Problem 1: Suppose \( f, g : \mathbb{R} \to \mathbb{R} \) are \( C^2 \) functions, and let \( F : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( F(t, x) = f(x + t) + g(x - t) \). Prove that \( F \) is \( C^2 \) on \( \mathbb{R}^2 \) and satisfies the wave equation on \( \mathbb{R}^2 \):

\[
\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} = 0.
\]

Solution: Notice that the functions \( h_1 : \mathbb{R}^2 \to \mathbb{R} : (x, t) \mapsto x + t \) and \( h_2 : \mathbb{R}^2 \to \mathbb{R} : (x, t) \mapsto x - t \) are both \( C^2 \). Indeed, their gradients are \((1, 1)\) and \((1, -1)\), which are constant and hence continuously differentiable. It follows that \( f \circ h_1 \) and \( g \circ h_2 \) are also both \( C^2 \), hence their sum \( F \) is \( C^2 \) as well.

To compute the second derivative of \( F \) with respect to \( t \), we use the chain rule twice:

\[
\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial t} \left( f'(x + t) - g'(x - t) \right) = f''(x + t) + g''(x - t).
\]

Here, since \( f \) and \( g \) are functions from \( \mathbb{R} \) to \( \mathbb{R} \), it makes sense to use their derivatives as usual. Similarly,

\[
\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial t} \left( f'(x + t) + g'(x - t) \right) = f''(x + t) + g''(x - t).
\]

It follows that

\[
\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} = 0.
\]

The way you should picture this is as follow: if you take the graph of the function \( f(x) \), and make it move towards the left as time varies (i.e. you consider \( f(x + t) \)), the movement described satisfies the wave equation. It is like taking a piece of rope, moving it violently with your hand, and watching the pattern you created travel away from you. For the \( g(x - t) \) part, it’s similar, except that the wave travels towards the right as time increases. Furthermore, this equation is linear: you can add \( f(x + t) \) and \( g(x - t) \), which are both solutions, to get another solution. So, if two waves travelling in opposite directions collide together, they will just pass through each other and keep travelling.
**Problem 2:** For a function $f \in C^2(\mathbb{R}^n)$ we define its Laplacian as

$$\Delta F(x) = \sum_{j=1}^{n} \frac{\partial^2 F(x)}{\partial x_j^2}.$$ 

(i) Show that in dimensions $n \geq 3$ the function $g(x) = 1/\|x\|^{n-2}$ satisfies the Laplace equation

$$\Delta g(x) = 0 \text{ for } x \neq 0.$$ 

(ii) Show that the function

$$G(t, x) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{\|x\|^2}{4t} \right), \quad t > 0, \ x \in \mathbb{R}^n,$$

satisfies the heat equation

$$\frac{\partial G(t, x)}{\partial t} = \Delta G(t, x), \ \text{for all } t > 0 \text{ and } x \in \mathbb{R}^n.$$ 

**Solution:**

(i) We compute the derivatives:

$$\frac{\partial g}{\partial x_j} = \frac{\partial}{\partial x_j} (x_1^2 + \cdots + x_n^2)^{(2-n)/2} = \frac{2-n}{2} (x_1^2 + \cdots + x_n^2)^{(2-n)/2-1} \cdot 2x_j = \frac{(2-n)x_j}{\|x\|^n}$$

$$\frac{\partial^2 g}{\partial x_j^2} = \frac{\partial}{\partial x_j} (2-n)x_j \cdot (x_1^2 + \cdots + x_n^2)^{-n/2}$$

$$= (2-n)(x_1^2 + \cdots + x_n^2)^{-n/2} + (2-n)x_j \cdot \left( -\frac{n}{2} \right) (x_1^2 + \cdots + x_n^2)^{-n/2-1} \cdot 2x_j$$

$$= (2-n)(x_1^2 + \cdots + x_n^2)^{-n/2} \left( 1 + x_j \cdot \left( -\frac{n}{2} \right) (x_1^2 + \cdots + x_n^2)^{-1} \cdot 2x_j \right)$$

$$= (2-n)(x_1^2 + \cdots + x_n^2)^{-n/2} \left( 1 - n \cdot \frac{x_j^2}{x_1^2 + \cdots + x_n^2} \right)$$
Now, summing over $j$, we get

$$
\Delta g(x) = \sum_{j=1}^{n} \frac{\partial^2 g}{\partial x_j^2} 
$$

$$
= \sum_{j=1}^{n} (2-n)(x_1^2 + \cdots + x_n^2)^{-n/2} \left( 1 - n \cdot \frac{x_j^2}{x_1^2 + \cdots + x_n^2} \right) 
$$

