Chapter 1

Maximum principle and the symmetry of solutions of elliptic equations

1.1 Act I. The maximum principle enters

We will have several main characters in this chapter: the maximum principle and the sliding and moving plane methods. The maximum principle and sliding will be introduced separately, and then blended together to study the symmetry properties of the solutions of elliptic equations. In this introductory section, we recall what the maximum principle is. This material is very standard and can be found in almost any undergraduate or graduate PDE text, such as the books by Evans [60], Han and Lin [82], and Pinchover and Rubinstein [121].

We will consider equations of the form

$$\begin{aligned} &\Delta u + F(x,u) = 0 \text{ in } \Omega, \\ &u = g \text{ on } \partial\Omega. \end{aligned} \tag{1.1.1}$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is its boundary. There are many applications where such problems appear. We will mention just two – one is in the realm of probability theory, where $u(x)$ is an equilibrium particle density for some stochastic process, and the other is in classical physics. In the physics context, one may think of $u(x)$ as the equilibrium temperature distribution inside the domain Ω. The temperature flux is proportional to the gradient of the temperature – this is the Fourier law, which leads to the term Δu in the overall heat balance (1.1.1). The term $F(x,u)$ corresponds to the heat sources or sinks inside Ω, while $g(x)$ is the (prescribed) temperature on the boundary $\partial\Omega$. The maximum principle reflects a basic observation known to any child – first, if $F(x,u) = 0$ (there are neither heat sources nor sinks), or if $F(x,u) \le 0$ (there are no heat sources but there may be heat sinks), the temeprature inside Ω may not exceed that on the boundary – without a heat source inside a room, you can not heat the interior of a room to a warmer temperature than its maximum on the boundary. The second observation is that if one considers two prescribed boundary conditions and heat sources such that

$$g_1(x) \le g_2(x) \text{ and } F_1(x,u) \le F_2(x,u),$$

then the corresponding solutions will satisfy $u_1(x) \le u_2(x)$ – stronger heating leads to warmer rooms. It is surprising how such mundane considerations may lead to beautiful mathematics.

The maximum principle in complex analysis

Most mathematicians first encounter the maximum principle in a complex analysis course. Recall that the real and imaginary parts of an analytic function $f(z)$ have the following property.

Proposition 1.1.1 *Let $f(z) = u(z) + iv(z)$ be an analytic function in a smooth bounded domain $\Omega \subset \mathbb{C}$, continuous up to the boundary Ω. Then $u(z) = Ref(z)$, $v(z) = Imf(z)$ and $w(z) = |f(z)|$ all attain their respective maxima over Ω on its boundary. In addition, if one of these functions attains its maximum inside Ω, it has to be equal identically to a constant in Ω.*

This proposition is usually proved via the mean-value property of analytic functions (which itself is a consequence of the Cauchy integral formula): for any disk $B(z_0, r)$ contained in Ω we have

$$f(z_0) = \int_0^{2\pi} f(z_0+re^{i\theta})\frac{d\theta}{2\pi},\;\; u(z_0) = \int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi},\;\; v(z_0) = \int_0^{2\pi} v(z_0+re^{i\theta})\frac{d\theta}{2\pi}, \quad (1.1.2)$$

and, as a consequence,

$$w(z) \le \int_0^{2\pi} w(z_0 + re^{i\theta})\frac{d\theta}{2\pi}. \quad (1.1.3)$$

It is immediate to see that (1.1.3) implies that if one of the functions u, v and w attains a local maximum at a point z_0 inside Ω, it has to be equal to a constant in a disk around z_0. Thus, the set where it attains its maximum is both open and closed, hence it is all of Ω and this function equals identically to a constant.

The above argument while incredibly beautiful and simple, relies very heavily on the rigidity of analytic functions that is reflected in the mean-value property. The same rigidity is reflected in the fact that the real and imaginary parts of an analytic function satisfy the Laplace equation

$$\Delta u = 0, \quad \Delta v = 0,$$

while $w^2 = u^2 + v^2$ is subharmonic: it satisfies

$$\Delta(w^2) \ge 0.$$

We will see next that the mean-value principle is associated to the Laplace equation and not analyticity in itself, and thus applies to harmonic (and, in a modified way, to subharmonic) functions in higher dimensions as well. This will imply the maximum principle for solutions of the Laplace equation in an arbitrary dimension. One may ask whether a version of the mean-value property also holds for the solutions of general elliptic equations rather than just for the Laplace equation – the answer is "yes if understood properly": the mean value property survives as the general elliptic regularity theory, an equally beautiful sister of the complex analysis which is occasionally misunderstood as "technical".

Interlude: a probabilistic connection digression

Apart from the aforementioned connection to physics and the Fourier law, a good way to understand how the Laplace equation comes about, as well as many of its properties, including the maximum principle, is via its connection to the Brownian motion. It is easy to understand in terms of the discrete equations, which requires only very elementary probability theory. Consider a system of many particles on the n-dimensional integer lattice $\mathbb{Z}^n$. They all perform a symmetric random walk: at each integer time $t = k$ each particle jumps (independently from the others) from its current site $x \in \mathbb{Z}^n$ to one of its $2n$ neighbors, $x \pm e_k$ (e_k is the unit vector in the direction of the x_k-axis), with equal probability $1/(2n)$. At each step we may also insert new particles, the average number of inserted (or eliminated) particles per unit time at each site is $F(x)$. Let now $u_m(x)$ be the average number of particles at the site x at time m. The balance equation for $u_{m+1}(x)$ is

$$u_{m+1}(x) = \frac{1}{2n}\sum_{k=1}^{n}[u_m(x+e_k) + u_m(x-e_k)] + F(x). \tag{1.1.4}$$

Exercise 1.1.2 Derive (1.1.4) by considering how particles may appear at the position x at the time $m+1$ – they either jump from a neighbor, or are inserted.

If the system is in an equilibrium, so that $u_{m+1}(x) = u_m(x)$ for all x, then $u(x)$ (dropping the subscript m) satisfies the discrete equation

$$\frac{1}{2n}\sum_{k=1}^{n}[u(x+e_k) + u(x-e_k) - 2u(x)] + F(x) = 0.$$

If we now take a small mesh size h, rather than have particles jump be of size one, the above equation becomes

$$\frac{1}{2n}\sum_{k=1}^{n}[u(x+he_k) + u(x-he_k) - 2u(x)] + F(x) = 0.$$

A Taylor expansion in h leads to

$$\frac{h^2}{2n}\sum_{k=1}^{n}\frac{\partial^2 u(x)}{\partial x_k^2} + F(x) = \text{lower order terms.}$$

Taking the source of the form $F(x) = h^2/(2n)G(x)$ – the small factor h^2 prevents us from inserting or removing too many particles, we arrive, in the limit $h \downarrow 0$, at

$$\Delta u + G(x) = 0. \tag{1.1.5}$$

In this model, we interpret $u(x)$ as the local particle density, and $G(x)$ as the rate at which the particles are inserted (if $G(x) > 0$), or removed (if $G(x) < 0$). When equation (1.1.5) is posed in a bounded domain Ω, we need to supplement it with a boundary condition, such as

$$u(x) = g(x) \text{ on } \partial\Omega.$$

This boundary condition means the particle density on the boundary is prescribed – the particles are injected or removed if there are "too many" or "too little" particles at the boundary, to keep $u(x)$ at the given prescribed value $g(x)$.

The mean value property for sub-harmonic and super-harmonic functions

We now return to the world of analysis. A function $u(x)$, $x \in \Omega \subset \mathbb{R}^n$ is harmonic if it satisfies the Laplace equation

$$\Delta u = 0 \text{ in } \Omega. \tag{1.1.6}$$

This is equation (1.1.1) with $F \equiv 0$, thus a harmonic function describes a heat distribution in Ω with neither heat sources nor sinks in Ω. We say that u is sub-harmonic if it satisfies

$$-\Delta u \leq 0 \text{ in } \Omega, \tag{1.1.7}$$

and it is super-harmonic if it satisfies

$$-\Delta u \geq 0 \text{ in } \Omega, \tag{1.1.8}$$

In other words, a sub-harmonic function satisfies

$$\Delta u + F(x) = 0, \text{ in } \Omega,$$

with $F(x) \leq 0$ – it describes a heat distribution in Ω with only heat sinks present, and no heat sources, while a super-harmonic function satisfies

$$\Delta u + F(x) = 0, \text{ in } \Omega,$$

with $F(x) \geq 0$ – it describes an equilibrium heat distribution in Ω with only heat sources present, and no sinks.

Exercise 1.1.3 Give an interpretation of the sub-harmonic and super-harmonic functions in terms of particle probability densities.

Note that any sub-harmonic function in one dimension is convex:

$$-u'' \leq 0,$$

and then, of course, for any $x \in \mathbb{R}$ and any $l > 0$ we have

$$u(x) \leq \frac{1}{2}\left(u(x+l) + u(x-l)\right), \text{ and } u(x) \leq \frac{1}{2l}\int_{x-l}^{x+l} u(y)dy.$$

The following generalization to sub-harmonic functions in higher dimensions shows that locally $u(x)$ is bounded from above by its spatial average. A super-harmonic function will be locally above its spatial average. A word on notation: for a set S we denote by $|S|$ its volume, and, as before, ∂S denotes its boundary.

Theorem 1.1.4 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $B(x,r)$ be a ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ contained in Ω. Assume that the function $u(x)$ is sub-harmonic, that is, it satisfies*

$$-\Delta u \leq 0, \tag{1.1.9}$$

for all $x \in \Omega$ and that $u \in C^2(\Omega)$. Then we have

$$u(x) \le \frac{1}{|B(x,r)|}\int_{B(x,r)} u dy, \quad u(x) \le \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u dS. \tag{1.1.10}$$

Next, suppose that the function $u(x)$ is super-harmonic:

$$-\Delta u \ge 0, \tag{1.1.11}$$

for all $x \in \Omega$ and that $u \in C^2(\Omega)$. Then we have

$$u(x) \ge \frac{1}{|B(x,r)|}\int_{B(x,r)} u dy, \quad u(x) \ge \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u dS. \tag{1.1.12}$$

Moreover, if the function u is harmonic: $\Delta u = 0$, then we have equality in both inequalities in (1.1.10).

One reason to expect the mean-value property is from physics – if Ω is a ball with no heat sources, it is natural to expect that the equilibrium temperature in the center of the ball may not exceed the average temperature over any sphere concentric with the ball. The opposite is true if there are no heat sinks (this is true for a super-harmonic function). Another explanation can be seen from the discrete version of inequality (1.1.9):

$$u(x) \le \frac{1}{2n}\sum_{j=1}^{n}(u(x+he_j) + u(x-he_j)).$$

Here, h is the mesh size, and e_j is the unit vector in the direction of the coordinate axis for x_j. This discrete equation says exactly that the value $u(x)$ is smaller than the average of the values of u at the neighbors of the point x on the lattice with mesh size h, which is similar to the statement of Theorem 1.1.4 (though there is no meaning to "nearest" neighbor in the continuous case).

