These are NOT complete solutions! You are always expected to write down all the details and justify everything. This document is meant to be a set of hints that you can use to solve the section problems.

1 Section 1

1. Yes: write a linear combination, and prove by hand that the coefficients must both be zero for the linear combination to be zero. For the second one: no, as $13\cdot(-2, -4)+2\cdot(13, 26) = 0$.

2. It is the $xz$-plane.

3. Write a linear combination, which has to be zero for every $x$. Evaluate at $x = 0$ and $x = \pi/2$ to prove that the coefficients must be zero.

4. Its span is not $\mathbb{R}$, because any linear combination of the $f_a$’s can be non zero at most at finitely many values of $x$, since linear combination are finite sums. Its span is the set of functions which are zero, except at finitely many points.

5. Write a linear combination, which has to be zero for every $x$. Evaluate at $x = a$ to show that the coefficient of $f_a$ must be zero.

6. A cover of $S \cup T$ must in particular cover $S$ and $T$. Use (HB) for $S$ and $T$ to extract finite subcollections that cover $S$ and $T$. Then, the union is a finite subcollection covering $S \cup T$.

7. Yes: apply the previous question twice.

8. No, the cover $\{(x - 1, x + 1) \mid x \in \mathbb{R}\}$ does not have any finite subcover.

9. No, the cover $\{(0,1/n) \mid n \geq 1\} \cup \{(0.5, 1.5)\}$ does not have any finite subcover. (The last set is to make sure we cover 1).

10. No, the cover $\{(-1/n,0) \cup (0,1/n) \mid n \geq 1\} \cup \{(0.5, 1.5), (-1.5, -0.5)\}$ does not have any finite subcover. (The last two sets is to make sure we cover $-1$ and 1).

11. A union or intersection of inductive sets is always inductive: each set contains 1, hence so does their union/intersection. Also, each set contains $x + 1$ for any $x$ in the set, hence so does the union/intersection.
12. Yes, a bijection is given by \( f(n) = 2n \).

13. Yes: write each rational as an irreducible fraction \( p/q \). Then place these as points in the first quadrant of the \( pq \)-plane (certain points will be missed, as they correspond to reducible fractions). Then draw a zigzag, starting from 0, and expanding towards the north-east.

14. Yes, just shift everyone the hotel up one room (so the person in room \( n \) goes to the room \( n + 1 \)). This leaves the room 1 empty for the traveller. If the traveller has 9 friends, just shift everyone up ten rooms; then the rooms 1 to 10 are empty.

15. Yes: ask everyone in your hotel to go to the room having twice their current room number. This leaves all the odd rooms empty; ask the people staying in the hotel that burnt to go to the room have twice their old room number minus 1.

16. Cover using the covering \( \{(x - 1, x + 1) \mid x \in \mathbb{R}\} \). It must have a finite subcover, indexed by \( x_1, \ldots, x_N \). Pick \( x_j \) with the largest absolute value; then your set must be contained in \((−|x_j|−1,|x_j|+1)\), hence it is bounded.

2 Section 2

1. Use Gauss elimination until you can express two variables in terms of the other ones. Then write the general form of the solution using only the "other ones". For example, I get

\[
\left( \frac{1}{2}y - w, y, -2y, w, 0 \right) = y \left( \frac{1}{2}, 1, -2, 0, 0 \right) + w(-1, 0, 0, 1, 0).
\]

These two vectors form a basis of the solution set: by the above their span is the solution set, and it’s easy to check that they are linearly independent.

2. \(|x + 2y + 3z| \leq \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \) when \((x, y, z)\) is on the sphere. Furthermore, the equality happens only when \((x, y, z) = \lambda(1, 2, 3)\) for some \(\lambda\). The fact that \((x, y, z)\) must be on the sphere fixed \(\lambda\), hence we know it is maximized at \(\frac{1}{\sqrt{14}}(1, 2, 3)\).

3. Suppose that \(a\sqrt{2} + b\sqrt{3} = 0\) for some rational numbers \(a\) and \(b\). First, show that we can assume that \(a\) and \(b\) are integer, and relatively prime. Then find a contradiction (like in the proof that \(\sqrt{2}\) is irrational) if \(a\) and \(b\) are not both zero.

