Math 61CM - Solutions to homework 10

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**Problem 1:** Suppose $A, B$ are similar $n \times n$ matrices, that is, there is an $n \times n$ invertible matrix $C$ with $B = C^{-1}AC$. Prove that $A, B$ have the same eigenvalues.

**Hint:** Show that for all $\lambda \in \mathbb{R}$ we have $\det(B - \lambda I) \equiv \det(A - \lambda I)$.

**Solution:** We have

$$\det(B - \lambda I) = \det(C^{-1}AC - \lambda C^{-1}C) = \det(C^{-1}) \det(A - \lambda I) \det(C) = \det(A - \lambda I)$$

since $\det(C^{-1}) \det(C) = \det(C^{-1}C) = \det(I) = 1$. It follows that if $\lambda_0$ is an eigenvalue of $A$, then $\det(A - \lambda_0 I) = 0$, hence $\det(B - \lambda_0 I) = 0$ as well, which means that $\lambda_0$ is an eigenvalue of $B$. Conversely, if $\lambda_0$ is an eigenvalue of $B$, then it is also an eigenvalue of $A$.

**Problem 2:** Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a $C^1(U)$ map. Show that if $\det Df(x) \neq 0$ for each $x \in U$, then $f(U)$ is open.

**Solution:** This is basically the statement of the inverse theorem. Let $y_0 \in f(U)$; this means that there exists $x_0 \in U$ such that $f(x_0) = y_0$. Since $\det Df(x_0) \neq 0$ by assumption, the matrix $Df(x_0)$ is invertible, hence by the inverse function theorem there exists open sets $V$ and $W$ such that $x_0 \in V$, $y_0 \in W$ and $f$ is a bijective map from $V$ to $W$ with $C^1$ inverse. It follows in particular that $W$ is an open set that contains $y_0$ and which is in the image of $f$. Therefore, since $y_0$ was arbitrary, we conclude that $f(U)$ is open.

**Problem 3:** Let $U$ be the annular region $\{x \in \mathbb{R}^2 : 1/2 < \|x\| < 1\}$. Prove that $U$ is open and give an example of a $C^1$ function $f : U \to U$ such that $\det Df(x) > 0$ for every $x \in U$ and such that $f$ is not one-to-one on $U$.

**Solution:** Let $x_0 \in U$; this means that $1/2 < \|x_0\| < 1$. Let $\delta = \min(\|x_0\| - 1/2, 1 - \|x_0\|) > 0$. Consider $B(x_0, \delta)$. Then for every $x \in B(x_0, \delta)$, we have by the triangle inequality that

$$\|x\| \leq \|x_0\| + \|x_0 - x\| < \|x_0\| + \delta \leq 1,$$

$$1/2 \leq \|x_0\| - \delta < \|x_0\| - \|x - x_0\| \leq \|x\|,$$

proving that $B(x_0, \delta) \subseteq U$. Since $x_0$ was arbitrary, this proves that $U$ is open.
Remark 0.1 To find the δ above, the best way is to draw a picture ☺

The idea for the second part of the question is simply to wrap this annular region around itself twice. This is certainly a local orientation-preserving diffeomorphism (meaning det $Df(x) > 0$, cf the inverse function theorem and the definition of orientation that you have not learned yet), but is not one-to-one since each point is covered twice. Now, let us find a formula for it, and prove the claims.

This map is very easy to describe in polar coordinates: it’s the map $(r, \theta) \mapsto (r, 2\theta)$. In cartesian coordinates, this gives

$$(x, y) = (r \cos \theta, r \sin \theta) \mapsto (r \cos(2\theta), r \sin(2\theta)) = (r \cos^2 \theta - r \sin^2 \theta, 2r \sin \theta \cos \theta).$$

Notice that

$$r \cos^2 \theta - r \sin^2 \theta = \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r} = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$$

$$2r \sin \theta \cos \theta = \frac{2r^2 \sin \theta \cos \theta}{r} = \frac{2xy}{\sqrt{x^2 + y^2}}.$$

It follows that our map is

$$f : U \to U : (x, y) \mapsto \left(\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}}\right). \quad \text{(1)}$$

It follows from the description in polar coordinates that this map preserves the coordinate $r$, and hence it maps $U$ to $U$. Furthermore, it is certainly a $C^1$ function, since it is a quotient of rational functions, and the denominator stays bounded away from 0 because $\sqrt{x^2 + y^2} > 1/2$ by definition of $U$.