Proof. We will only consider a sub-harmonic function, the super-harmonic functions are treated identically. Let us fix the point $x \in \Omega$ and define

$$\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(z) dS(z). \tag{1.1.13}$$

It is easy to see that, since $u(x)$ is continuous, we have

$$\lim_{r \downarrow 0} \phi(r) = u(x). \tag{1.1.14}$$

Therefore, we would be done if we knew that $\phi'(r) \ge 0$ for all $r > 0$ small enough so that that the ball $B(x,r)$ is contained in Ω. To this end, passing to the polar coordinates $z = x + ry$, with $y \in \partial B(0,1)$, we may rewrite (1.1.13) as

$$\phi(r) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+ry) dS(y).$$

Then, differentiating in r gives

$$\phi'(r) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} y \cdot \nabla u(x+ry)dS(y).$$

Going back to the z-variables leads to

$$\phi'(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{1}{r}(z-x)\cdot\nabla u(z)dS(z) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}dS(z).$$

Here, we used the fact that the outward normal to $B(x,r)$ at a point $z \in \partial B(x,r)$ is

$$\nu = (z-x)/r.$$

Using Green's formula

$$\int_U \Delta g dy = \int_U \nabla\cdot(\nabla g)dy = \int_{\partial U}(\nu\cdot\nabla g)dS = \int_{\partial U}\frac{\partial g}{\partial \nu}dS,$$

gives now

$$\phi'(r) = \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \Delta u(y)dy \ge 0.$$

It follows that $\phi(r)$ is a non-decreasing function of r, and then (1.1.14) implies that

$$u(x) \le \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} udS, \tag{1.1.15}$$

which is the second identity in (1.1.10).

In order to prove the first equality in (1.1.10) we use the polar coordinates once again:

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} udy = \frac{1}{|B(x,r)|}\int_0^r\left(\int_{\partial B(x,s)} udS\right)ds \ge \frac{1}{|B(x,r)|}\int_0^r u(x)n\alpha(n)s^{n-1}ds$$
$$= u(x)\alpha(n)r^n\frac{1}{\alpha(n)r^n} = u(x).$$

We used above two facts: first, the already proved identity (1.1.15) about averages on spherical shells, and, second, that the area of an $(n-1)$-dimensional unit sphere is $n\alpha(n)$, where $\alpha(n)$ is the volume of the n-dimensional unit ball. Now, the proof of (1.1.10) is complete. The proof of the mean-value property for super-harmonic functions works identically. □

The maximum principle for the Laplacian

The first consequence of the mean value property is the maximum principle that says that a sub-harmonic function attains its maximum over any domain on the boundary and not inside the domain. From the physical point of view this is, again, obvious – a sub-harmonic function is nothing but the heat distribution in a room without heat sources, hence it is very natural that it attains its maximum on the boundary (the walls of the room). In one dimension this claim is also familiar: a sub-harmonic function of a one-dimensional variable is convex, and, of course, a smooth convex function does not have any local maxima.

Theorem 1.1.5 (**The maximum principle**) *Let $u(x)$ be a sub-harmonic function in a connected domain Ω and assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then*

$$\max_{x\in\bar{\Omega}} u(x) = \max_{y\in\partial\Omega} u(y). \tag{1.1.16}$$

Moreover, if $u(x)$ achieves its maximum at a point x_0 in the interior of Ω, then $u(x)$ is identically equal to a constant in Ω. Similarly, if $v \in C^2(\Omega) \cap C(\bar{\Omega})$ is a super-harmonic function in Ω, then

$$\min_{x\in\bar{\Omega}} v(x) = \min_{y\in\partial\Omega} v(y), \tag{1.1.17}$$

and if $v(x)$ achieves its minimum at a point x_0 in the interior of Ω, then $v(x)$ is identically equal to a constant in Ω.

Proof. Again, we only treat the case of a sub-harmonic function. Suppose that $u(x)$ attains its maximum at an interior point $x_0 \in \Omega$, and set

$$M = u(x_0).$$

Then, for any $r > 0$ sufficiently small (so that the ball $B(x_0, r)$ is contained in Ω), we have

$$M = u(x) \leq \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u dy \leq M,$$

with the equality above holding only if $u(y) = M$ for all y in the ball $B(x_0, r)$. Therefore, the set S of points where $u(x) = M$ is open. Since $u(x)$ is continuous, this set is also closed. Since S us both open and closed in Ω, and Ω is connected, it follows that $S = \Omega$, hence $u(x) = M$ at all points $x \in \Omega$. □

We should note the particularly simple proof above only applies to the Laplacian itself but the maximum principle applies to much more general elliptic operators than the Laplacian. In particular, already in this chapter, we will deal with slightly more general operators than the Laplacian, of the form

$$Lu = \Delta u(x) + c(x)u. \tag{1.1.18}$$

In order to anticipate that this issue is not totally trivial, consider the following exercise.

Exercise 1.1.6 Consider the boundary value problem

$$-u'' - au = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

with a given non-negative function $f(x)$, and a constant $a \geq 0$. Show that if $a < \pi^2$, then the function $u(x)$ is positive on the interval $(0, 1)$.

The reader may observe that $a = \pi^2$ is the leading eigenvalue of the operator $Lu = -u''$ on the interval $0 < x < 1$ with the boundary conditions $u(0) = u(1) = 0$. This transition will be generalized to much more general operators later on.

1.2 Act II. The moving plane method

1.2.1 The isoperimeteric inequality and sliding

We now bring in our second set of characters, the moving plane and sliding methods. As an introduction, we show how the sliding method can work alone, without the maximum principle. Maybe the simplest situation when the sliding idea proves useful is in an elegant proof of the isoperimetric inequality given by X. Cabré in [30] (see also [31]). The isoperimetric inequality says that among all domains of a given volume the ball has the smallest surface area.

Theorem 1.2.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Then,*

$$\frac{|\partial\Omega|}{|\Omega|^{(n-1)/n}} \geq \frac{|\partial B_1|}{|B_1|^{(n-1)/n}}, \tag{1.2.1}$$

where B_1 is the open unit ball in $\mathbb{R}^n$, $|\Omega|$ denotes the measure of Ω and $|\partial\Omega|$ is the perimeter of Ω (the $(n-1)$-dimensional measure of the boundary of Ω). In addition, equality in (1.2.1) holds if and only if Ω is a ball.

A technical aside: the area formula

The proof will use the area formula, a generalization of the usual change of variables formula in the multi-variable calculus. The latter says that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth one-to-one map (a change of variables), then

$$\int_{\mathbb{R}^n} g(x)Jf(x)dx = \int_{\mathbb{R}^n} g(f^{-1}(y))dy. \tag{1.2.2}$$

Here, Jf is the Jacobian of the map f:

$$Jf(x) = \left|\det\left(\frac{\partial f_i}{\partial x_j}\right)\right|.$$

For general maps we have

Theorem 1.2.2 *(Area formula) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each function $g \in L^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} g(x)Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x\in f^{-1}\{y\}} g(x)\right] dy. \tag{1.2.3}$$

Note that if f is Lipschitz then it is differentiable almost everywhere by the Rademacher theorem [61], thus the Jacobian is defined almost everywhere as well. We will not prove the area formula here – see [61] for the proof. We will use the following corollary.

Corollary 1.2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each measurable set $A \subset \mathbb{R}^n$ we have*

$$|f(A)| \leq \int_A Jf(x)dx. \tag{1.2.4}$$

Proof. For a given set S we define its characteristic function as

$$\chi_S(x) = \begin{cases} 1, & \text{for } x \in S, \\ 0, & \text{for } x \notin S, \end{cases}$$

We use the area formula with $g(x) = \chi_A(x)$:

$$\begin{aligned} \int_A Jf(x)dx &= \int_{\mathbb{R}^n} \chi_A(x)Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} \chi_A(x) \right] dy \\ &= \int_{\mathbb{R}^n} [\#x \in A : \ f(x) = y]\, dy \geq \int_{\mathbb{R}^n} \chi_{f(A)}(y)dy = |f(A)|, \end{aligned}$$

and we are done. □

A more general form of this corollary is the following.

Corollary 1.2.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with the Jacobian Jf. Then, for each nonnegative function $p \in L^1(\mathbb{R}^n)$ and each measurable set A, we have*

$$\int_{f(A)} p(y)dy \leq \int_A p(f(x))Jf(x)dx. \tag{1.2.5}$$

Proof. The proof is as in the previous corollary. This time, we apply the area formula to the function $g(x) = p(f(x))\chi_A(x)$:

$$\begin{aligned} &\int_A p(f(x))Jf(x)dx = \int_{\mathbb{R}^n} \chi_A(x)p(f(x))Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} \chi_A(x)p(f(x)) \right] dy \\ &= \int_{\mathbb{R}^n} [\#x \in A : \ f(x) = y]\, p(y)dy \geq \int_{f(A)} p(y)dy, \end{aligned}$$

and we are done. □

The proof of the isoperimetric inequality

We now proceed with Cabré's proof of the isoperimetric inequality in Theorem 1.2.1.

Step 1: sliding. Let $v(x)$ be the solution of the Neumann problem

$$\begin{aligned} &\Delta v = k, \quad \text{in } \Omega, \\ &\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega. \end{aligned} \tag{1.2.6}$$

Here, ν is the outward normal at the boundary. Integrating the first equation above and using the boundary condition, we obtain

$$k|\Omega| = \int_\Omega \Delta v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = |\partial\Omega|.$$

Hence, solution exists only if

$$k = \frac{|\partial\Omega|}{|\Omega|}. \tag{1.2.7}$$

It is a classical result (see [79], for example) that with this particular value of k there exist infinitely many solutions that differ by addition of an arbitrary constant. We let v be any of them. As Ω is a smooth domain, v is also smooth.

Let Γ_v be the lower contact set of v, that is, the set of all $x \in \Omega$ such that the tangent hyperplane to the graph of v at x lies below that graph in all of $\bar{\Omega}$. More formally, we define

$$\Gamma_v = \{x \in \Omega : \ v(y) \geq v(x) + \nabla v(x) \cdot (y - x) \text{ for all } y \in \bar{\Omega}.\} \tag{1.2.8}$$

The crucial observation is that

$$B_1 \subset \nabla v(\Gamma_v). \tag{1.2.9}$$

Here, B_1 is the open unit ball centered at the origin.

Exercise 1.2.5 Explain why (1.2.9) is trivial in one dimension.