4. (a) We have to show that the right-hand side is an upper bound for \(S \cup T\); this is easy. To show that it is the lowest lower bound, assume that there is a lower one; then it must be smaller than sup \(S\) or sup \(T\), which is not possible.

(b) The right-hand side is certainly an upper bound for the intersection: anything in \(S \cap T\) must be either in \(S\) or in \(T\). So it has to be larger than the supremum, since the supremum is the lowest upper bound.

(c) \(S = \{0, 1\}\) and \(T = \{0, 2\}\).
5. Given $\varepsilon > 0$, we need to show that $|s_n t_n|$ becomes smaller than $\varepsilon$. Since $t_n$ is bounded, $|t_n| < B$ for some large $B$ and for every $n$. Then, pick $N$ in the definition of $\lim s_n = 0$ that makes $|s_n|$ smaller than $\varepsilon/B$ for $n \geq N$. Then, for $n \geq N$, $|s_n t_n|$ must be smaller than $\varepsilon$.

6. $(\lambda u + \mu v)^n + (\lambda u + \mu v) = \lambda(u^n + u) + \mu(v^n + v) = 0$ if $u$ and $v$ are in the set. Two linearly independent elements are $\sin(x)$ and $\cos(x)$ (see section 1).

7. (a) Take the sequence $1, 3, 14, 1, 3, 14, 1, 3, 14, 1, 3, 14, \cdots$.
   (b) Take the sequence $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, \cdots$.
   (c) See question 13 of section 1.
   (d) Let $f : N \rightarrow \mathbb{Q}$ be the bijective map of the previous question. Then take $x_n = f(n)$. This sequence contains all the rational numbers. So, given a real number, there exists a rational closer than $1/2$ away from it. Then there exists a rational further in the sequence closer than $1/3$ from it, etc. This somewhat uses problem 7 of homework 2.

8. (a) Just a computation. Bound $h(n)$ by some $\alpha$, and apply the hint.
   (b) $g(n) = g(1)^n = e^n$ by point (a). Also, $g(p) = g(p/q)^q$, hence $g(p/q) = g(p)^1/q = e^{p/q}$, proving it for rational numbers. Since $e^x$ is continuous, if $g$ also is, then it must be equal to $e^x$. Indeed, two continuous functions coinciding on a dense set must coincide everywhere.
   (c) Write
   \[ g(x) - 1 = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n - 1 = \lim_{n \to \infty} \frac{x}{n} \left(\left(1 + \frac{x}{n}\right)^{n-1} + \cdots + \left(1 + \frac{x}{n}\right)^0\right), \]
   and notice that an $x$ appears. Bound the rest by $g(x)$.

3 Section 3

1. It follows from $\langle \lambda v + \mu w, s \rangle = \lambda \langle v, s \rangle + \mu \langle w, s \rangle$, which is zero for all $s \in S$ if $v, w \in S^\perp$.

2. Something in the span of $S$ must be orthogonal to all the vectors which are orthogonal to the vectors in $S$ (this is almost tautological).

3. See the proof of Lemma 7.7.

4. See the proof of Lemma 7.8.

5. The points 3 and 4 together imply that the union of a basis of $W$ and of a basis of $W^\perp$ is a basis (by 3 it’s linearly independent, and by 4 it’s generating). So the dimension of $W^\perp$ must be $n - k$.

6. By 2, we know that $\text{span}(S) \subseteq (S^\perp)^\perp$. Notice that $\text{span}(S)^\perp = S^\perp$. If the dimension of $\text{span}(S)$ is $k$, then the dimension of $S^\perp$ is $n - k$, so the dimension of $(S^\perp)^\perp$ is $n - (n - k) = k$. So we have an inclusion of a $k$-dimensional vector space into a $k$-dimensional vector space. This implies that they must be equal.

3
7. Just draw a plane (through 0) and a line (through 0) that’s orthogonal to it.

8. For $\delta = \frac{1}{n}$, compute $f(x + 1/n) - f(x)$, and notice that if $x$ is not bounded, this will never always be smaller than a fixed $\varepsilon$, no matter how small $\delta$ is (i.e. how large $n$ is).

