) be a continuous function defined on an interval \([c_1, d_1]\) and \( g \) be a continuous function defined on an interval \([c, d]\), such that \( g(x) \in [c_1, d_1] \) for all \( x \in [c, d] \), so that the composition \((f \circ g)(x) = f(g(x))\) is defined for all \( x \in [c, d] \). Show that if \( f \) is continuous on \([c - 1, d_1]\) and \( g \) is continuous on \([c, d]\), then \( f \circ g \) is continuous on \([c, d]\).

be two functions such that the composition

Exercise 4.7 Prove Proposition 4.6, pay special attention to (iii).

Theorem 4.8 (Intermediate Value Theorem) Let \( f \) be a continuous function on a closed interval \([a, b]\) such that \( f(a) \) and \( f(b) \) have different signs, that is, \( f(a)f(b) \leq 0 \). Then there exists \( c \in [a, b] \) such that \( f(c) = 0 \).

Proof. If either \( f(a) = 0 \) or \( f(b) = 0 \), we are done, so let us assume without loss of generality that \( f(a) < 0 \) and \( f(b) > 0 \). Let us denote \( I_1 = [a, b] \) and divide \( I_1 \) in half. If the value of \( f \) at the mid-point is zero, we are done, otherwise, one of the two closed intervals has the same property: the signs of \( f \) at the two end-points are different. Let us call that interval \( I_2 = [a_1, b_1] \), divide it at its half-point, and continue. The process terminates if the value of \( f \) at one of the mid-points will be zero, which means we are done. Otherwise, we get an infinite sequence of nested intervals \( I_k = [a_k, b_k] \) such that \( f(a_k) \) and \( f(b_k) \) have different signs. We also know that \( |I_k| = |a - b|/2^{k-1} \) goes to zero as \( k \to +\infty \). The Nested Intervals Lemma implies that there is a unique point \( c \in [a, b] \) in the intersection of all \( I_k \). Let \( a'_k \) be the endpoint of \( I_k \) such that \( f(a'_k) < 0 \) and \( b'_k \) be the endpoint of \( I_k \) such that \( f(b'_k) > 0 \). Moreover, we have \( a'_k \to c \) and \( b'_k \to c \) as \( k \to +\infty \). As \( c \in [a, b] \), the function \( f \) is continuous at \( c \). Continuity of \( f \) at \( c \) implies that

\[
f(c) = \lim_{k \to \infty} f(a'_k), \quad f(c) = \lim_{k \to \infty} f(b'_k).
\]

The first equality above implies that \( f(x) \leq 0 \) and the second implies that \( f(c) \geq 0 \). It follows that \( f(c) = 0 \).

Corollary 4.9 Let \( f \) be a continuous function on a closed interval \([a, b]\) such that \( f(a) < A \) and \( f(b) > A \). Then there exists \( c \in [a, b] \) such that \( f(c) = A \).

Theorem 4.10 A continuous function on a closed interval \([a, b]\) is bounded and attains both its maximum and its minimum on \([a, b]\).
**Proof.** For every point $x \in [a, b]$ there exists an open interval $I_x$ containing $x$ such that $f(x) - 1 \leq f(y) \leq f(x) + 1$ for all $y \in I_x$ such that $y \in [a, b]$. The open intervals $I_x$ cover the closed interval $[a, b]$, hence there exists a finite sub-collection $I_{x_1}, \ldots, I_{x_N}$ that also covers $[a, b]$. Let

$$M = 1 + \max(f(x_1), \ldots, f(x_N)),$$

and

$$m = -1 + \min(f(x_1), \ldots, f(x_N)).$$

As the intervals $I_{x_1}, \ldots, I_{x_N}$ cover $[a, b]$, we know that every point $z \in [a, b]$ belongs to some $I_{x_k}$. It follows that

$$m - 1 \leq f(z) \leq M + 1,$$

for all $z \in [a, b]$, hence the function $f$ is bounded on $[a, b]$.

To see that the maximum and minimum are attained, let $M = \sup_{x \in [a, b]} f(x)$ and assume that $f(x) \neq M$ for all $x \in [a, b]$. Then the function $g(x) = 1/M - f(x)$ is continuous on $[a, b]$, thus, by what we have just proved, it is bounded: there exists $K$ so that $g(x) \leq K$ for all $x \in [a, b]$. But then we have

$$M - f(x) \geq \frac{1}{K}, \text{ for all } x \in [a, b],$$

so that

$$f(x) \leq M - \frac{1}{K}, \text{ for all } x \in [a, b],$$

which contradicts the fact that $M = \sup_{x \in [a, b]} f(x)$. Thus, there has to exist some $c \in [a, b]$ such that $f(c) = M$, hence $f$ attains its maximum on $[a, b]$. The proof that $f$ attains its minimum on $[a, b]$ is almost verbatim the same. $\square$

**Exercise 4.11** Prove that a continuous function on a closed interval $[a, b]$ is bounded and attains its minimum on $[a, b]$.