The geometric reason for this is as follows: take any $p \in B_1$ and consider the graphs of the functions

$$r_c(y) = p \cdot y + c.$$

We will now slide this plane upward – we will start with a "very negative" c, and start increasing it, moving the plane up. Note that there exists $M > 0$ so that if $c < -M$, then

$$r_c(y) < v(y) - 100 \text{ for all } y \in \bar{\Omega},$$

that is, the plane is below the graph in all of Ω. On the other hand, possibly after increasing M further, we may ensure that if $c > M$, then

$$r_c(y) > v(y) + 100 \text{ for all } y \in \bar{\Omega},$$

in other words, the plane is above the graph in all of Ω. Let then

$$\alpha = \sup\{c \in \mathbb{R} : \ r_c(y) < v(y) \text{ for all } y \in \bar{\Omega}\} \tag{1.2.10}$$

be the largest c so that the plane lies below the graph of v in all of Ω. It is easy to see that the plane $r_\alpha(y) = p \cdot y + \alpha$ has to touch the graph of v: there exists a point $y_0 \in \bar{\Omega}$ such that $r_\alpha(y_0) = v(y_0)$ and

$$r_\alpha(y) \leq v(y) \text{ for all } y \in \bar{\Omega}. \tag{1.2.11}$$

Furthermore, the point y_0 can not lie on the boundary $\partial\Omega$ since $|p| < 1$. Indeed, for all $y \in \partial\Omega$ we have

$$\left|\frac{\partial r_c}{\partial \nu}\right| = |p \cdot \nu| \leq |p| < 1 \text{ and } \frac{\partial v}{\partial \nu} = 1.$$

This means that if $r_c(y) = v(y)$ for some c, and y is on the boundary $\partial\Omega$, then there is a neighborhood $U \in \Omega$ of y such that $r_c(y) > v(y)$ for all $y \in U$. Comparing to (1.2.11), we see that $c \neq \alpha$, hence it is impossible that $y_0 \in \partial\Omega$. Thus, y_0 is an interior point of Ω, and, moreover, the graph of $r_\alpha(y)$ is the tangent plane to v at y_0. In particular, we have $\nabla v(y_0) = p$, and (1.2.11) implies that y_0 is in the contact set of v: $y_0 \in \Gamma_v$. We have now shown the inclusion (1.2.9): $B_1 \subset \nabla v(\Gamma_v)$. Note that the only information about the function $v(x)$ we have used so far is the Neumann boundary condition

$$\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega,$$

but not the Poisson equation for v in Ω.

Step 2: using the area formula. A trivial consequence of (1.2.9) is that

$$|B_1| \leq |\nabla v(\Gamma_v)|. \tag{1.2.12}$$

Now, we will apply Corollary 1.2.3 to the map $\nabla v: \Gamma_v \to \nabla v(\Gamma_v)$. The Jacobian of this map is $|\det[D^2 v]|$.

Exercise 1.2.6 Show that if Γ_v is the contact set of a smooth function $v(x)$, then $\det[D^2 v]$ is non-negative for $x \in \Gamma_v$, and, moreover, all eigenvalues of $D^2 v$ are nonnegative on Γ_v.

As $\det[D^2 v]$ is non-negative for $x \in \Gamma_v$, we conclude from Corollary 1.2.3 and (1.2.12) that

$$|B_1| \leq |\nabla v(\Gamma_v)| \leq \int_{\Gamma_v} \det[D^2 v(x)]dx. \tag{1.2.13}$$

It remains to notice that by the classical arithmetic mean-geometric mean inequality applied to the (nonnegative) eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix $D^2 v(x)$, $x \in \Gamma_v$ we have

$$\det[D^2 v(x)] = \lambda_1 \lambda_2 \dots \lambda_n \leq \left(\frac{\lambda_1 + \lambda_2 + \dots + \lambda_n}{n}\right)^n. \tag{1.2.14}$$

However, by a well-known formula from linear algebra,

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = \mathrm{Tr}[D^2 v],$$

and, moreover, $\mathrm{Tr}[D^2 v]$ is simply the Laplacian Δv. This gives

$$\det[D^2 v(x)] \leq \left(\frac{\mathrm{Tr}[D^2 v]}{n}\right)^n = \left(\frac{\Delta v}{n}\right)^n \text{ for } x \in \Gamma_v. \tag{1.2.15}$$

Recall that v is the solution of (1.2.6):

$$\begin{aligned} &\Delta v = k, \quad \text{in } \Omega, \\ &\frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega. \end{aligned} \tag{1.2.16}$$

with

$$k = \frac{|\partial\Omega|}{|\Omega|}.$$

Going back to (1.2.13), we deduce that

$$|B_1| \leq \int_{\Gamma_v} \det[D^2 v(x)]dx \leq \int_{\Gamma_v} \left(\frac{\Delta v}{n}\right)^n dx \leq \left(\frac{k}{n}\right)^n |\Gamma_v| = \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Gamma_v| \leq \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Omega|.$$

In addition, for the unit ball we have $|\partial B_1| = n|B_1|$, hence the above implies

$$\frac{|\partial B_1|^n}{|B_1|^{n-1}} \leq \frac{|\partial\Omega|^n}{|\Omega|^{n-1}}, \tag{1.2.17}$$

which is nothing but the isoperimetric inequality (1.2.1).

In order to see that the inequality in (1.2.17) is strict unless Ω is a ball, we observe that it follows from the above argument that for the equality to hold in (1.2.17) we must have equality in (1.2.14), and, in addition, Γ_v has to coincide with Ω. This means that for each $x \in \Omega$ all eigenvalues of the matrix $D^2v(x)$ are equal to each other. That is, $D^2v(x)$ is a multiple of the identity matrix for each $x \in \Omega$.

Exercise 1.2.7 Show that if $v(x)$ is a smooth function such that

$$\frac{\partial^2 v(x)}{\partial x_i^2} = \frac{\partial^2 v(x)}{\partial x_j^2},$$

for all $1 \leq i, j \leq n$ and $x \in \Omega$, and

$$\frac{\partial^2 v(x)}{\partial x_i \partial x_j} = 0,$$

for all $i \neq j$ and $x \in \Omega$, then there exists $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$, so that

$$v(x) = b\left[(x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_n - a_n)^2\right] + c, \tag{1.2.18}$$

for all $x \in \Omega$.

Our function $v(x)$ does satisfy the assumptions of Exercise 1.2.7, hence it must be of the form (1.2.18). Finally, the boundary condition $\partial v/\partial\nu = 1$ on $\partial\Omega$ implies that Ω is a ball centered at the point $a \in \mathbb{R}^n$. $\Box$

1.3 Act III. Their first meeting

The maximum principle returns, and we study it in a slightly greater depth. At the end of this act the maximum principle and the moving plane method are introduced to each other.

The Hopf lemma and the strong maximum principle

We now generalize the maximum principle to slightly more general operators than the Laplacian, to allow for a zero-order term. Let us begin with the following exercises.

Exercise 1.3.1 Show that if the function $u(x)$ satisfies an ODE of the form

$$u'' + c(x)u = 0, \quad a < x < b, \tag{1.3.1}$$

and $u(x_0) = 0$ for some $x_0 \in (a, b)$, and the function $c(x)$ is continuous on $[a, b]$, then u can not attain its maximum (or minimum) over the interval (a, b) at the point x_0 unless $u \equiv 0$.

This exercise is relatively easy – one has to think about the initial value problem for (1.3.1) with the data $u(x_0) = u'(x_0) = 0$. Now, look at the next exercise, which is slightly harder.

Exercise 1.3.2 Show that, once again, in one dimension, if $u(x)$, $x \in \mathbb{R}$ satisfies a differential inequality of the form

$$u'' + c(x)u \geq 0, \quad a < x < b,$$

the function $c(x)$ is continuous on $[a, b]$, and $u(x_0) = 0$ for some $x_0 \in (a, b)$ then u can not attain its maximum over the interval (a, b) at the point x_0 unless $u \equiv 0$.

The proof of the strong maximum principle relies on the Hopf lemma which guarantees that the point on the boundary where the maximum is attained is not a critical point of u.

Theorem 1.3.3 *(The Hopf Lemma) Let $B = B(y, r)$ be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$, and assume that $c(x) \leq 0$ in B. Suppose that a function $u \in C^2(B) \cap C(B \cup x_0)$ is a sub-solution, that is, it satisfies*

$$\Delta u + c(x)u \geq 0 \text{ in } B,$$

and that $u(x) < u(x_0)$ for any $x \in B$ and $u(x_0) \geq 0$. Then, we have $\dfrac{\partial u}{\partial \nu}(x_0) > 0$.

Proof. We may assume without loss of generality that B is centered at the origin: $y = 0$. We may also assume that $u \in C(\bar{B})$ and that $u(x) < u(x_0)$ for all $x \in \bar{B} \backslash \{x_0\}$ – otherwise, we would simply consider a smaller ball $B_1 \subset B$ that is tangent to B at x_0.

The idea is to modify u to turn it into a strict sub-solution of the form

$$w(x) = u(x) + \varepsilon h(x).$$

We also need w to inherit the other properties of u: it should attain its maximum over $\bar{B}$ at x_0, and we need to have $w(x) < w(x_0)$ for all $x \in B$. In addition, we would like to have

$$\frac{\partial h}{\partial \nu} < 0 \text{ on } \partial B,$$

so that the inequality

$$\frac{\partial w}{\partial \nu}(x_0) \geq 0$$

would imply

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

An appropriate choice is

$$h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2},$$

in a smaller domain

$$\Sigma = B \cap B(x_0, r/2).$$

Observe that $h > 0$ in B, $h = 0$ on ∂B (thus, h attains its minimum on ∂B – unlike u which attains its maximum there), and, in addition:

$$\begin{aligned}\Delta h + c(x)h &= e^{-\alpha|x|^2}\left[4\alpha^2|x|^2 - 2\alpha n + c(x)\right] - c(x)e^{-\alpha r^2} \\ &\geq e^{-\alpha|x|^2}\left[4\alpha^2|x|^2 - 2\alpha n + c(x)\right] \geq e^{-\alpha|x|^2}\left[4\alpha^2\frac{|r|^2}{4} - 2\alpha n + c(x)\right] > 0,\end{aligned}$$

for all $x \in \Sigma$ for a sufficiently large $\alpha > 0$. Hence, we have a strict inequality

$$\Delta w + c(x) w > 0, \text{ in } \Sigma, \tag{1.3.2}$$

for all $\varepsilon > 0$. Note that $w(x_0) = u(x_0) \geq 0$, thus the maximum of w over Σ is non-negative. Suppose that w attains this maximum at an interior point x_1, and $w(x_1) \geq 0$. As $\Delta w(x_1) \leq 0$ and $c(x_1) \leq 0$, it follows that

$$\Delta w(x_1) + c(x_1) w(x_1) \leq 0,$$

which is a contradiction to (1.3.2). Thus, w may not attain a non-negative maximum inside Σ but only on the boundary. We now show that if $\varepsilon > 0$ is sufficiently small, then w attains this maximum only at x_0. Indeed, as $u(x) < u(x_0)$ in B, we may find δ, so that

$$u(x) < u(x_0) - \delta \text{ for } x \in \partial\Sigma \cap B.$$

Take ε so that

$$\varepsilon h(x) < \delta \text{ on } \partial\Sigma \cap B,$$

then

$$w(x) < u(x_0) = w(x_0) \text{ for all } x \in \partial\Sigma \cap B.$$

On the other hand, for $x \in \partial\Sigma \cap \partial B$ we have $h(x) = 0$ and

$$w(x) = u(x) < u(x_0) = w(x_0).$$

We conclude that $w(x)$ attains its non-negative maximum in $\bar{\Sigma}$ at x_0 if ε is sufficiently small. This implies

$$\frac{\partial w}{\partial \nu}(x_0) \geq 0,$$

and, as a consequence

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial h}{\partial \nu}(x_0) = \varepsilon \alpha r e^{-\alpha r^2} > 0.$$

This finishes the proof. □

The next theorem is an immediate consequence of the Hopf lemma.