9. By the calculation of the previous point, if we can bound $x$, we can always find $n$ that makes $f(x + 1/n) - f(x)$ smaller than $\varepsilon$ for every $x$ in that bounded set. Also, see Theorem 4.19 in the notes.

10. Any point $(x, y)$ in that rectangle must have a positive distance to the boundary, which you can write explicitly in terms of the distance of $x$ to $a$ and $b$ and the distance of $y$ to $b$ and $c$.

11. Same as before, but for every point outside the rectangle (using Theorem 5.6 in the notes).

12. If $U$ is a union of balls, then any point in $U$ is in one of the balls, and since the balls are open, we can find a smaller ball around that point contained in the ball. Conversely, if $U$ is open, then for every $x$ in $U$ there is a small ball $B_x$ around $x$ that is included in $U$. Show then that $U = \bigcup_{x \in U} B_x$.

13. An open one: $\mathbb{R}^2 \setminus \{0\}$. A non-open one: $\{(x, 0) \mid x > 0\}$.

4 Section 4

1. (a) The $j^{th}$ column of $\text{rref}A$ is $\text{rref}A(e_j)$. Similarly, the $j^{th}$ column of $A$ is $A(e_j)$.

(b) The $k^{th}$ column of $\text{rref}A$ can be written as a linear combination $\sum_{i=1}^{Q} c_{ik} f_i$ of the first $Q$ columns. This is clear from the form of the matrix $\text{rref}A$.

(c) The $j^{th}$ column of $\text{rref}A$ is therefore $\text{rref}A(e_j) = \sum_{i=1}^{Q} c_{ik} f_i$, which is equal to $\sum_{i=1}^{Q} c_{ik} \text{rref}A(e_i)$ since $\text{rref}A(e_i) = f_i$ for $i \leq Q$. So

$$\text{rref}A(e_j) = \sum_{i=1}^{Q} c_{ik} \text{rref}A(e_i),$$

hence,

$$\text{rref}A \left( e_j - \sum_{i=1}^{Q} c_{ik} e_i \right) = 0.$$

Since $\text{Ker} (\text{rref}A) = \text{Ker} (A)$, this implies

$$A \left( e_j - \sum_{i=1}^{Q} c_{ik} e_i \right) = 0,$$

which means that $A(e_j)$ is a linear combination of $A(e_i)$ for $i \leq Q$. 

4
2. Similar argument: if a linear combination of $A(e_i)$ is zero, then since $\text{Ker (rref}_A) = \text{Ker (}_A\)$, the same linear combination of $\text{rref}_A(e_i)$ is zero. But these are equal to $f_i$, which are linearly independent.

3. $C(A)$ has a basis formed by $Q$ columns. The other columns of rref$A$ correspond to free variables, which form a basis of the nullspace. So $C(A) + N(A)$ is the total number of columns of $A$, which is $n$.

4. Just plug in.

5. Take a solution, substract $x_0$ from it, and notice that the result is in $\text{Ker (}_A\)$. 

6. Do as if you wanted to find $\text{Ker (}_A\)$, except that you add a column to remember the coefficients on the right-hand side.

7. (a) For any $x$, we have $\lim_{y \to 0} f(x, y) = 0$. Similarly, for any $y$, we have $\lim_{x \to 0} f(x, y) = 0$. So, the iterated limits are both 0.

(b) $f(x, x) = 1/2$ for every $x$, so the limit as $x$ goes to 0 is also $1/2$. This proves that there are points arbitrarily close to 0 on which $f$ takes value $1/2$. By the previous point, it’s also the case for points where $f$ equals 0. So the limit can not exist.

8. We get $f(r \cos \theta, r \sin \theta) = \frac{\sin(2\theta)}{2}$. Notice that $\theta = 0$ corresponds to the $x$-axis, where $f = 0$. Also, $\theta = \pi/2$ corresponds to the $y$-axis, where $f$ is also 0. And $\theta = \pi/4$ corresponds to the diagonal $x = y$, where we know that $f = 1/2$. So this formula recovers the values of the previous point.