$$
= (2-n)(x_1^2 + \cdots + x_n^2)^{-n/2} \sum_{j=1}^{n} \left( 1 - n \cdot \frac{x_j^2}{x_1^2 + \cdots + x_n^2} \right) 
$$

$$
= (2-n)(x_1^2 + \cdots + x_n^2)^{-n/2} \left( n - n \cdot \sum_{j=1}^{n} \frac{x_j^2}{x_1^2 + \cdots + x_n^2} \right) 
$$

$$
= 0.
$$

(ii) As before, we compute the derivatives:

$$
\frac{\partial G}{\partial t} = \frac{1}{(4\pi)^{n/2}} \frac{-n}{2} \frac{1}{t^{n/2+1}} \exp \left( -\frac{\|x\|^2}{4t} \right) + \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{\|x\|^2}{4t} \right) \frac{\|x\|^2}{4t^2} 
$$

$$
= \frac{1}{(4\pi)^{n/2}} \frac{-n}{2} \frac{1}{t^{n/2+1}} \exp \left( -\frac{\|x\|^2}{4t} \right) \cdot \left( -\frac{n}{2} + \frac{\|x\|^2}{4t} \right) 
$$

$$
\frac{\partial G}{\partial x_i} = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{\|x\|^2}{4t} \right) \frac{-2x_i}{4t} 
$$

$$
\frac{\partial^2 G}{\partial x_i^2} = \frac{1}{(4\pi)^{n/2}} \frac{1}{t^{n/2+1}} \exp \left( -\frac{\|x\|^2}{4t} \right) \cdot \left( -\frac{1}{2} + \frac{x_i^2}{4t} \right) 
$$

Hence

$$
\Delta G = \sum_{i=1}^{n} \frac{\partial^2 G}{\partial x_i^2} = \sum_{i=1}^{n} \frac{1}{(4\pi)^{n/2}} \frac{1}{t^{n/2+1}} \exp \left( -\frac{\|x\|^2}{4t} \right) \cdot \left( -\frac{1}{2} + \frac{x_i^2}{4t} \right) 
$$

$$
= \frac{1}{(4\pi)^{n/2}} \frac{1}{t^{n/2+1}} \exp \left( -\frac{\|x\|^2}{4t} \right) \cdot \left( -\frac{n}{2} + \frac{\|x\|^2}{4t} \right) = \frac{\partial G}{\partial t}.
$$
**Problem 3:** Suppose that \( f(x, y) = \frac{1}{3}(x^3 + y^3) - x^2 - 2y^2 - 3x + 3y \). Find the critical points (i.e. the points where \( \nabla f = 0 \)) of \( f \), and discuss whether \( f \) has a local maximum/minimum at these points. (Justify any claims that you make by proof or by referring to a result proved in lecture.)

**Solution:** We compute the gradient of \( f \):

\[
\nabla f(x, y) = (x^2 - 2x - 3, y^2 - 4y + 3),
\]

which is 0 when \( x \in \{-1, 3\} \) and \( y \in \{1, 3\} \). So the four critical points are \((-1, 1), (-1, 3), (3, 1)\) and \((3, 3)\). Now, we compute the Hessian of \( f \), which is the matrix of second derivatives:

\[
Hf = \begin{pmatrix}
2x - 2 & 0 \\
0 & 2y - 4
\end{pmatrix}.
\]

- At the point \((-1, 1)\), this matrix is \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}, which is negative definite (because it has only negative eigenvalues). Hence \((-1, 1)\) is a local maximum.
- At the point \((-1, 3)\), this matrix is \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}, which is neither positive nor negative definite. Hence we can not conclude.
- At the point \((3, 1)\), this matrix is \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}, which is neither positive nor negative definite. Hence we can not conclude.
- At the point \((3, 3)\), this matrix is \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, which is positive definite (because it has only positive eigenvalues). Hence \((-1, 1)\) is a local minimum.

**Problem 4:** Prove that the degree \( n \) polynomials \( u, v \) obtained by taking the real and imaginary parts of \((x + iy)^n\) are harmonic: 

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \equiv 0
\]

on \( \mathbb{R}^2 \) in both cases \( p = u, p = v \). Hint: Thus \((x + iy)^n = u(x, y) + iv(x, y)\) where \( u, v \) are real-valued polynomials in the variables \( x, y \) (e.g. when \( n = 2 \), \( u(x, y) = x^2 - y^2 \) and \( v(x, y) = 2xy \)). Start by showing that 

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

**Solution:** Notice first that \( u \) and \( v \) are infinitely differentiable, since \( u \) and \( v \) are polynomials. The first part follows directly from the hint:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} = \frac{\partial}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial}{\partial y} \frac{\partial v}{\partial x} = 0,
\]
since \( v \) is \( C^2 \) and hence the order in which we take derivatives does not matter. A similar computation shows that \( v \) is also harmonic.