Theorem 1.3.4 *(The strong maximum principle) Assume that $c(x) \leq 0$ in Ω, and the function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\Delta u + c(x) u \geq 0,$$

and attains its maximum over $\bar{\Omega}$ at a point x_0. In this case, if $u(x_0) \geq 0$, then $x_0 \in \partial\Omega$ unless u is a constant. If the domain Ω has the internal sphere property, and $u \not\equiv const$, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Proof. Let $M = \sup_{\bar{\Omega}} u(x)$ and define the set $\Sigma = \{x \in \Omega : \ u(x) = M\}$, where the maximum is attained. We need to show that either Σ is empty or $\Sigma = \Omega$. Assume that Σ is non-empty but $\Sigma \neq \Omega$, and choose a point $p \in \Omega \backslash \Sigma$ such that

$$d_0 = d(p, \Sigma) < d(p, \partial\Omega).$$

Consider the ball $B_0 = B(p, d_0)$ and let $x_0 \in \partial B_0 \cap \partial\Sigma$. Then we have

$$\Delta u + c(x)u \geq 0 \text{ in } B_0,$$

and

$$u(x) < u(x_0) = M,\ M \geq 0 \text{ for all } x \in B_0.$$

The Hopf Lemma implies that

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the normal to B_0 at x_0. However, x_0 is an internal maximum of u in Ω and hence $\nabla u(x_0) = 0$. This is a contradiction. □

Now, we may state the strong comparison principle – note that we do not make any assumptions on the sign of the function $c(x)$ here.

Theorem 1.3.5 *(The strong comparison principle) Assume that $c(x)$ is a bounded function, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\Delta u + c(x)u \geq 0. \tag{1.3.3}$$

If $u \leq 0$ in Ω then either $u \equiv 0$ in Ω or $u < 0$ in Ω. Similarly, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\Delta u + c(x)u \leq 0 \text{ in } \Omega, \tag{1.3.4}$$

with $u \geq 0$ in Ω, with a bounded function $c(x)$. Then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

Proof. If $c(x) \leq 0$, this follows directly from the strong maximum principle. In the general case, as $u \leq 0$ in Ω, the inequality (1.3.3) implies that, for any $M > 0$ we have

$$\Delta u + c(x)u - Mu \geq -Mu \geq 0.$$

However, if $M > \|c\|_{L^\infty(\Omega)}$ then the zero order coefficient satisfies

$$c_1(x) = c(x) - M \leq 0,$$

hence we may conclude, again from the strong maximum principle that either $u < 0$ in Ω or $u \equiv 0$ in Ω. The proof in the case (1.3.4) holds is identical. □

Separating sub- and super-solutions

A very common use of the strong maximum principle is to re-interpret it as the "untouchability" of a sub-solution and a super-solution of a linear or nonlinear problem – the basic principle underlying what we will see below. Assume that the functions $u(x)$ and $v(x)$ satisfy

$$-\Delta u \leq f(x, u), \quad -\Delta v \geq f(x, v) \text{ in } \Omega. \tag{1.3.5}$$

We say that $u(x)$ is a sub-solution, and $v(x)$ is a super-solution. Assume that, in addition, we know that

$$u(x) \leq v(x) \text{ for all } x \in \Omega, \tag{1.3.6}$$

that is, the sub-solution sits below the super-solution. In this case, we are going to rule out the possibility that they touch inside Ω (they can touch on the boundary, however): there can not be an $x_0 \in \Omega$ so that $u(x_0) = v(x_0)$. Indeed, if the function $f(x, s)$ is differentiable (or Lipschitz) in s, the quotient

$$c(x) = \frac{f(x, u(x)) - f(x, v(x))}{u(x) - v(x)}$$

is a bounded function, and the difference $w(x) = u(x) - v(x)$ satisfies

$$\Delta w + c(x)w \geq 0 \text{ in } \Omega. \tag{1.3.7}$$

As $w(x) \leq 0$ in all of Ω, the strong maximum principle implies that either $w(x) \equiv 0$, so that u and v coincide, or $w(x) < 0$ in Ω, that is, we have a strict inequality: $u(x) < v(x)$ for all $x \in \Omega$. In other words, a sub-solution and a super-solution can not touch at a point – this very simple principle will be extremely important in what follows.

Let us illustrate an application of the strong maximum principle, with a cameo appearance of the sliding method in a disguise as a bonus. Consider the boundary value problem

$$-u'' = e^u, \quad 0 < x < L, \tag{1.3.8}$$

with the boundary condition

$$u(0) = u(L) = 0. \tag{1.3.9}$$

If we think of $u(x)$ as a temperature distribution, then the boundary condition means that the boundary is "cold". On the other hand, the positive term e^u is a "heating term", which competes with the cooling by the boundary. A nonnegative solution $u(x)$ corresponds to an equilibrium between these two effects. We would like to show that if the length of the interval L is sufficiently large, then no such equilibrium is possible – the physical reason is that the boundary is too far from the middle of the interval, so the heating term wins. This absence of an equilibrium is interpreted as an explosion, and this model was introduced exactly in that context in late 30's-early 40's. It is convenient to work with the function $w = u + \varepsilon$, which satisfies

$$-w'' = e^{-\varepsilon} e^w, \quad 0 < x < L, \tag{1.3.10}$$

with the boundary condition

$$w(0) = w(L) = \varepsilon. \tag{1.3.11}$$

Consider a family of functions

$$v_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right), \quad \lambda \geq 0, \quad 0 < x < L.$$

These functions satisfy (for any $\lambda \geq 0$)

$$v_\lambda'' + \frac{\pi^2}{L^2} v_\lambda = 0, \quad v_\lambda(0) = v_\lambda(L) = 0. \tag{1.3.12}$$

Therefore, if L is so large that

$$\frac{\pi^2}{L^2} s \leq e^{-\varepsilon} e^s, \quad \text{for all } s \geq 0,$$

we have

$$w'' + \frac{\pi^2}{L^2} w \leq 0, \tag{1.3.13}$$

that is, w is a super-solution for (1.3.12). In addition, when $\lambda > 0$ is sufficiently small, we have

$$v_\lambda(x) \leq w(x) \text{ for all } 0 \leq x \leq L. \tag{1.3.14}$$

Let us now start increasing λ until the graphs of v_λ and w touch at some point:

$$\lambda_0 = \sup\{\lambda : \; v_\lambda(x) \leq w(x) \text{ for all } 0 \leq x \leq L.\} \tag{1.3.15}$$

The difference

$$p(x) = v_{\lambda_0}(x) - w(x)$$

satisfies

$$p'' + \frac{\pi^2}{L^2} p \geq 0,$$

and $p(x) \leq 0$ for all $0 < x < L$. In addition, there exists x_0 such that $p(x_0) = 0$, and, as

$$v_\lambda(0) = v_\lambda(L) = 0 < \varepsilon = w(0) = w(L),$$

it is impossible that $x_0 = 0$ or $x_0 = L$. We conclude that $p(x) \equiv 0$, which is a contradiction. Hence, no solution of (1.3.8)-(1.3.9) may exist when L is sufficiently large.

In order to complete the picture, the reader may look at the following exercise.

Exercise 1.3.6 Show that there exists $L_1 > 0$ so that a nonnegative solution of (1.3.8)-(1.3.9) exists for all $0 < L < L_1$, and does not exist for all $L > L_1$.

The maximum principle for narrow domains

Before we allow the moving plane method to return, we describe the maximum principle for narrow domains, which is an indispensable tool in this method. Its proof will utilize the "ballooning method" we have seen in the analysis of the explosion problem. As we have discussed, the usual maximum principle in the form "$\Delta u + c(x)u \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ implies either $u \equiv 0$ or $u < 0$ in Ω" can be interpreted physically as follows. If u is the temperature distribution then the boundary condition $u \leq 0$ on $\partial\Omega$ means that "the boundary is cold". At the same time, the term $c(x)u$ can be viewed as a heat source if $c(x) \geq 0$ or as a heat sink if $c(x) \leq 0$. The conditions $u \leq 0$ on $\partial\Omega$ and $c(x) \leq 0$ together mean that both the boundary is cold and there are no heat sources – therefore, the temperature is cold everywhere, and we get $u \leq 0$. On the other hand, if the domain is such that each point inside Ω is "close to the boundary" then the effect of the cold boundary can dominate over a heat source, and then, even if $c(x) \geq 0$ at some (or all) points $x \in \Omega$, the maximum principle still holds.

Mathematically, the first step in that direction is the maximum principle for narrow domains. We use the notation $c^+(x) = \max[0, c(x)]$.

Theorem 1.3.7 *(The maximum principle for narrow domains) There exists $d_0 > 0$ that depends on the L^∞-norm $\|c^+\|_\infty$ so that if there exists a unit vector e such that $|(y-x)\cdot e| < d_0$*

for all $(x,y) \in \Omega$ then the maximum principle holds for the operator $\Delta + c(x)$. That is, if a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$$\Delta u(x) + c(x)u(x) \geq 0 \text{ in } \Omega, \tag{1.3.16}$$

and $u \leq 0$ on $\partial\Omega$ then either $u \equiv 0$ or $u < 0$ in Ω.

The main observation here is that in a narrow domain we need not assume $c \leq 0$ – but "the largest possible narrowness", depends, of course, on the size of the positive part $c^+(x)$ that competes against it.