9. We get 

$$f(r \cos \theta, r \sin \theta) = \frac{1}{1 + r \cos \theta}$$

after simplifying, which goes to 1 as $r \to 0$ (independently of the value of $\theta$). Proving it in terms of $\varepsilon$ and $\delta$ is just a rephrasing of this argument.

5 Section 5

1. If the columns are linearly, express one of them as a linear combination of the others. Then use multilinearity of $\mathcal{D}$, and see that each term of this sum is 0 because $\mathcal{D}$ has a repeated entry. For the rows, since $\det A = \det A^T$, we can reduce to the previous point, as the rows of $A$ are the columns of $A^T$.

2. Use multilinearity and the fact that if a matrix has a repeated column, its determinant is 0. For the rows, use $\det A = \det A^T$.

3. Add 100 times the first column and 10 times the second column to the third column. Then, you see the numbers 195, 247 and 403 appear in the third column. Factor out a 13 using multilinearity of the determinant, to get that this determinant is 13 times the determinant of a matrix with integer coefficients.
4. (a) If $\sigma = \tau_1 \cdots \tau_k$, then $\sigma^{-1} = \tau_k \cdots \tau_1$ (this uses the fact that the inverse of a transposition is the transposition itself). So if we can write $\sigma$ as the product of $k$ transpositions, then we can also write $\sigma^{-1}$ as the product of $k$ transpositions.

(b) Just use the definition of the determinant, and the fact that the $ij$-entry of $A$ is the $ji$-entry of $A^T$.

(c) Basically, replace each permutation in the sum for $\det A$ by its inverse. Since we sum over all permutations, we also sum over all their inverses, so it does not change the sum. Now, notice that the sum you get is exactly the sum for $\det A^T$. See page 67 of Simon’s book for more details.

5. (a) Just check that the three conditions of Definition 7.1 in the notes are satisfied. You may have to break up into cases, depending on whether or not $x$ and $y$ lie on the same line through the origin.

(b) If $p$ is the origin, then it’s a ball of radius $r$ centered at the origin. Otherwise, suppose that $p$ is not the origin. If $r$ is smaller than the Euclidean distance from $r$ to the origin, then it’s a segment of total length $2r$ centered at $p$ on the line through $p$ and the origin. Otherwise, if $a$ is the Euclidean distance between $p$ and the origin, then it’s a ball of radius $r - a$ through the origin, along with a segment from the origin, going through $p$, and ending at distance $r$ from $p$.

This is called the Parisian distance because the stereotype (which is true to a certain extend) is that in France, in order to take the train from anywhere to anywhere else, you have to go through Paris. Then this distance is the distance between cities if you can only take the train.

6. Take $\varepsilon = 1/2$. Then for any $\delta > 0$, the point $x + \delta/2$ is at distance $\delta/2$ of $x$ (for the Euclidean distance on the domain), but its image is a point which is different from $x$, hence its distance (in the image) from $x$ is $1$, which is larger than $\varepsilon = 1/2$.

7. (a) If the interval is $[0, 1]$, $f$ must achieve its minimum $m > 0$. Then any function $g$ such that $\|f - g\|_\infty < m$ must also be positive, as its minimum is at least $m - \|f - g\|_\infty$ by definition of the norm $\|\cdot\|_\infty$. So we found an open ball of positive radius around $f$ in $A$. Since $f$ was arbitrary, this means that $A$ is open.

(b) Now, on $(0, 1)$, the infimum could be 0, so the previous argument does not work. Notice that $f(x) = x$ is in $B$, but for any $r > 0$, the function $g(x) = x - r$ is at distance $r$ of $f$, but is not in $B$. So the complement of $B$ is not closed, hence $B$ is not open.

8. Take a function $f_n$ as in the hint, with a straight line from $(1/2, 1)$ to $(1/2 + 1/n, 0)$. Then it is easy to see that this is a Cauchy sequence for the norm $\|\cdot\|_1$, but the only function it could converge to in the norm $\|\cdot\|_\infty$ would be the function which is 1 on $[0, 1/2]$ and 0 on $(1/2, 1]$, which is not a continuous function, and hence does not exist in $E$. Notice that this sequence is not a Cauchy sequence for the norm $\|\cdot\|_\infty$ since given $n$, we can always find $m$ much larger such that $\|f_n - f_m\|_\infty > 1/2$. 