Now, $f$ is not one-to-one because $f(0.75,0) = (0.75,0)$ and $f(-0.75,0) = (0.75,0)$ as well. You can see this from the formulae above, or just from the fact that ”$f$ wraps $U$ around itself twice”. Finally, we need to check that det $Df(x) > 0$. We simply compute:

$$Df(x) = \begin{pmatrix}
\frac{x(x^2 + 3y^2)}{\sqrt{x^2 + y^2}} & \frac{y(3x^2 + y^2)}{\sqrt{x^2 + y^2}} \\
\frac{2y^3}{\sqrt{x^2 + y^2}} & \frac{2x^3}{\sqrt{x^2 + y^2}}
\end{pmatrix}.$$

It follows that

$$\det Df(x) = \frac{2x^4(x^2 + 3y^2)}{(x^2 + y^2)^3} + \frac{2y^4(3x^2 + y^2)}{(x^2 + y^2)^3} = \frac{2x^6 + 6x^4y^2 + 6x^2y^4 + 2y^6}{(x^2 + y^2)^3} = 2 > 0.$$

Remark 0.2 It is reassuring that the result is 2: since this map wraps $U$ around itself twice, it should multiply the volumes by 2. And indeed, $\det Df(x) = 2$, so it does (this will become more clear in MATH62CM - see the change of variable formula for integrals).

The point of this question is that the assumptions of the inverse function theorem apply here: there exists always a small open set in the domain and a small open set in the target such that $f$ induces a bijection between these two open sets. However, this bijection is not global, since $f$ is not a bijection from $U$ to $U$.

Remark 0.3 Another way we could have obtained the formula (1) for $f$ above is the following: we could consider the map $\mathbb{C} \to \mathbb{C} : z \mapsto z^2$, as this indeed wraps everything two times around, but it changes the absolute value (more precisely, it squares it). So, in order not to modify the absolute
value (so that $U$ gets mapped to itself), we can consider the complex function $\mathbb{C} \rightarrow \mathbb{C} : z \mapsto \frac{z^2}{|z|^2}$. This division by $|z|$ explains the denominator in (1). The numerator is explained by the fact that $z^2 = (x + iy)^2 = x^2 + 2ixy + (iy)^2 = x^2 - y^2 + 2ixy$: the real part is $x^2 - y^2$, and the imaginary part is $2xy$. These are exactly the numerators in (1).

**Problem 4:** Let $A$ be an $n \times n$ invertible matrix and let $E = \{x \in \mathbb{R}^n : \|Ax\| \leq 1\}$. Prove that $E$ is an ellipsoid: there exists an orthogonal matrix $Q$ and positive real numbers $\lambda_1, \ldots, \lambda_n$ so that

$$E = Q\{y : y \in \mathbb{R}^n \text{ and } \sum_{j=1}^n \lambda_j y_j^2 \leq 1\}.$$

**Hint:** Start by checking that $\|Ax\| \leq 1$ if and only if $(x \cdot (A^TAx)) \leq 1$, and think about applying the Spectral Theorem to $A^TA$.

**Solution:** We have

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^T(Ax) = x^TA^TAx = x \cdot (A^TAx),$$

hence $\|Ax\| \leq 1$ if and only if $(x \cdot (A^TAx)) \leq 1$.

Now, $A^TA$ is certainly a symmetric matrix since $(A^TA)^T = A^T(A^T)^T = A^TA$. Hence we can apply the spectral theorem, which says that there exists a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ and real numbers $\lambda_1, \ldots, \lambda_n$ such that $A^TAv_i = \lambda_i v_i$ for every $i$. Let $Q$ be the $n \times n$ matrix whose $i$th column is given by the coordinates of $v_i$ in the standard basis.

Notice that $Q$ is an orthonormal matrix: we have $Q^TQ = I$ since the $v_1, \ldots, v_n$ form an orthonormal basis of $\mathbb{R}^n$ by the spectral theorem. Indeed, the $(i, j)$-spot of $Q^TQ$ is given in coordinates by the inner product of $v_i$ with $v_j$. It follows in particular that $Q$ is invertible, since $Q^TQ = I$ implies $\det(Q)^2 = 1$, hence in particular $\det(Q) \neq 0$. Also, the equality $Q^TQ = I$ means that $Q^T = Q^{-1}$.

We claim furthermore that, in the standard basis of $\mathbb{R}^n$, we have

$$A^TA = QDQ^{-1},$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$ 

It suffices to check it on the basis $v_1, \ldots, v_n$. We have $Q^{-1}v_i = e_i$ by the definition of $Q$: the matrix $Q$ maps $e_i$ to $v_i$, hence $Q^{-1}$ maps $v_i$ to $e_i$. The matrix $D$ then maps $e_i$ to $\lambda_i e_i$, which itself is mapped back to $\lambda_i v_i$ by $Q$. It follows that

$$A^TAv_i = \lambda_i v_i = QDQ^{-1}v_i$$

for every $i$, which proves the identity above since $v_1, \ldots, v_n$ forms a basis of $\mathbb{R}^n$. 