Proof. Note that, according to the strong maximum principle, it is sufficient to show that $u(x) \leq 0$ in Ω. For the sake of contradiction, suppose that

$$\sup_{x\in\Omega} u(x) > 0. \tag{1.3.17}$$

Without loss of generality we may assume that e is the unit vector in the direction x_1, and that

$$\bar{\Omega} \subset \{0 < x_1 < d\}.$$

Suppose that d is so small that

$$c(x) \leq \pi^2/d^2, \quad \text{for all } x \in \Omega, \tag{1.3.18}$$

and consider the function

$$w(x) = \sin\big(\frac{\pi x_1}{d}\big).$$

It satisfies

$$\Delta w + \frac{\pi^2}{d^2} w = 0, \tag{1.3.19}$$

and $w(x) > 0$ in $\bar{\Omega}$, in particular

$$\inf_{\bar{\Omega}} w(x) > 0. \tag{1.3.20}$$

A consequence of the above is

$$\Delta w + c(x)w \leq 0, \tag{1.3.21}$$

so that $w(x)$ is a super-solution to (1.3.16), while $u(x)$ is a sub-solution. Given $\lambda \geq 0$, let us set $w_\lambda(x) = \lambda w(x)$. As a consequence of (1.3.20), there exists $\Lambda > 0$ so large that

$$\Lambda w(x) > u(x) \text{ for all } x \in \Omega.$$

We are going to push w_λ down until it touches $u(x)$: set

$$\lambda_0 = \inf\{\lambda : \; w_\lambda(x) > u(x) \text{ for all } x \in \Omega.\}$$

Note, that, because of (1.3.17), we know that $\lambda_0 > 0$. The difference

$$v(x) = u(x) - w_{\lambda_0}(x)$$

satisfies

$$\Delta v + c(x)v \geq 0.$$

The difference between $u(x)$, which satisfies the same inequality, and $v(x)$ is that we know already that $v(x) \leq 0$ – hence, we may conclude from the strong maximum principle again that either $v(x) \equiv 0$, or $v(x) < 0$ in Ω. As $w_\lambda(x) > 0$ on $\partial\Omega$, the former contradicts the boundary condition on $u(x)$. It follows that $v(x) < 0$ in Ω. As $v(x) < 0$ also on the boundary $\partial\Omega$, there exists $\varepsilon_0 > 0$ so that

$$v(x) < -\varepsilon_0 \text{ for all } x \in \bar{\Omega},$$

that is,

$$u(x) + \varepsilon_0 < w_{\lambda_0}(x) \text{ for all } x \in \bar{\Omega}.$$

But then we may choose $\lambda' < \lambda_0$ so that we still have

$$w_{\lambda'}(x) > u(x) \text{ for all } x \in \Omega.$$

This contradicts the minimality of λ_0. Thus, it is impossible that $u(x) > 0$ for some $x \in \Omega$, and we are done. □

The maximum principle for small domains

The maximum principle for narrow domains can be extended, dropping the requirement that the domain is narrow and replacing it by the condition that the domain has a small volume. We begin with the following lemma, a simple version of the Alexandrov-Bakelman-Pucci maximum principle, which measures how far from the maximum principle a force can push the solution.

Lemma 1.3.8 *(The baby ABP Maximum Principle) Assume that $c(x) \leq 0$ for all $x \in \Omega$, and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\Delta u + c(x)u \geq f \text{ in } \Omega, \tag{1.3.22}$$

and $u \leq 0$ on $\partial\Omega$. Then

$$\sup_\Omega u \leq C \text{diam}(\Omega) \left\| f^- \right\|_{L^n(\Omega)}, \tag{1.3.23}$$

with the constant C that depends only on the dimension n (but not on the function $c(x) \leq 0$).

Proof. The idea is very similar to what we have seen in the proof of the isoperimetric inequality. If $M := \sup_\Omega u \leq 0$, then there is nothing to prove, hence we assume that $M > 0$. As $u(x) \leq 0$ on $\partial\Omega$, the maximum is achieved at an interior point $x_0 \in \Omega$, so that $M = u(x_0)$. The function $v = -u^+$, satisfies $v \leq 0$ in Ω, $v \equiv 0$ on $\partial\Omega$ and

$$-M = \inf_\Omega v = v(x_0) < 0.$$

Let Γ be the lower contact set of the function v, defined as in (1.2.8): the collection of all points $x \in \Omega$ such that the graph of v lies above the tangent plane at x. As $v \leq 0$ in Ω, we must have $v < 0$ on Γ. Hence v is smooth on Γ, and

$$\Delta v = -\Delta u \leq -f(x) + c(x)u \leq -f(x), \text{ for } x \in \Gamma, \tag{1.3.24}$$

as $c(x) \le 0$ and $u(x) \ge 0$ on Γ. The analog of the inclusion (1.2.9) that we will now prove is

$$B(0; M/d) \subset \nabla v(\Gamma), \tag{1.3.25}$$

with $d = \text{diam}(\Omega)$ and $B(0, M/d)$ the open ball centered at the origin of radius M/d. One way to see that is by sliding: let $p \in B(0; M/d)$ and consider the hyperplane that is the graph of

$$z_k(x) = p \cdot x - k.$$

Clearly, $z_k(x) < v(x)$ for k sufficiently large. As we decrease k, sliding the plane up, let $\bar{k}$ be the first value when the graphs of $v(x)$ and $z_{\bar{k}}(x)$ touch at a point x_1. Then we have

$$v(x) \ge z_{\bar{k}}(x) \text{ for all } x \in \Omega.$$

If x_1 is on the boundary $\partial\Omega$ then $v(x_1) = z_{\bar{k}}(x_1) = 0$, and we have

$$p \cdot (x_0 - x_1) = z_{\bar{k}}(x_0) - z_{\bar{k}}(x_1) \le v(x_0) - 0 = -M,$$

whence $|p| \ge M/d$, which is a contradiction. Therefore, x_1 is an interior point, which means that $x_1 \in \Gamma$ (by the definition of the lower contact set), and $p = \nabla v(x_1)$. This proves the inclusion (1.3.25).

Mimicking the proof of the isoperimetric inequality we use the area formula (c_n is the volume of the unit bal in $\mathbb{R}^n$):

$$c_n \left(\frac{M}{d}\right)^n = |B(0; M/d)| \le |\nabla v(\Gamma)| \le \int_\Gamma |\det(D^2 v(x))| dx. \tag{1.3.26}$$

Now, as in the aforementioned proof, for every point x in the contact set Γ, the matrix $D^2 v(x)$ is non-negative definite, hence (note that (1.3.24) implies that $f(x) \le 0$ on Γ)

$$|\det[D^2 v(x)]| \le \left(\frac{\Delta v}{n}\right)^n \le \frac{(-f(x))^n}{n^n}. \tag{1.3.27}$$

Integrating (1.3.27) and using (1.3.26), we get

$$M^n \le \frac{(\text{diam}(\Omega))^n}{c_n n^n} \int_\Gamma |f^-(x)|^n dx, \tag{1.3.28}$$

which is (1.3.23). □

An important consequence of Lemma 1.3.8 is a maximum principle for a domain with a small volume [6].

Theorem 1.3.9 *(The maximum principle for domains of a small volume) Let a function* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfy*

$$\Delta u(x) + c(x)u(x) \ge 0 \text{ in } \Omega,$$

and assume that $u \le 0$ *on* $\partial\Omega$. *Then there exists a positive constant* δ *which depends on the spatial dimension* n, *the diameter of* Ω, *and* $\|c^+\|_{L^\infty}$, *so that if* $|\Omega| \le \delta$ *then* $u \le 0$ *in* Ω.

Proof. If $c \le 0$ then $u \le 0$ by the standard maximum principle. In general, assume that $u^+ \not\equiv 0$, and write $c = c^+ - c^-$. We have

$$\Delta u - c^- u \ge -c^+ u.$$

Lemma 1.3.8 implies that (with a constant C that depends only on the dimension n)

$$\sup_\Omega u \le C\mathrm{diam}(\Omega)\|c^+u^+\|_{L^n(\Omega)} \le C\mathrm{diam}(\Omega)\|c^+\|_\infty|\Omega|^{1/n}\sup_\Omega u \le \frac{1}{2}\sup_\Omega u,$$

when the volume of Ω is sufficiently small:

$$|\Omega| \le \frac{1}{(2C\mathrm{diam}(\Omega)\|c^+\|_\infty)^n}. \tag{1.3.29}$$

We deduce that $\sup_\Omega u \le 0$ contradicting the assumption $u^+ \not\equiv 0$, Hence, we have $u \le 0$ in Ω under the condition (1.3.29). □

1.4 Act IV. Dancing together

We will now use a combination of the maximum principle (mostly for small domains) and the moving plane method to prove some results on the symmetry of the solutions to elliptic problems. We show just the tip of the iceberg – a curious reader will find many other results in the literature, the most famous being, perhaps, the De Giorgi conjecture, a beautiful connection between geometry and applied mathematics.

1.4.1 The Gidas-Ni-Nirenberg theorem

The following result on the radial symmetry of non-negative solutions is due to Gidas, Ni and Nirenberg. It is a basic example of a general phenomenon that positive solutions of elliptic equations tend to be monotonic in one form or other. We present the proof of the Gidas-Ni-Nirenberg theorem from [23]. The proof uses the moving plane method combined with the maximum principles for narrow domains, and domains of small volume.

Theorem 1.4.1 *Let $B_1 \in \mathbb{R}^n$ be the unit ball, and $u \in C(\bar{B}_1) \cap C^2(B_1)$ be a positive solution of the Dirichlet boundary value problem*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } B_1, \\ &u = 0 \ \text{on } \partial B_1, \end{aligned} \tag{1.4.1}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then, the function u is radially symmetric in B_1 and

$$\frac{\partial u}{\partial r}(x) < 0 \text{ for } x \ne 0.$$

To address an immediate question the reader may have, we give the following simple exercise.

Exercise 1.4.2 Show that the conclusion that a function u satisfying (1.4.1) is radially symmetric is false in general without the assumption that the function u is positive. Hint: you may have to learn a little more about the Bessel functions and spherical harmonics.

The proof of Theorem 1.4.1 is based on the following lemma, which applies to general domains with a planar symmetry, not just balls.