**Exercise 4.12** (i) Give an example a continuous function on the open interval $(0, 1)$ that is unbounded. (ii) Give an example a continuous function on the open interval $(0, 1)$ that is bounded but does not attain its maximum or minimum on $(0, 1)$. (iii) Give an example a discontinuous function on the closed interval $[0, 1]$ that is unbounded. (iv) Give an example a discontinuous function on the closed interval $[0, 1]$ that is bounded but does not attain its maximum or minimum on $[0, 1]$.

### 4.3 Uniform continuity

**Definition 4.13** A function $f$ is uniformly continuous on a set $E$ if for every $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in E$ such that $|x - y| < \delta$.

**Exercise 4.14** Show that the function $f(x) = \sin(1/x)$ is continuous but not uniformly continuous on the open interval $(0, 1)$.

**Proposition 4.15** A uniformly continuous function defined on a bounded set $E$ is bounded.

**Proof.** Assume that $f$ is unbounded on $(a, b)$. Then there exists a sequence of points $x_k \in (a, b)$ such that

$$|f(x_{k+1})| \geq |f(x_k)| + k.$$
It follows, in particular, that
\[ |f(x_m) - f(x_k)| > k \text{ for any } k \text{ and } m > k. \] (4.1)

Uniform continuity of \( f \) implies that there exists \( \delta_0 > 0 \) so that \( |f(x) - f(y)| < 1 \) for all \( x, y \in E \) such that \( |x - y| < \delta_0 \). The sequence \( x_k \) is bounded, hence it has a convergent subsequence \( x_{n_k} \). Therefore, the sequence \( x_{n_k} \) is Cauchy. In particular, there exists \( N \) such that \( |x_{n_k} - x_{m_k}| < \delta_0 \) for all \( k, m \geq N \). But then \( |f(x_{n_k}) - f(x_{m_k})| < 1 \) for all \( k, m \geq N \), which contradicts (4.1). Thus, the function \( f \) has to be bounded on \( E \). □

**Theorem 4.16** A function that is continuous on a closed interval \([a, b]\) is uniformly continuous on that interval.

**Proof.** Let \( f \) be continuous on \([a, b]\). Then, given \( \varepsilon > 0 \) for each \( x \in [a, b] \) there exists \( \delta_x > 0 \) such that \( |f(x) - f(y)| < \varepsilon/10 \) for all \( y \) such that \( |y - x| < \delta_x/2 \). The open intervals \( I_x = (x - \delta_x/2, x + \delta_x/2) \) cover the closed interval \([a, b]\). The Heine-Borel lemma implies that there exists a finite sub-collection \( I_{x_1}, \ldots, I_{x_N} \) of such intervals that also covers \([a, b]\). Let then \( \delta = (1/2) \min(\delta_{x_1}, \ldots, \delta_{x_N}) \) and consider any \( x, y \in [a, b] \) such that \( |x - y| < \delta \). Since the intervals \( I_{x_1}, \ldots, I_{x_N} \) cover \([a, b]\), there exists \( x_k \) such that \( x \in I_{x_k} \), which means that \( |x_k - x| < \delta_{x_k}/2 \). Then we also have
\[ |y - x_k| \leq |y - x| + |x - x_k| < \delta + \frac{\delta_{x_k}}{2} \leq \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \delta_{x_k}. \]

Hence, both \( x \) and \( y \) belong to \((x_k - \delta_{x_k}, x_k + \delta_{x_k})\). It follows that
\[ |f(x) - f(x_k)| < \frac{\varepsilon}{10}, \quad |f(y) - f(x_k)| < \frac{\varepsilon}{10}. \]

The triangle inequality now implies that
\[ |f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(y) - f(x_k)| < \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon. \]

Thus, we have show that for any \( \varepsilon > 0 \) we can find \( \delta > 0 \) so that for any \( x, y \in [a, b] \) such that \( |x - y| < \delta \) we have \( |f(x) - f(y)| < \varepsilon \), which means that the function \( f \) is uniformly continuous on \([a, b]\). □

