Math 61CM Homework # 5

Due at TA session on Friday, October 26.

1. Let \( V \) be a subspace of \( \mathbb{R}^n \) and let \( P_V : \mathbb{R}^n \to \mathbb{R}^n \) be the orthogonal projection of \( \mathbb{R}^n \) onto \( V \) as defined in lecture. [Hint: remember that for any subspace \( U \) and \( u \in \mathbb{R}^n \), \( u = P_U(u) + P_{U^⊥}(u) \).]

(i) If \( I \) is the identity transformation of \( \mathbb{R}^n \) (i.e. \( I(x) = x \), for all \( x \in \mathbb{R}^n \)), prove that \( I - P_V = P_{V^⊥} \).

(ii) When is the column space of \( A \) “reflection in the line \( y = x \)?”

2. Let \((x_1, y_1), \ldots, (x_n, y_n)\) be \( n \) points in \( \mathbb{R}^2 \). This question is about linear regression, which means finding real numbers \( r, t \) such that the line \( y = rx + t \) is the best linear approximation to \((x_1, y_1), \ldots, (x_n, y_n)\), in the sense that the total square error

\[
E = \sum_{i=1}^{n} (y_i - (rx_i + t))^2
\]

is as small as possible.

(i) Let

\[
A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}^T, \quad y = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T.
\]

Show that \((r, t)\) achieves a minimum value of \( E \) if and only if \( A \begin{pmatrix} r \\ t \end{pmatrix} = P_{C(A)}(y) \), where \( C(A) \) is the column space of \( A \).

(ii) When is \( N(A) = \{0\} \) (i.e. when is the solution \((r, t)\) unique)?

3. Suppose \( A \) is an \( m \times n \) real matrix. Prove:

(a) \( A^T A \) is positive semi-definite, and

(b) \( A^T A \) is positive definite if \( N(A) = \{0\} \).

Here we use the following terminology: an \( n \times n \) matrix \( B = (b_{ij}) \) is positive semi-definite if \( x^T B x \geq 0 \) for all \( x \in \mathbb{R}^n \) and \( B \) is positive definite if \( x^T B x > 0 \) for all \( x \in \mathbb{R}^n \setminus \{0\} \). Notice that

\[
x^T B x = \sum_{i,j=1}^{n} b_{ij} x_i x_j,
\]

such an expression is called a quadratic form. Similarly, for a linear map \( T \in \mathcal{L}(V, V) \) where \( V \) is a finite dimensional inner product space, one says that \( T \) is positive semi-definite if \( T x \cdot x \geq 0 \) for all \( x \in V \), and one says that \( T \) is positive definite if \( T x \cdot x > 0 \) for all \( x \in V \setminus \{0\} \).

4. (i) Let \( \theta \in [0, 2\pi) \) and let \( T \) be the linear transformation of \( \mathbb{R}^2 \) defined by \( T(x) = Q(\theta)x \), where \( Q(\theta) \) is the \( 2 \times 2 \) matrix \( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \). Prove that if \( x = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix} \) (with \( r \geq 0 \) and \( \alpha \in [0, 2\pi) \))

then \( T(x) = \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} \). With the aid of a sketch, give a geometric interpretation of this.

(ii) What is the matrix of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) which takes the point \((x, y)\) to its “reflection in the line \( y = x \)” i.e. the transformation \( T(x, y) = (y, x) \).

Caution: In part (ii) we are writing points in \( \mathbb{R}^2 \) as row vectors, but in order to represent \( T \) in terms of matrix multiplication you should first rewrite everything in terms of column vectors.
5. Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots \supseteq K_n \supseteq K_{n+1} \supseteq \cdots$ be a nested sequence of sets in $\mathbb{R}^k$, so that each $K_n$ is a non-empty compact set. Show that there exists a point $x \in \mathbb{R}^k$ that belongs to all sets $K_m$. Hint: consider a sequence $x_n$ so that $x_1 \in K_1$, $x_2 \in K_2$, $\ldots$, $x_n \in K_n$, $\ldots$ Show that $x_n$ has a convergent subsequence and that the limit belongs to all $K_n$.

6. Let $f$ be a continuous real-valued function on $\mathbb{R}^n$ and $c \in \mathbb{R}$. Show that

(i) The set $\{x \in \mathbb{R}^n : f(x) < c\}$ is open.
(ii) The set $\{x \in \mathbb{R}^n : f(x) > c\}$ is open.
(iii) The set $\{x \in \mathbb{R}^n : f(x) \leq c\}$ is closed.
(iv) The set $\{x \in \mathbb{R}^n : f(x) \geq c\}$ is closed.
(v) The set $\{x \in \mathbb{R}^n : f(x) = c\}$ is closed.

(vi) Assume, in addition that for any $M > 0$ there exists $R > 0$ so that $|f(x)| > M$ for all $x \notin B(0, R)$. Show that the set $\{x \in \mathbb{R}^n : |f(x)| \leq c\}$ is compact.

7. Show that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous if and only if given any open set $U \in \mathbb{R}^n$, the pre-image $f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}$ is an open set.

8. Show that the image $f(E)$ of a connected set $E \subset \mathbb{R}^n$ under a continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is a connected set in $\mathbb{R}^m$. 