Lemma 1.4.3 *Let Ω be a bounded domain that is convex in the x_1-direction and symmetric with respect to the plane $\{x_1 = 0\}$. Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ be a positive solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } \Omega, \\ &u = 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.4.2}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then, the function u is symmetric with respect to x_1 and

$$\frac{\partial u}{\partial x_1}(x) < 0 \text{ for any } x \in \Omega \text{ with } x_1 > 0.$$

Proof of Theorem 1.4.1. Theorem 1.4.1 follows immediately from Lemma 1.4.3. Indeed, Lemma 1.4.3 implies that $u(x)$ is decreasing in any given radial direction, since the unit ball is symmetric with respect to any plane passing through the origin. It also follows from the same lemma that $u(x)$ is invariant under a reflection with respect to any hyperplane passing through the origin – this trivially implies that u is radially symmetric. □

Proof of Lemma 1.4.3

We use the coordinate system $x = (x_1, y) \in \Omega$ with $y \in \mathbb{R}^{n-1}$. We will prove that

$$u(x_1, y) < u(x_1^*, y) \text{ for all } x_1 > 0 \text{ and } -x_1 < x_1^* < x_1. \tag{1.4.3}$$

This, obviously, implies monotonicity in x_1 for $x_1 > 0$. Next, letting $x_1^* \to -x_1$, we get the inequality

$$u(x_1, y) \le u(-x_1, y) \text{ for any } x_1 > 0.$$

Changing the direction, we get the reflection symmetry: $u(x_1, y) = u(-x_1, y)$.

We now prove (1.4.3). Given any $\lambda \in (0, a)$, with $a = \sup_\Omega x_1$, we take the "moving plane"

$$T_\lambda = \{x_1 = \lambda\},$$

and consider the part of Ω that is "to the right" of T_λ:

$$\Sigma_\lambda = \{x \in \Omega : \; x_1 > \lambda\}.$$

Finally, given a point x, we let x_λ be the reflection of $x = (x_1, x_2, \dots, x_n)$ with respect to T_λ:

$$x_\lambda = (2\lambda - x_1, x_2, \dots, x_n).$$

Consider the difference

$$w_\lambda(x) = u(x) - u(x_\lambda) \text{ for } x \in \Sigma_\lambda.$$

The mean value theorem implies that w_λ satisfies

$$\Delta w_\lambda = f(u(x_\lambda)) - f(u(x)) = \frac{f(u(x_\lambda)) - f(u(x))}{u(x_\lambda) - u(x)} w_\lambda = -c(x,\lambda) w_\lambda$$

in Σ_λ. This is a recurring trick: the difference of two solutions of a semi-linear equation satisfies a "linear" equation with an unknown function c. However, we know a priori that the function c is bounded:

$$|c(x)| \leq \mathrm{Lip}(f), \text{ for all } x \in \Omega. \tag{1.4.4}$$

The boundary $\partial\Sigma_\lambda$ consists of a piece of $\partial\Omega$, where $w_\lambda = -u(x_\lambda) < 0$ and of a part of the plane T_λ, where $x = x_\lambda$, thus $w_\lambda = 0$. Summarizing, we have

$$\begin{aligned} &\Delta w_\lambda + c(x,\lambda) w_\lambda = 0 \text{ in } \Sigma_\lambda \\ &w_\lambda \leq 0 \text{ and } w_\lambda \not\equiv 0 \text{ on } \partial\Sigma_\lambda, \end{aligned} \tag{1.4.5}$$

with a bounded function $c(x,\lambda)$. As the function $c(x,\lambda)$ does not necessarily have a definite sign, we may not directly apply the comparison principle and immediately conclude from (1.4.5) that

$$w_\lambda < 0 \text{ inside } \Sigma_\lambda \text{ for all } \lambda \in (0,a). \tag{1.4.6}$$

Nevertheless, using the moving plane method, we will be able to show that (1.4.6) holds. This implies in particular that w_λ assumes its maximum (equal to zero) over $\bar{\Sigma}_\lambda$ along T_λ. The Hopf lemma implies then

$$\left.\frac{\partial w_\lambda}{\partial x_1}\right|_{x_1=\lambda} = 2 \left.\frac{\partial u}{\partial x_1}\right|_{x_1=\lambda} < 0.$$

Given that λ is arbitrary, we conclude that

$$\frac{\partial u}{\partial x_1} < 0, \text{ for any } x \in \Omega \text{ such that } x_1 > 0.$$

Therefore, it remains only to show that $w_\lambda < 0$ inside Σ_λ to establish monotonicity of u in x_1 for $x_1 > 0$. Another consequence of (1.4.6) is that

$$u(x_1, x') < u(2\lambda - x_1, x') \text{ for all } \lambda \text{ such that } x \in \Sigma_\lambda,$$

that is, for all $\lambda \in (0, x_1)$, which is the same as (1.4.3).

In order to show that $w_\lambda < 0$ one would like to apply the maximum principle to the boundary value problem (1.4.5). However, as we have mentioned, a priori the function $c(x,\lambda)$ does not have a sign, so the usual maximum principle may not be used. On the other hand, there exists δ_c such that the maximum principle for narrow domains holds for the operator

$$Lu = \Delta u + c(x)u,$$

and domains of the width not larger than δ_c in the x_1-direction. Note that δ_c depends only on $\|c\|_{L^\infty}$ that is controlled in our case by (1.4.4). Moreover, when λ is sufficiently close to a:

$$a - \delta_c < \lambda < a,$$

the domain Σ_λ does have the width in the x_1-direction which is smaller than δ_c. Thus, for such λ the maximum principle for narrow domains implies that $w_\lambda < 0$ inside Σ_λ. This is because $w_\lambda \leq 0$ on $\partial\Sigma_\lambda$, and $w_\lambda \not\equiv 0$ on $\partial\Sigma_\lambda$.

Let us now decrease λ (move the plane T_λ to the left, hence the name "the moving plane" method), and let (λ_0, a) be the largest interval of values so that $w_\lambda < 0$ inside Σ_λ for all $\lambda \in (\lambda_0, a)$. If $\lambda_0 = 0$, that is, if we may move the plane T_λ all the way to $\lambda = 0$, while keeping (1.4.6) true, then we are done – (1.4.6) follows. Assume, for the sake of a contradiction, that $\lambda_0 > 0$. Then, by continuity, we still know that

$$w_{\lambda_0} \leq 0 \text{ in } \Sigma_{\lambda_0}.$$

Moreover, w_{λ_0} is not identically equal to zero on $\partial\Sigma_{\lambda_0}$. The strong comparison principle implies that

$$w_{\lambda_0} < 0 \text{ in } \Sigma_{\lambda_0}. \tag{1.4.7}$$

We will show that then

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \tag{1.4.8}$$

for sufficiently small $\varepsilon < \varepsilon_0$. This will contradict our choice of λ_0 (unless $\lambda_0 = 0$).

Here is the key step and the reason why the maximum principle for domains of small volume is useful for us here: choose a compact set K in Σ_{λ_0}, with a smooth boundary, which is "nearly all" of Σ_{λ_0}, in the sense that

$$|\Sigma_{\lambda_0} \backslash K| < \delta/2$$

with $\delta > 0$ to be determined. Inequality (1.4.7) implies that there exists $\eta > 0$ so that

$$w_{\lambda_0} \leq -\eta < 0 \text{ for any } x \in K.$$

By continuity, there exits $\varepsilon_0 > 0$ so that

$$w_{\lambda_0-\varepsilon} < -\frac{\eta}{2} < 0 \text{ for any } x \in K, \tag{1.4.9}$$

for $\varepsilon \in (0, \varepsilon_0)$ sufficiently small. Let us now see what happens in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. As far as the boundary is concerned, we have

$$w_{\lambda_0-\varepsilon} \leq 0$$

on $\partial\Sigma_{\lambda_0-\varepsilon}$ – this is true for $\partial\Sigma_\lambda$ for all $\lambda \in (0, a)$, and, in addition,

$$w_{\lambda_0-\varepsilon} < 0 \text{ on } \partial K,$$

because of (1.4.9). We conclude that

$$w_{\lambda_0-\varepsilon} \leq 0 \text{ on } \partial(\Sigma_{\lambda_0-\varepsilon} \backslash K),$$

and $w_{\lambda_0-\varepsilon}$ does not vanish identically on $\partial(\Sigma_{\lambda_0-\varepsilon} \backslash K)$. Choose now δ (once again, solely determined by $\|c\|_{L^\infty(\Omega)}$), so small that we may apply the maximum principle for domains of small volume in domains of volume less than δ. When ε is sufficiently small, we have $|\Sigma_{\lambda_0-\varepsilon} \backslash K| < \delta$. Applying this maximum principle to the function $w_{\lambda_0-\varepsilon}$ in the domain $\Sigma_{\lambda_0-\varepsilon} \backslash K$, we obtain

$$w_{\lambda_0-\varepsilon} \leq 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \backslash K.$$

The strong maximum principle implies that

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \backslash K.$$

Putting two and two together we see that (1.4.8) holds. This, however, contradicts the choice of λ_0. The proof of the Gidas-Ni-Nirenberg theorem is complete. □

1.4.2 The sliding method: moving sub-solutions around

The sliding method differs from the moving plane method in that one compares translations of a function rather than its reflections with respect to a plane. One elementary but beautiful application of the sliding method allows to extend lower bounds obtained on a solution of a semi-linear elliptic equation in one part of a domain to a different part by moving a sub-solution around the domain and observing that it may never touch a solution. This is a very simple and powerful tool in many problems.

Lemma 1.4.4 *Let u be a positive function in an open connected set D satisfying*

$$\Delta u + f(u) \le 0 \ \textit{in } D$$

with a Lipschitz function f. Let B be a ball with its closure $\bar{B} \subset D$, and suppose z is a function in $\bar{B}$ satisfying

$$\begin{aligned} & z \le u \ \textit{in } B \\ & \Delta z + f(z) \ge 0, \ \textit{wherever } z > 0 \ \textit{in } B \\ & z \le 0 \ \textit{on } \partial B. \end{aligned}$$

Then for any continuous one-parameter family of Euclidean motions (rotations and translations) $A(t)$, $0 \le t \le T$, so that $A(0) = \mathrm{Id}$ and $A(t)\bar{B} \subset D$ for all t, we have

$$z^t(x) := z(A(t)^{-1}x) < u(x) \ \textit{in } B^t := A(t)B. \tag{1.4.10}$$

Proof. The rotational invariance of the Laplace operator implies that the function z^t satisfies

$$\begin{aligned} & \Delta z^t + f(z^t) \ge 0, \ \text{wherever } z^t > 0 \text{ in } B^t \\ & z^t \le 0 \text{ on } \partial B^t. \end{aligned}$$

Thus the difference $w^t = z^t - u$ satisfies

$$\Delta w^t + c^t(x) w^t \ge 0 \text{ wherever } z^t > 0 \text{ in } B^t, \tag{1.4.11}$$

with c^t bounded in B^t, where, as before,

$$c^t(x) = \begin{cases} \dfrac{f(z^t(x)) - f(u(x))}{z^t(x) - u(x)}, \ \text{if } z^t(x) \ne u(x) \\ \qquad 0, \ \text{otherwise.} \end{cases}$$

In addition, $w^t < 0$ on ∂B^t.