5 Open and closed sets in \( \mathbb{R}^n \)

Let us recall that the distance between two points \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) is
\[ d(x, y) = \left[ (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 \right]^{1/2}, \]
also often denoted as
\[ ||x - y|| = \left[ (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 \right]^{1/2}. \]

We will use the notation \( B(x, r) \) for an ball centered at a point \( x \in \mathbb{R}^n \) of radius \( r \):
\[ B(x, r) = \{ y \in \mathbb{R}^n : d(x, y) < r \}, \]
and the closed ball as
\[ \overline{B}(x, r) = \{ y \in \mathbb{R}^n : d(x, y) \leq r \}. \]

**Definition 5.1** A set \( U \subset \mathbb{R}^n \) is open if for every \( x \in U \) there is a ball \( B(x, r) \) that is contained in \( U \).
**Definition 5.2** A point \( x \in \mathbb{R}^n \) is a limit point of a set \( G \) if for any open ball \( B(x, r) \) there exists a point \( y \in B(x, r) \) such that \( y \in G \) and \( y \neq x \).

**Definition 5.3** A set \( U \subset \mathbb{R}^n \) is closed if all limit points of \( U \) are contained in \( U \).

**Theorem 5.4** A set \( C \subset \mathbb{R}^n \) is closed if and only if its complement \( U = \mathbb{R}^n \setminus C \) is open.

**Proof.** Let \( C \) be a closed set and \( x \in U = \mathbb{R}^n \setminus C \). As \( C \) is closed, we know that \( x \) is not a limit point of \( C \). Thus, there exists a ball \( B(x, r) \) such that there is no point from \( C \), except, possibly, \( x \), in \( B(x, r) \). As \( x \notin C \), \( B(x, r) \) contains no points from \( C \), hence \( B(x, r) \subset U \). This shows that \( U \) is an open set.

Next, assume that \( U = \mathbb{R}^n \setminus C \) be an open set, and let \( x \) be a limit point of \( C \). We will show that \( x \in C \). Indeed, if \( x \notin C \), then \( x \in U \) and, as \( U \) is open, there exists a ball \( B(x, r) \) that is contained in \( U \). This contradicts the assumption that \( x \) is a limit point of \( C \). Thus, \( C \) contains all its limit points and is closed. \( \square \)

**Theorem 5.5** (i) The union of any collection of open sets is itself an open set.

(ii) The intersection of any collection of closed sets is a closed set.

(iii) The intersection \( \bigcap_{i=1}^N U_i \) of a finite collection of open sets \( U_1, \ldots, U_N \) is an open set.

(iv) The union \( \bigcup_{i=1}^N F_i \) of a finite collection of closed sets \( F_1, \ldots, F_N \) is a closed set.

**Proof.** Let us prove (i). Let \( U_\alpha, \alpha \in A \) (here, \( A \) is just some set of indices, finite or infinite) be a collection of open sets and let \( x \in U = \bigcup_{\alpha \in A} U_\alpha \). Then there exists some \( U_\alpha \) with \( \alpha \in A \) such that \( x \in U_\alpha \). As \( U_\alpha \) is an open set, there exists \( r > 0 \) such that \( B(x, r) \) is contained in \( U_\alpha \). But then \( B(x, r) \subset U \), hence \( U \) is an open set. Note that (i) is equivalent to (iii) because of Theorem 5.4, hence (iii) is also proven.

To prove (ii), let \( x \in U = \bigcap_{i=1}^N U_i \) of a finite collection of open sets \( U_1, \ldots, U_N \). As all \( U_i \) are open, there exist \( r_1, \ldots, r_N > 0 \) such that \( B(x, r_k) \in U_k \). Let \( r = \min(r_1, \ldots, r_N) \), then \( B(x, r) \in U \), hence \( U \) is open. Note that (ii) is equivalent to (iv) because of Theorem 5.4, hence (iv) is also proven. \( \square \)

Here is a bunch of related definitions.

**Definition 5.6** (i) A point \( x \) is an interior point of a set \( U \subset \mathbb{R}^n \) if there is a ball \( B(x, r) \) that is contained in \( U \).

(ii) A point \( x \) is a boundary point of a set \( U \subset \mathbb{R}^n \) if for any \( r > 0 \) the ball \( B(x, r) \) contains both points in \( U \) and not in \( U \). The boundary of \( U \) is denoted as \( \partial U \).

(iii) The closure \( \bar{U} \) of a set \( U \) is the union of \( U \) and the set of its limit points.

**Exercise 5.7** Show that \( \bar{U} \) is the union of \( U \) and the boundary \( \partial U \).