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Now, notice that

\[ |Ax| \leq 1 \iff x^T A^T A x \leq 1 \]
\[ \iff x^T QDQ^{-1} x \leq 1 \]
\[ \iff y^T D y \leq 1 \]
\[ \iff \sum_{j=1}^{n} \lambda_j y_j^2 \leq 1 \]

where \( y = Q^{-1} x \) (recall that \( Q^{-1} = Q^T \)). It follows that

\[ x \in E \iff \|Ax\| \leq 1 \iff x = Qy \text{ for } y \in \{y : y \in \mathbb{R}^n \text{ and } \sum_{j=1}^{n} \lambda_j y_j^2 \leq 1\}, \]

which proves the claim.

**Remark 0.4** The idea of the proof above is to write an explicit change of basis for the matrix \( A \): the matrix \( D \) is the matrix of the linear transformation defined by \( A \) in the basis \( v_1, \ldots, v_n \). See any course on linear algebra for this.

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**Problem 5:** The ODE local existence and uniqueness theorem proved in lecture shows that there is \( \delta > 0 \) such that the ODE \( x' = x, \quad x(t_0) = x_0 \) has a unique \( C^1 \) solution on \([t_0 - \delta, t_0 + \delta]\). In fact, notice that the proof of the ODE local existence and uniqueness theorem shows that the solution is unique on \([t_0 - \delta'_1, t_0 + \delta'_2]\) for any \( 0 \leq \delta'_1, \delta'_2 \leq \delta \) (not both 0). (Existence on a smaller interval being an automatic consequence of existence on a larger!)

(i) Suppose that \( 0 \leq \delta_1, \delta_2 \) (and one of them > 0), where possibly \( \delta_1, \delta_2 > \delta, \delta \) given by the theorem. Show that there is at most one solution of the ODE on \([t_0 - \delta_1, t_0 + \delta_2]\).

**Hint:** Suppose \( x, y \) solve the ODE on this interval, and, say \( x(t) \neq y(t) \) for some \( t > t_0 \). Let \( T = \inf\{t > T_0 : x(t) \neq y(t)\} \). Now use the ODE existence and uniqueness theorem with initial data imposed at \( T \). Also, note that this part of the problem is valid for all ODEs to which the basic existence and uniqueness theorem applies.

(ii) By examining the proof of the theorem, show that \( \delta \) can be taken independent of \( t_0 \) (as well as \( x_0 \)). Use this to show that in fact \( x' = x, \quad x(0) = 1 \) has a unique solution \( x \in C^1(\mathbb{R}) \).

**Hint:** First solve on \([-\delta, \delta]\), then on \([0, 2\delta]\), etc.

(iii) Show that this solution is in fact \( C^\infty \), and its Taylor series around any point converges to it.

(iv) Show that the solution satisfies \( x(t + s) = x(t)x(s), \quad t, s \in \mathbb{R} \).

**Hint:** For fixed \( s \in \mathbb{R} \), show that both sides solve the same ODE with the same initial condition at \( t = 0 \).

(v) Calculate explicitly the iteration sequence given by the contraction mapping

\[ T x(t) = 1 + \int_0^t x(\tau) \, d\tau, \]

starting with \( x_0(t) = 1 \). (This is the iteration if you solve \( x' = x \) with \( x(0) = 1 \) and apply the proof of the contraction mapping theorem with \( x_0(t) = 1 - a \) reasonable first guess as it satisfies the initial condition.)
Solution:
(i) First of all, notice that for two continuous functions $f$ and $g$, the set \( \{ x \in \mathbb{R} \mid f(x) = g(x) \} \) is always closed, as it is \((f - g)^{-1}(0)\) and \(0\) is a closed set and \(f - g\) a continuous function. Therefore \( \{ x \in \mathbb{R} \mid f(x) \neq g(x) \} \) is open, and therefore it can not contain its infimum.

As suggested, suppose \( x, y \) solve the ODE on this interval (in particular they are continuous, since they are differentiable), and, say \( x(t) + y(t) \) for some \( t > t_0 \) (the case where \( x(t) - y(t) \) for \( t < t_0 \) is similar). Let \( T = \inf \{ t > t_0 : x(t) = y(t) \} \). It follows from that above that \( x(T) = y(T) \), otherwise the set \( \{ t > t_0 : x(t) = y(t) \} \) would not be open. Let us call \( a = x(T) = y(T) \). Now, \( x \) and \( y \) are also solutions of \( x' = x \) with \( x(T) = a \). By the uniqueness result of the theorem for ODEs, there two functions must be equal on a small interval \([T, T + \varepsilon]\) at least. But this contradicts the definition of \( T \). Therefore, there is at most one solution on \([t_0, t_0 + \delta_2]\). As we said above, the case about \([t_0 - \delta_1, t_0]\) is similar.