We now argue by contradiction. Suppose that there is a first t so that the graph of z^t touches the graph of u at a point x_0 – such t exists by continuity. Then, for that t, we still have $w^t \le 0$ in B^t, but also $w^t(x_0) = 0$. As $u > 0$ in D, and $z^t \le 0$ on ∂B^t, the point x_0 has to be inside B^t, which means that z^t satisfies

$$\Delta z^t + f(z^t) \ge 0$$

in the whole component G of the set of points in B^t where $z^t > 0$ that contains x_0. Thus, w^t satisfies (1.4.11) in G, and, in addition, $w^t \le 0$ and $w^t(x_0) = 0$. The comparison principle implies that $w^t \equiv 0$ in G. In particular, we have $w^t(\widetilde{x}) = 0$ for all $\widetilde{x} \in \partial G$. But then

$$z^t(\widetilde{x}) = u(\widetilde{x}) > 0 \text{ on } \partial G,$$

which contradicts the fact that $z^t = 0$ on ∂G. Hence the graph of z^t may not touch that of u and (1.4.10) follows. □

Lemma 1.4.4 is often used to "slide around" a sub-solution that is positive somewhere to show that solution itself is uniformly positive. We will use it repeatedly when we talk about the reaction-diffusion equations later on. Here is an exercise (to which we will return later) on how it can be applied.

Exercise 1.4.5 Let $u(x) > 0$ be a positive bounded solution of the equation

$$u_{xx} + u - u^2 = 0$$

on the real line $x \in \mathbb{R}$. Show that if L is sufficiently large and $\lambda > 0$ is sufficiently small, then the function

$$z_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right)$$

satisfies

$$\frac{\partial^2 z_\lambda}{\partial x^2} + z_\lambda - z_\lambda^2 \ge 0, \quad 0 < x < L,$$

and $u(x) \ge z_\lambda(x)$ for all $0 \le x \le L$. Use this to conclude that

$$\inf_{x \in \mathbb{R}} u(x) > 0.$$

Try to strengthen this result to prove that $u(x) \equiv 1$.

1.4.3 Monotonicity for the Allen-Cahn equation in $\mathbb{R}^n$

Our next example, taken from the paper [17] by Berestycki, Hamel and Monneau, shows one analog of the Gidas-Ni-Nirenberg theorem in the whole space $\mathbb{R}^n$. Recall that for the latter result we have considered a semi-linear elliptic equation in a ball with the Dirichlet boundary conditions, which are compatible with radially symmetric solutions, and have shown that the only possible non-negative solutions are, indeed, radially symmetric. In the whole space we will impose boundary conditions that allow solutions to depend on just one variable, say, x_n, and will show that any solution satisfying these boundary conditions depends only on x_n.

We consider solutions of

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^n \tag{1.4.12}$$

which satisfy $|u| \leq 1$ together with the asymptotic conditions

$$u(x', x_n) \to \pm 1 \text{ as } x_n \to \pm\infty \text{ uniformly in } x' = (x_1, \ldots, x_{n-1}). \tag{1.4.13}$$

We assume that f is a smooth (actually, just assuming that f is Lipschitz would be sufficient) function on $[-1, 1]$, and there exists $\delta > 0$ so that

$$f \text{ is non-increasing on } [-1, -1 + \delta] \text{ and on } [1 - \delta, 1], \text{ and } f(\pm 1) = 0. \tag{1.4.14}$$

The standard example to keep in mind is $f(u) = u - u^3$. In that case, (1.4.12) is known as the Allen-Cahn equation. Such problems appear in many applications, ranging from biology and combustion to the differential geometry, as a very basic model of a diffusive connection between two stable states. The main feature of the nonlinearity is that the corresponding time-dependent ODE

$$\frac{du}{dt} = f(u) \tag{1.4.15}$$

has two stable solutions $u \equiv -1$ and $u \equiv 1$. Solutions of the partial differential equation (1.4.12), on the other hand, describe the diffusive transitions between regions in space where u is close to the equilibrium $u \equiv -1$ and those where u is close to $u \equiv 1$.

In one dimension, this is simply the ODE

$$u_0'' + f(u_0) = 0, \quad x \in \mathbb{R}, \tag{1.4.16}$$

with the boundary conditions

$$u_0(\pm\infty) = \pm 1. \tag{1.4.17}$$

This equation may be solved explicitly: multiplying (1.4.16) by u_0' and integrating from $-\infty$ to x, using the boundary conditions, leads to

$$\frac{1}{2}(u_0')^2 + F(u_0) = 0, \quad u_0(\pm\infty) = \pm 1. \tag{1.4.18}$$

Here, we have defined

$$F(s) = \int_{-1}^{s} f(u)du. \tag{1.4.19}$$

Letting $x \to +\infty$ in (1.4.18) we see that a necessary condition for a solution of (1.4.18) to exist is that $F(1) = 0$, or

$$\int_{-1}^{1} f(u)du = 0. \tag{1.4.20}$$

Exercise 1.4.6 Show that the solutions of (1.4.16)-(1.4.17) are unique, up to a translation in the x-variable – note that if $u_0(x)$ is a solution to (1.4.16)-(1.4.17), then so is $\widetilde{u}(x) = u_0(x+\xi)$, for any $\xi \in \mathbb{R}$.

Our goal is to show that the asymptotic conditions (1.4.13) imply that the positive solutions of (1.4.12) are actually one-dimensional.

Theorem 1.4.7 *Let u be any solution of (1.4.12)-(1.4.13) such that $|u| \le 1$. Then it has the form $u(x', x_n) = u_0(x_n)$ where u_0 is a solution of*

$$u_0'' + f(u_0) = 0 \text{ in } \mathbb{R}, \; u_0(\pm\infty) = \pm 1. \tag{1.4.21}$$

Moreover, u is increasing with respect to x_n. Finally, such solution is unique up to a translation.

Without the uniformity assumption in (1.4.13), that is, imposing simply

$$u(x', x_n) \to \pm 1 \text{ as } x_n \to \pm\infty, \tag{1.4.22}$$

this problem is known as "the weak form" of the De Giorgi conjecture, and was resolved by Savin [127] who showed that all solutions are one-dimensional in $n \le 8$, and del Pino, Kowalczyk and Wei [51] who showed that non-planar solutions exist $n \ge 9$. Their work is well beyond the scope of this chapter.

Note that (1.4.22), without the uniformity condition for the limits at infinity as in (1.4.13), does not imply that u depends only on the variable x_n. For example, any function of the form $u(x) = u_0(e \cdot x)$, where $e \in \mathbb{S}^{n-1}$ is a fixed vector with $|e| = 1$ and $e_n > 0$, and u_0 is any solution of (1.4.21), satisfies both (1.4.12) and (1.4.22). It will not, however, satisfy the uniformity assumption in (1.4.13). The additional assumption of uniform convergence at infinity made here makes this question much easier than the weak form of the De Giorgi conjecture. Nevertheless, the proof of Theorem 1.4.7 is both non-trivial and instructive. The full De Giorgi conjecture is that any solution of (1.4.14) in dimension $n \le 8$ with $f(u) = u - u^3$ (without imposing any boundary conditions on u at all) such that $-1 \le u \le 1$ is one-dimensional. It is still open in this generality, to the best of our knowledge. The motivation for the conjecture comes from the study of the minimal surfaces in differential geometry but we will not discuss this connection here.

A maximum principle in an unbounded domain

For the proof, we will need a version of the maximum principle for unbounded domains, interesting in itself.

Lemma 1.4.8 *Let D be an open connected set in $\mathbb{R}^n$, possibly unbounded. Assume that $\bar{D}$ is disjoint from the closure of an infinite open (solid) cone Σ. Suppose that a function $z \in C(\bar{D})$ is bounded from above and satisfies*

$$\begin{aligned} &\Delta z + c(x) z \ge 0 \text{ in } D \\ &z \le 0 \text{ on } \partial D. \end{aligned} \tag{1.4.23}$$

with some continuous function $c(x) \le 0$, then $z \le 0$.

Proof. If the function $z(x)$ would, in addition, vanish at infinity:

$$\limsup_{|x|\to+\infty} z(x) = 0, \tag{1.4.24}$$

then the proof would be easy. Indeed, if (1.4.24) holds then we can find a sequence $R_n \to +\infty$ so that

$$\sup_{\bar{D}\cap\{|x|=R_n\}} z(x) \le \frac{1}{n}. \tag{1.4.25}$$

The usual maximum principle applied in the bounded domain $D_n = D \cap B(0; R_n)$ implies then that $z(x) \le 1/n$ in D_n since this inequality holds on ∂D_n. Letting $n \to \infty$ gives

$$z(x) \le 0 \text{ in } D.$$

Our next task is to reduce the case of a bounded function z to (1.4.24). To do this, we will construct a harmonic function $g(x) > 0$ in D such that

$$|g(x)| \to +\infty \text{ as } |x| \to +\infty. \tag{1.4.26}$$

Since g is harmonic, the ratio $\sigma = z/g$ will satisfy a differential inequality in D:

$$\Delta\sigma + \frac{2}{g}\nabla g \cdot \nabla\sigma + c\sigma \ge 0. \tag{1.4.27}$$

This is similar to (1.4.23) but now σ does satisfy the asymptotic condition

$$\limsup_{x\in D, |x|\to\infty} \sigma(x) \le 0,$$

uniformly in $x \in D$. Moreover, $\sigma \le 0$ on ∂D. Hence one may apply the above argument to the function $\sigma(x)$, and conclude that $\sigma(x) \le 0$, which, in turn, implies that $z(x) \le 0$ in D.

Exercise 1.4.9 Note that we have brazenly applied the maximum principle above to the operator in the left side of (1.4.27), while we have previously only proved it for operators of the form $\Delta + c(x)$, with $c(x) \le 0$. To remedy this, consider a function ϕ which satisfies an inequality of the form

$$\Delta\phi + b(x) \cdot \nabla\phi + c(x)\phi \ge 0 \tag{1.4.28}$$

in a bounded domain D with $c(x) \le 0$. Show that ϕ can not attain a positive maximum inside D. Hint: mimic the proof of the strong maximum principle.

In order to construct such harmonic function $g(x)$ in D, the idea is to decrease the cone Σ to a cone $\widetilde{\Sigma}$ and to consider the principal eigenfunction $\psi > 0$ of the spherical Laplace-Beltrami operator in the region $G = \mathbb{S}^{n-1}\backslash\widetilde{\Sigma}$ with $\psi = 0$ on ∂G:

$$\begin{aligned} &\Delta_S\psi + \mu\psi = 0, \quad \psi > 0 \text{ in } G, \\ &\psi = 0 \text{ on } \partial G. \end{aligned}$$

Here, Δ_S is simply the restriction of the standard Laplacian operator to functions of the angular variable only (independent of the radial variable). Existence of such an eigenvalue that corresponds to a positive eigenfunction follows from the general spectral theory of elliptic operators. We do not expect the reader to be familiar with this theory, but for the moment, in order to keep the flow of the presentation, we simply ask to take for granted that such principal eigenvalue with a positive eigenfunction exists and is unique, or consult [60].