The second part is more interesting. These equations are called the Cauchy-Riemann equations; we will shortly explain where they come from. Given a function \( f : \mathbb{R}^2 \to \mathbb{C} \), we say that it is \textit{holomorphic} if it is derivable as a function of the complex variable \( z = x + iy \). In other words, the limit \( \lim_{w \to 0} \frac{f(z+w)-f(z)}{w} \) exists, where \( w \) is a complex number going to 0. This means that we can replicate all of calculus using complex numbers instead of real numbers; in particular, we can define a complex differential matrix. Hence the derivative should be linear as a map of vector space over the complex numbers, and so it should commute with the multiplication by \( i \). When we take such a holomorphic function \( f \), and just consider it as a function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) with components \( u \) and \( v \), its derivative matrix

\[
Df = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}
\]

should therefore commute with the matrix

\[
J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

of the multiplication by \( i \) (which is just a counterclockwise rotation of 90 degrees). When we write the equation \( JDf = DfJ \) in coordinates, we get that this is equivalent to the Cauchy-Riemann equations above. Anyways, you are welcome to come to my office hours if you want to hear more about this (although this is not math61cm material; it is covered in any course on complex analysis).

Let us compute the derivative of \( f(x, y) = (x + iy)^n \) in the \( x \)-direction:

\[
\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{(x+h+iy)^n - (x+iy)^n}{h} = n(x+iy)^{n-1},
\]

as can be seen after expanding the first term \((x+h+iy)^n = ((x+iy)+h)^n\). Similarly,

\[
\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{(x+i(y+t))^n - (x+iy)^n}{h} = ni(x+iy)^{n-1}.
\]

This means that

\[
\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.
\]

Notice that this is consistent with the paragraph above: the \( y \)-direction is \( i \) times the \( x \)-direction, and \( \frac{\partial f}{\partial y} \) is \( i \) times \( \frac{\partial f}{\partial x} \). Now, remembering that \( f = u + iv \), we can write

\[
\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}.
\]

Therefore

\[
\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = i \frac{\partial f}{\partial x} = i \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = -\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x}.
\]

Identifying the real and imaginary parts of this last equation, we get

\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.
\]

Notice that you can also probably compute explicit expressions for \( u \) and \( v \) by hand, and check the claimed relations explicitly.
Problem 5: A function \( f : U \to \mathbb{R} \) defined on an open set \( U \subset \mathbb{R}^m \) is called homogeneous if we have \( f(\lambda x) = \lambda^n f(x) \) for all \( \lambda \in \mathbb{R} \) and \( x \in \mathbb{R}^m \) such that both \( x \) and \( \lambda x \) are in \( U \). We say that \( f \) is locally homogeneous of degree \( n \) if for every \( x_0 \in U \) there exists \( r > 0 \) so that \( B(x_0, r) \subset U \) and \( f(\lambda x) = \lambda^n f(x) \) for all \( x \) such that \( \|x - x_0\| < r \) and \( \lambda x \in B(x_0, r) \).

(i) Prove that if \( U \) is a convex set then any locally homogeneous function is homogeneous.

(ii) Determine the degree of homogeneity of the following functions in their natural domains of definition:

\[
\begin{align*}
 f(x_1, \ldots, x_m) & = x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n, \\
 g(x_1, x_2, x_3) & = \frac{x_1 x_2 + x_2 x_3}{x_1 x_2 x_3}, \\
 p(x_1, x_2, x_3) & = x_1^2 x_2^2 x_3. 
\end{align*}
\]

(iii) Show that if \( f \) is homogeneous of degree \( n \) then it satisfies the identity

\[
x_1 \frac{\partial f(x)}{\partial x_1} + x_2 \frac{\partial f(x)}{\partial x_2} + \cdots + x_n \frac{\partial f(x)}{\partial x_n} = n f(x)
\]

Hint: differentiate the identity \( f(tx) = t^n f(x) \) with respect to \( t \in \mathbb{R} \).

(iv) Show that if a function \( f \) satisfies

\[
x_1 \frac{\partial f(x)}{\partial x_1} + x_2 \frac{\partial f(x)}{\partial x_2} + \cdots + x_n \frac{\partial f(x)}{\partial x_n} = n f(x)
\]

in a domain \( U \) then \( f \) is locally homogeneous in \( U \). Hint: show that the function \( \phi(t) = t^{-n} f(tx) \) is defined for each \( x \in U \) for \( t \) close to \( t = 1 \) and is a constant in an interval around \( t = 1 \).