(ii) In the proof of the theorem, we see that it suffices that \( M\delta < 1 \) (page 69 of Lenya’s notes, third line), where \( M \) is given by the Lipshitz constant of \( f \). This does not depend on the point \( x_0 \). Recall by the way that this fact is used to prove that the operator \( A \) is a contraction; the fixed point theorem is then applied to \( A \) to find a solution of the ODE.

It then follows that it has a unique solution on \( \mathbb{R} \). First, the theorem tells use that it has a unique solution \( y_0 \) on \([-\delta, \delta]\). Then, apply the theorem again with the initial condition \( x(\delta) = y(\delta) \): it gives us a unique solution \( y_1 \) on \([0, 2\delta]\). Since both \( y_0 \) and \( y_1 \) are solutions on \([0, \delta]\), they have to be equal. It follows that the functions \( y_0 \) and \( y_1 \) define a function on \([-\delta, 2\delta]\) that is a solution to the ODE. Then, apply the theorem again with the initial condition \( x(2\delta) = y_1(2\delta) \): it gives us a unique solution \( y_2 \) on \([\delta, 3\delta]\). Keep doing this way, and also in the negatives, such that we have a solution on the whole of \( \mathbb{R} \).

(iii) The equation is \( x' = x \). By the theorem, we are assured that the solution is at least differentiable. But its derivative \( x' \) must be equal to \( x \), by the equation. So the derivative must be continuous and differentiable as well, meaning that \( x \) is twice differentiable. It follows by the equation that \( x'' = (x')' = x' = x \), so \( x'' \) must itself be continuous and differentiable, hence \( x \) is three times differentiable.

More generally, if \( x^{(k)} \) is the \( k \)th derivative of \( x \), we have that \( x^{(k)} = x \), hence it is continuous and differentiable, hence \( x \) is \( k + 1 \) times differentiable. It follows that \( x \) is \( C^\infty \).

Applying the Theorem 9.10 of the notes, we see that for any \( r > 0 \), the Taylor series of \( x \) converges to \( x \) on \((-r, r)\) for all \( r > 0 \). Indeed, all the derivatives of \( x \) at \( 0 \) are equal to \( 1 \), hence we need to check that for every \( r > 0 \), there exists \( C > 0 \) such that \( \frac{r^n}{n!} < C \) for all \( n \). Let us compute

\[
\frac{r^{n+1}/(n+1)!}{r^n/n!} = \frac{r^{n+1}n!}{r^n(n+1)!} = \frac{r}{n+1}.
\]

This is smaller than \( 1 \) as soon as \( n > r - 1 \). It follows that for fixed \( r \), the sequence \( r^n/n! \) attains its maximum around \( n = \lfloor r \rfloor \), and hence is bounded by some \( C > 0 \). So we can apply the Theorem 9.10 with any \( r > 0 \). It follows that the Taylor series converges to any \( x \in \mathbb{R} \). By the way, the Taylor series is \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) (I’m taking another variable, because the function itself is already called \( x \)).

(iv) Let \( f \) be a solution. Fix \( s \in \mathbb{R} \), and consider the function \( y_1(t) = f(t + s) \) and the function \( y_2(t) = f(t)f(s) \). Then these two functions are solutions of \( x' = x \) and \( x(0) = f(s) \). It follows from
the point (ii) that they must be equal for all \( t \in \mathbb{R} \). Since \( s \) was arbitrary, this proves the claimed relation.

(v) By the Theorem 11.2, the sequence \( x_0, T(x_0), T^2(x_0), \ldots \) should converge to a fixed point of \( T \) (where, by the way, \( T \) is the operator \( A \) from the proof of the existence and uniqueness theorem for ODEs). We compute:

\[
x_0(t) = 1 \\
x_1(t) = Tx_0(t) = 1 + \int_0^t 1 d\tau = 1 + t \\
x_2(t) = Tx_1(t) = 1 + \int_0^t (1 + \tau) d\tau = 1 + t + \frac{t^2}{2}
\]

Let us prove by induction that \( x_k(t) = \sum_{n=0}^{k} \frac{t^n}{n!} \). This is, by the above, true for \( k = 0, 1 \) and 2. Now assume it is true for \( k \), and we will prove it for \( k + 1 \). We have

\[
x_{k+1}(t) = T x_k(t) = 1 + \int_0^t x_k(\tau) d\tau = 1 + \int_0^t \sum_{n=0}^{k} \frac{\tau^n}{n!} d\tau = 1 + \sum_{n=0}^{k} \frac{1}{n!} \int_0^t \tau^n d\tau = 1 + \sum_{n=0}^{k} \frac{1}{n!} \frac{t^{n+1}}{n+1} = \sum_{n=0}^{k+1} \frac{t^n}{n!},
\]

as claimed.

**Remark 0.5** Notice that everything checks out: this sequence \( x_k(t) \) is just the sequence of the partial sums of the Taylor series of the solution, which indeed converges to the solution by the point (iv) above.