Exercise 1.4.10 Show that $\mu > 0$.

Going to the polar coordinates $x = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{n-1}$, we now define the function

$$g(x) = r^\alpha \psi(\xi), \quad x \in D,$$

with

$$\alpha(n + \alpha - 2) = \mu.$$

This choice of α makes the function g be harmonic:

$$\Delta g = \frac{\partial^2 g}{\partial r^2} + \frac{n-1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\Delta_S g = [\alpha(\alpha - 1) + \alpha(n-1) - \mu] r^{\alpha-2}\Psi = 0.$$

Moreover, as $\mu > 0$, we have $\alpha > 0$, and it is easy to see that there exists $c_0 > 0$ such that $\psi(x) \geq c_0$ for all $x \in D$. Thus (1.4.26) also holds, and the proof is complete. □

We will need the following corollary that we will use for half-spaces.

Corollary 1.4.11 *Let f be a Lipschitz continuous function, non-increasing on $[-1, -1+\delta]$ and on $[1-\delta, 1]$ for some $\delta > 0$. Assume that u_1 and u_2 satisfy*

$$\Delta u_i + f(u_i) = 0 \text{ in } \Omega$$

and are such that $|u_i| \leq 1$. Assume furthermore that $u_2 \geq u_1$ on $\partial\Omega$ and that either $u_2 \geq 1-\delta$ or $u_1 \leq -1+\delta$ in Ω. If $\Omega \subset \mathbb{R}^n$ is an open connected set so that $\mathbb{R}^n \backslash \bar{\Omega}$ contains an open infinite cone then $u_2 \geq u_1$ in Ω.

Proof. Assume, for instance, that $u_2 \geq 1 - \delta$, and set $w = u_1 - u_2$. Then

$$\Delta w + c(x) w = 0 \text{ in } \Omega$$

with

$$c(x) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}.$$

Note that $c(x) \leq 0$ if $w(x) \geq 0$. Indeed, if $w(x) \geq 0$, then

$$u_1(x) \geq u_2(x) \geq 1 - \delta.$$

As, in addition, we know that $u_1 \leq 1$, and f is non-increasing on $[1-\delta, 1]$, it follows that $f(u_1(x)) \leq f(u_2(x))$, and thus $c(x) \leq 0$. Hence, if the set $G = \{w > 0\}$ is not empty, we may apply the maximum principle of Lemma 1.4.8 to the function w in G (note that $w \leq 0$ on ∂G), and conclude that $w \leq 0$ in G giving a contradiction. □

Proof of Theorem 1.4.7

We are going to prove that

$$u \text{ is increasing in any direction } \nu = (\nu_1, \dots, \nu_n) \text{ with } \nu_n > 0. \tag{1.4.29}$$

This will mean that

$$\frac{1}{\nu_n}\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} \alpha_j \frac{\partial u}{\partial x_j} > 0$$

for any choice of $\alpha_j = \nu_j/\nu_n$. It follows that all $\partial u/\partial x_j = 0$, $j = 1, \ldots, n-1$, so that u depends only on x_n, and, moreover, $\partial u/\partial x_n > 0$. Hence, (1.4.29) implies the conclusion of Theorem 1.4.7 on the monotonicity of the solution.

We now prove (1.4.29). Monotonicity in the direction ν can be restated as

$$u^t(x) \geq u(x), \text{ for all } t \geq 0 \text{ and all } x \in D, \tag{1.4.30}$$

where $u^t(x) = u(x + t\nu)$ are the shifts of the function u in the direction ν. We start the sliding method with a very large t. The uniformity assumption in the boundary condition (1.4.13) implies that there exists a real $a > 0$ so that

$$u(x', x_n) \geq 1 - \delta \text{ for all } x_n \geq a,$$

and

$$u(x', x_n) \leq -1 + \delta \text{ for all } x_n \leq -a.$$

Take $t \geq 2a/\nu_n$, then the functions u and u^t are such that

$$\begin{aligned} u^t(x', x_n) &\geq 1 - \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and all } x_n \geq -a \\ u(x', x_n) &\leq -1 + \delta \quad \text{for all } x' \in \mathbb{R}^{n-1} \text{ and all } x_n \leq -a, \end{aligned} \tag{1.4.31}$$

and, in particular,

$$u^t(x', -a) \geq u(x', -a) \text{ for all } x' \in \mathbb{R}^{n-1}. \tag{1.4.32}$$

Hence, we may apply Corollary 1.4.11 separately in the half-spaces $\Omega_1 = \{(x', x_n) : \ x_n \leq -a\}$ and $\Omega_2 = \{(x', x_n) : \ x_n \geq -a\}$. In both cases, we conclude that $u^t \geq u$ and thus

$$u^t \geq u \text{ in all of } \mathbb{R}^n \text{ for } t \geq 2a/\nu_n.$$

Following the philosophy of the sliding method, we start to decrease t, and let

$$\tau = \inf\{t > 0, \ u^t(x) \geq u(x) \text{ for all } x \in \mathbb{R}^n\}.$$

By continuity, we still have $u^\tau \geq u$ in $\mathbb{R}^n$. Note that (1.4.30) is equivalent to $\tau = 0$, and we show this by contradiction. If $\tau > 0$, there are two possibilities.

Case 1. Suppose that

$$\inf_{D_a}(u^\tau - u) > 0, \quad D_a = \mathbb{R}^{n-1} \times [-a, a]. \tag{1.4.33}$$

The function u is globally Lipschitz continuous – the reader may either accept that this follows from the standard elliptic estimates [60], or do the following exercise.

Exercise 1.4.12 Let $u(x)$ be a uniformly bounded solution ($|u(x)| \le M$ for all $x \in \mathbb{R}^n$) of an equation of the form

$$-\Delta u = F(u)$$

in $\mathbb{R}^n$, with a differentiable function $F(u)$. Show that there exists a constant $C > 0$ which depends on the function F so that $|\nabla u(x)| \le CM$ for all $x \in \mathbb{R}^n$. Hint: fix $y \in \mathbb{R}^n$, and let $\chi(x)$ be a smooth cut-off function supported in the ball B centered at y of radius $r = 1$. Write an equation for the function $v(x) = \chi(x)u(x)$ of the form

$$-\Delta v = g,$$

with the function g that depends on u, F and χ, use the Green's function of the Laplacian to bound $\nabla v(y)$, and deduce a uniform bound on $\nabla u(y)$. Make sure you see why you need to pass from u to v.

The Lipschitz continuity of u together with assumption (1.4.33) implies that there exists $\eta_0 > 0$ so that for all $\tau - \eta_0 < t < \tau$ we still have

$$u^t(x', x_n) > u(x', x_n) \text{ for all } x' \in \mathbb{R}^{n-1} \text{ and for all } -a \le x_n \le a. \tag{1.4.34}$$

As $u(x', x_n) \ge 1 - \delta$ for all $x_n \ge a$, we know that

$$u^t(x', x_n) \ge 1 - \delta \text{ for all } x_n \ge a \text{ and } t > 0. \tag{1.4.35}$$

We may then apply Corollary 1.4.11 in the half-spaces $\{x_n > a\}$ and $\{x_n < -a\}$ to conclude that

$$u^{\tau-\eta}(x) \ge u(x)$$

everywhere in $\mathbb{R}^n$ for all $\eta \in [0, \eta_0]$. This contradicts the choice of τ. Thus, the case (1.4.33) is impossible.

Case 2. Suppose that

$$\inf_{D_a}(u^\tau - u) = 0, \quad D_a = \mathbb{R}^{n-1} \times [-a, a]. \tag{1.4.36}$$

This would be a contradiction to the maximum principle if we could conclude from (1.4.36) that the graphs of u^τ and u touch at an internal point. This, however, is not clear, as there may exist a sequence of points ξ_k with $|\xi_k| \to +\infty$, such that $u^\tau(\xi_k) - u(\xi_k) \to 0$, without the graphs ever touching. In order to deal with this issue, we will use the usual trick of moving "the interesting part" of the domain to the origin and passing to the limit. We know from (1.4.36) that there exists a sequence $\xi_k \in D_a$ so that

$$u^\tau(\xi_k) - u(\xi_k) \to 0 \text{ as } k \to \infty. \tag{1.4.37}$$

Let us re-center: set

$$u_k(x) = u(x + \xi_k).$$

Differentiating the equation for u, we may bootstrap the claim of Exercise 1.4.12 to conclude that u is uniformly bounded in $C^3(\mathbb{R}^n)$, thus so is the sequence $u_k(x)$. The Ascoli-Arzela theorem implies that $u_k(x)$ converge along a subsequence to a function $u_\infty(x)$, uniformly on compact sets, together with the first two derivatives. The limit satisfies

$$\Delta u_\infty + f(u_\infty) = 0, \tag{1.4.38}$$

and, in addition, we have, because of (1.4.37):

$$u_\infty^\tau(0) = u_\infty(0),$$

and also

$$u_\infty^\tau(x) \geq u_\infty(x), \quad \text{for all } x \in \mathbb{R}^n,$$

because $u_k^\tau \geq u_k$ for all k. As both u_∞ and u_∞^τ satisfy (1.4.38), the strong maximum principle implies that $u_\infty^\tau = u_\infty$, that is,

$$u_\infty(x + \tau\nu) = u_\infty(x) \text{ for all } x \in \mathbb{R}^n.$$

In other words, the function u_∞ is periodic in the ν-direction. However, as all ξ_k lie in D_a, their n-th components are uniformly bounded $|(\xi_k)_n| \leq a$. Therefore, when we pass to the limit we do not lose the boundary conditions in x_n: the function u_∞ must satisfy the boundary conditions (1.4.13). This is a contradiction to the above periodicity. Hence, this case is also impossible, and thus $\tau = 0$. This proves monotonicity of $u(x)$ in x_n and the fact that u depends only on x_n: $u(x) = u(x_n)$.

In order to prove the uniqueness of such solution, assuming there are two such solutions u and v, one repeats the sliding argument above but applied to the difference

$$w^\tau(x_n) = u(x_n + \tau) - v(x_n).$$

Exercise 1.4.13 Use this sliding argument to show that there exists $\tau \in \mathbb{R}^n$ such that

$$u(x_n + \tau) = v(x_n) \text{ for all } x_n \in \mathbb{R},$$

showing uniqueness of such solution, up to a shift.

This completes the proof. □