6
6 Section 6

1. The inverse is
   \[
   \begin{pmatrix}
   3/4 & 1/2 & 1/4 \\
   1/2 & 1 & 1/2 \\
   1/4 & 1/2 & 3/4 \\
   \end{pmatrix}.
   \]

2. The inverse of this matrix is
   \[
   \frac{1}{ad - bc} \begin{pmatrix}
   d & -b \\
   -c & a \\
   \end{pmatrix}.
   \]
   Write \( k = ad - bc \). In order for this matrix to have integer coefficient, we need \( k \) to divide \( a, b, c \) and \( d \). Write \( a = a'k \), \( b = b'k \), \( c = c'k \) and \( d = d'k \), we get \( k = (a'd' - b'c')k^2 \), so \( (a'd' - b'c')k = 1 \). The only divisors of 1 in \( \mathbb{Z} \) are 1 and -1, so we need \( k = +1 \) or \( k = -1 \). And if \( k = \pm 1 \), then the inverse above certainly has integer coefficients.

3. \( B \) is invertible, because exchanging two rows does not change \( N(A) \), which is \( \{0\} \). And \( B^{-1} \) is \( A^{-1} \) with the first two columns exchanged. You can see this from either method above: in the first one, just play with the coefficients; in the second one, since we only perform row operations, they don’t modify the columns, so it’s easy to keep track of the right part of the matrix. But a maybe easier way is the following: \( B = CA \), where \( C \) is the transformation that permutes \( e_1 \) and \( e_2 \). So \( B^{-1} = A^{-1}C^{-1} = A^{-1}C \) since \( C \) is its own inverse. And this last equation means that \( e_1 \) is mapped to \( e_2 \) by \( C \), which is mapped to the second column of \( A \) by \( A \), and similarly \( e_2 \) is mapped to the first column of \( A \). This means that we interchanged the first two columns of \( A \).

4. (a) Take the basis \( \{1, x, x^2\} \) of that subspace of polynomials. The second basis vector \( x \) is already orthogonal to 1. Then, applying Gram-Schmidt to \( x^2 \) gives \( x^2 - \frac{1}{3} \). Dividing by their lengths, we get \( \frac{1}{\sqrt{2}} \), \( \frac{3}{\sqrt{2}} \), \( \frac{3}{\sqrt{8}} (x^2 - \frac{1}{3}) \).
   
   (b) Just use the Lemma 5.2 in the book, since now we have an orthonormal basis.

5. (a) It is equal to the identity matrix.

   (b) We have \( QQ^T = I \), so \( Q^T \) is a right inverse for \( Q \). But right inverses of \( n \times n \) matrices are also left inverses (homework 4, problem 3), so \( Q^TQ = I \). So \( (Q^T)(Q^T)^T = I \), which means that the rows of \( Q^T \) are orthonormal. But the rows of \( Q^T \) are the columns of \( Q \), so we are done.

6. Just draw on the picture the graph of \( f \), and a bunch of rectangles over \([n, n+1]\) of height \( a_n = f(n) \). Compare the two to see that if the sequence converges, then so does the integral: the area of the rectangles is larger than the area under the graph. For the other implication, draw a bunch of rectangles over \([n-1, n]\) of height \( a_n = f(n) \), and notice that the area of the rectangles is smaller than the area under the graph.

7. By the previous problem, this is equivalent to the convergence of \( \int_1^\infty x^{-p} \, dx = \left[ \frac{-1}{(p-1)x^{p-1}} \right]_1^\infty \), which converges only if \( p > 1 \) (so that the denominators goes to \( \infty \) if \( x \to \infty \)).
8. Using the fact that \( \lim_{n \to \infty} \frac{\sqrt{n}}{n} = 1 \), we see that \( R = 1 \). So by the Theorem this converges absolutely for \( x < -2 \), and it diverges for \( x \in (\infty, -2) \cup (0, \infty) \). If \( x = 0 \) then the series is \( \sum_n 1/n \), which diverges. If \( x = -2 \) then the series is \( \sum_n (-1)^n/n \) which converges (homework 6, problem 3.(i)) but not absolutely.