Solution:

(i) Let \( y \in U \), and let \( I_y \) be the closed segment between 0 and \( y \). This is a compact subset of \( \mathbb{R}^m \), which is entirely contained in \( U \) as \( U \) is convex. For any point \( z \in I_y \), there exists an open ball \( B(z, r_z) \) around \( z \) such that \( f(\lambda x) = \lambda^n f(x) \) whenever both \( x \) and \( \lambda x \) are in \( B(z, r_z) \). The collection \( \{ B(z, r_z) \cap I_y \mid z \in I_y \} \) is therefore a covering of \( I_y \) by open sets (of \( I_y \)). By compactness of \( I_y \), there exists a finite number \( z_1, \ldots, z_N \) of points of \( I_y \) such that the associated balls cover \( I_y \). Take one ball that covers 0; then the relation is true for all \( x \) and \( \lambda x \) in the ball. Remove that ball from the interval; there must be another ball that covers the first point from that new interval, in which the relation must be satisfied. Furthermore, these two balls must intersect, by openness, hence the relation holds in the union of the two. Keep going this way, and we find that the relation holds for all \( x \) and \( \lambda x \) in \( I_y \). Since \( y \) was arbitrary to start with, this shows that \( f \) is homogeneous.

(ii) \( f \) is homogeneous of degree 2, \( g \) is homogeneous of degree \(-1\) and \( p \) is homogeneous of degree 150: just replace \( x \) by \( \lambda x \), and see what power of \( \lambda \) pops out when applying the function.
(iii) Consider the identity \( f(tx) = t^n f(x) \), which must be true for all \( t \) close enough to 1 by definition of homogeneity. Differentiate it with respect to \( t \): the right-hand side becomes \( nt^{n-1} f(x) \). For the left hand side, we need to use the chain rule.

Fix \( x = (x_1, \cdots, x_n) \), and define \( G : \mathbb{R} \rightarrow \mathbb{R}^n : t \mapsto (tx_1, \cdots, tx_n) \). Then, the function \( h : t \mapsto f(tx) \) is \( f \circ G \). We can differentiate it using the chain rule:

\[
Dh(t) = Df(G(t))DG(t).
\]

Here \( Df \) is the gradient of \( f \), i.e.

\[
Df(x) = \nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right).
\]

and \( DG(t) \) is the column matrix of partial derivatives of \( G \), i.e.

\[
DG(t) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.
\]

Multiplying them together, we get

\[
Dh(t) = \left( \frac{\partial f(G(x))}{\partial x_1}, \frac{\partial f(G(x))}{\partial x_2}, \ldots, \frac{\partial f(G(x))}{\partial x_n} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \frac{\partial f(tx)}{\partial x_1} + \cdots + x_n \frac{\partial f(tx)}{\partial x_n}.
\]

By the computation above, we get

\[
x_1 \frac{\partial f(tx)}{\partial x_1} + \cdots + x_n \frac{\partial f(tx)}{\partial x_n} = nt^{n-1} f(x).
\]

Taking \( t = 1 \), we get the relation we want.

(iv) The function \( \phi \) is certainly defined for \( t \) close to 1, and is therefore differentiable in both \( t \) and \( x \) since \( f \) is, since \( (x,t) \mapsto tx \) is and since division by \( t^n \) is. Derive it with respect to \( t \):

\[
\phi'(t) = -nt^{-n-1} f(tx) + t^{-n} \left( x_1 \frac{\partial f(tx)}{\partial x_1} + \cdots + x_n \frac{\partial f(tx)}{\partial x_n} \right),
\]

where we used the computation from part (iii) for the second term.

For \( t \) close to 1, we have

\[
tx_1 \frac{\partial f(tx)}{\partial x_1} + tx_2 \frac{\partial f(tx)}{\partial x_2} + \cdots + tx_n \frac{\partial f(tx)}{\partial x_n} = nf(tx)
\]

by assumption. Therefore,

\[
\phi'(t) = -nt^{-n-1} f(tx) + t^{-n}nt^{-1} f(tx) = 0.
\]

It follows that \( \phi \) is constant around \( t = 1 \), and therefore \( t^{-n} f(tx) = f(x) \) for all \( t \) close to 1. This means that \( f(tx) = t^n f(x) \) for all \( t \) close to 1, hence it is locally homogeneous.