7 Section 7

1. \( \det(T - \lambda I) \) is a polynomial in \( \lambda \), whose roots are 1, 2 and 5. The nullspace of \( T - 1 \cdot I \) is generated by \((1, -1, 1)\), the nullspace of \( T - 2 \cdot I \) is generated by \((1, 0, 1)\) and the nullspace of \( T - 5 \cdot I \) is generated by \((1, 1, 2)\).

2. The spectral theorem gives us an orthonormal basis \( \{f_i\} \) of eigenvectors. Define \( S f_i = f_i \), and extend by linearity. It suffices to check that it is an isometry on basis vectors, and we have \( \delta_{ij} = \langle f_i, f_j \rangle = \langle S f_i, S f_j \rangle \) and \( \delta_{ij} = \langle e_i, e_j \rangle \), so it’s ok.

3. The spectral theorem gives us an orthonormal basis \( \{f_i\} \) of eigenvectors. Notice that \( 0 \leq \langle Tf_i, f_i \rangle = \langle \lambda_i f_i, f_i \rangle = \lambda_i \|f_i\|^2 = \lambda_i \), where the first inequality follows from the fact that \( T \) is semi-positive definite. So the \( \lambda_i \)'s are positive; define \( S \) by \( S f_i = \sqrt{\lambda_i} f_i \).

4. \( Df = \begin{pmatrix} 2x & -2 & 4z^3 \\ y \cos(xy) + e^z & x \cos(xy) & xe^z \end{pmatrix} \). At \( x_0 = (1, 2, 3) \), this is

\[
Df(1, 2, 3) = \begin{pmatrix} 2 & -2 & 108 \\ 2 \cos(2) + e^3 & \cos(2) & e^3 \end{pmatrix}.
\]

Using the theorem, the directional derivative of \( f \) at \( x_0 = (1, 2, 3) \) in the direction \( v = (-2, 3, 1) \) is

\[
Df(1, 2, 3) \cdot (-2, 3, 1)^T = \begin{pmatrix} 98 \\ -\cos(2) - e^3 \end{pmatrix}.
\]

5. We get \( \nabla f(x) = \frac{x}{\|x\|} \). This is a radial vector field of norm 1.

6. Let \( v \) be tangent to \( f^{-1}(c) \) at \( x_0 \); this means that \( f(x_0) = c \) and that the directional derivative of \( f \) at \( x_0 \) in the direction \( v \) is 0. By the theorem, this is equal to \( Df(x_0)v \). But \( Df(x_0) = \nabla f(x_0) \) since \( f \) takes values in \( \mathbb{R} \), so this means that the inner product of the gradient and \( v \) is 0. Since this holds for any \( v \) tangent to \( f^{-1}(c) \), this means that \( \nabla f(x_0) \) is orthogonal to \( f^{-1}(c) \).

7. For \( f(x) = \|x\| \), we get that \( f^{-1}(c) \) is a sphere centered at 0 of radius \( c \). And \( \nabla f \), which is radial, is indeed orthogonal to all spheres centered at 0.
8 Section 8

1. (a) 

\[ 3x^2 - 12xy + xz + 7y^2 = 3 \left( x - 2y + \frac{z}{6} \right)^2 - 5y^2 - \frac{z^2}{12} + 2yz \]

\[ = 3 \left( x - 2y + \frac{z}{6} \right)^2 - 5 \left( y - \frac{z}{5} \right)^2 - \frac{z^2}{12} + \frac{z^2}{5} \]

\[ = 3 \left( x - 2y + \frac{z}{6} \right)^2 - 5 \left( y - \frac{z}{5} \right)^2 + \frac{7z^2}{60} \]

So in coordinates \( x' = x - 2y + \frac{z}{6}, y' = y - \frac{z}{5} \) and \( z' = z \), this quadratic form is

\[ 3x'^2 - 5y'^2 + \frac{7}{60}z'^2. \]

The coordinates \( x, y, z \) are in the standard basis \( e_1, e_2, e_3 \), and the coordinates \( x', y', z' \) in some other basis \( u_1, u_2, u_3 \). Inverting the relations above, we have \( x = x' + 2y' + \frac{17}{30}z', y = y' + \frac{2z'}{5}, z = z' \). Since we must have

\[ xe_1 + ye_2 + ze_3 = x'u_1 + y'u_2 + z'u_3, \]

taking \( x' = 1 \) and \( y' = z' = 0 \), we find \( u_1 = e_1 \). With \( y' = 1 \) and \( x' = z' = 0 \), we find \( u_2 = 2e_1 + e_2 \). And with \( z' = 1 \) and \( x' = y' = 0 \), we get \( u_3 = \frac{17}{30}e_1 + \frac{1}{5}e_3 + e_3 \). In that basis, the quadratic form is diagonal.

(b) Similar.

2. Let \( h : \mathbb{R}^2 \to \mathbb{R}^2 : (r, \theta) \mapsto (r \cos \theta, r \sin \theta) \). By the chain rule, we have to compute

\[ D_{(1,1)}(F \circ h)(\sqrt{2}, \pi/4) = DF(h(\sqrt{2}, \pi/4))Dh(\sqrt{2}, \pi/4) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ = DF(1, 1)Dh(\sqrt{2}, \pi/4) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ = (1 \ 1) \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ = \sqrt{2}, \]

since \( Dh = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} \).

9 Section 9

1. Let us construct it iteratively. \( a_1 \neq 1 \) means that we remove the central open third of the interval \([0, 1]\); let us call \( C_1 \) the limit. Then \( a_2 \neq 1 \) means that we remove the central open third of each of the two intervals that \( C_1 \) is made of; call the result \( C_2 \). Then \( a_3 \neq 1 \) means that again we remove the central open third of each interval that \( C_2 \) is made of. And we keep going. Then \( \mathcal{C} = \bigcap_n C_n \).
2. Since each $C_n$ is closed, their intersection is closed.

3. See solutions to the homework. It is uncountable because it can be put in bijection with infinite sequences composed of 0’s and 1’s, which are uncountable by a diagonal argument. It has measure 0 because it is contained in $C_n$ for every $n$, which has measure $(2/3)^n$, which goes to 0 as $n$ goes to $\infty$.

4. If $x$ has two expansions, one of them ends with a tail of 2’s. Then its image ends with a tail of 1’s. We can also compute the image of the expansion that does not end with a tail of 2’s; its image will end with a tail of 0’s. Comparing the two expansions directly, we see that they define the same number.

5. Let $x, y \in [0, 1]$. If $x > y$, there must be an $N$ such that $x_N > y_N$. If there are 1’s before the $N^{th}$ spot, then the images coincide by definition of $f$. Otherwise, by inspection of the three cases $(x_N = 1, y_N = 0), (x_N = 2, y_N = 0)$ and $(x_N = 2, y_N = 1)$, we see that $f(x) \geq f(y)$.

It is also clearly onto; an inverse of each number can be constructed explicitly. Finally, a function that is monotone and onto must be continuous (use question 1 of homework 9, for example).

6. An interval of the complement of the Cantor set corresponds to all the numbers that have a 1 at a fixed place in their expansion, and no 1 before that. By definition of the function, it is constant on each interval.

7. Notice that $g$ is continuous, since $f$ is. Furthermore, it is strictly increasing, since $f$ is non-decreasing and $x \mapsto x$ is strictly increasing. Hence, $g$ is injective. Also, $g(0) = 0$ and $g(1) = 2$, so by the intermediate value theorem it must be surjective.

8. We must show that the image of an open set is an open set, or equivalently that the image of a closed set is a closet set. Take a closed set of $[0, 1]$. It is closed and bounded, hence it is compact. So its image is also compact because $g$ is continuous, hence it is closed, and we are done.

To see that the image of a compact set if compact: cover the image by open sets; take the preimage of everything. Then the preimages of the open sets cover the compact set, so there is a finite subcover. The corresponding open sets in the image will then cover the image of the compact set, which is therefore compact because we started with an arbitrary cover.

Remark For the remark, just wrap $[0, 1)$ around a circle. It is bijective and continuous, but the inverse is not continuous because it "cuts" the circle.