Lecture notes for Math 256B, Version 2015

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April 26, 2015

Nothing found here is original except for a few mistakes and misprints here and there. These notes are simply a record of what I cover in class, to spare the students the necessity of taking the lecture notes. The readers should consult the original books for a better presentation and context. We plan to follow the following books: C. Doering and J. Gibbon "Applied Analysis of the Navier-Stokes Equations", A. Majda and A. Bertozzi "Vorticity and Incompressible Flow", P. Constantin and C. Foias "The Navier-Stokes Equations", as well as lecture notes by Vladimir Sverak on the mathematical fluid dynamics that can be found on his website.

1 The derivation of the Navier-Stokes and Euler equations

The equations of motion of a fluid come from three considerations: conservation of mass, Newton's second law and the material properties. The state of the fluid is characterized by its density $\rho(t,x)$ and fluid velocity $u(t,x)$, and our first task is to derive the equations that govern their evolution.

The continuity equation

Each fluid particle is following a trajectory governed by the fluid velocity $u(t,x)$:

$$\frac{dX(t,\alpha)}{dt} = u(t, X(\alpha,t)), \quad X(0,\alpha) = \alpha. \tag{1.1}$$

Here, α is the starting position of the particle, and is sometimes called "the label", and the inverse map $A_t : X(t,\alpha) \to \alpha$ is called the "back-to-the-labels" map. If the flow $u(t,x)$ is sufficiently smooth, the forward map $\alpha \to X(t,\alpha)$ should preserve the mass. Let us first assume that $\rho(t,x) = \rho_0$ is a constant, and see what we can deduce from the mass preservation. In the constant density case, mass preservation is equivalent to the conservation of the volume. That is, if $V_0 \subset \mathbb{R}^d$ $(d = 2,3)$ is an initial volume, then the set

$$V(t) = \{X(t,\alpha) : \ \alpha \in V_0\}$$

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should have the same volume as V_0. In order to quantify this property, let us define the Jacobian

$$J(t,\alpha) = \det(\frac{\partial X_i(t,\alpha)}{\partial \alpha_j}).$$

Volume preservation means that $J(t,\alpha) \equiv 1$. As $J(0,\alpha) \equiv 1$, this condition is equivalent to $dJ/dt \equiv 0$. The full matrix $H_{ij}(t,\alpha) = \dfrac{\partial X_i(t,\alpha)}{\partial \alpha_j}$ obeys the evolution equation

$$\frac{dH_{ij}}{dt} = \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k} \frac{\partial X_k}{\partial \alpha_j}, \tag{1.2}$$

that is,

$$\frac{dH}{dt} = (\nabla u)H, \tag{1.3}$$

with $(\nabla u)_{ik} = \dfrac{\partial u_i}{\partial x_k}$. In order to find dJ/dt, we consider a general $n \times n$ matrix $A_{ij}(t)$ and decompose, for each $i = 1, \ldots, n$ fixed:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} M_{ij} A_{ij}.$$

Note that the minors $M_{ij'}$, for all $1 \le j \le n$, do not depend on the matrix element A_{ij}, hence

$$\frac{\partial}{\partial A_{ij}}(\det A) = (-1)^{i+j} M_{ij}.$$

We conclude that

$$\frac{d}{dt}(\det A) = \sum_{i,j=1}^{n} (-1)^{i+j} M_{ij} \frac{dA_{ij}}{dt}.$$

Recall also that $(A^{-1})_{ij} = (1/\det A)(-1)^{i+j} M_{ji}$, meaning that

$$\sum_{j=1}^{n} (-1)^{j+i} M_{ij} A_{kj} = (\det A)\delta_{ik}.$$

We apply this now to the matrix $H_{ij} = (\dfrac{\partial X_i(t,\alpha)}{\partial \alpha_j})$,

$$\frac{dJ}{dt} = \sum_{i,j=1}^{n} (-1)^{i+j} M_{ij} \frac{d}{dt}(\frac{\partial X_i(t,\alpha)}{\partial \alpha_j}),$$

and

$$J\delta_{ik} = \sum_{j=1}^{n} (-1)^{j+i} M_{ij} \frac{\partial X_k}{\partial \alpha_j}. \tag{1.4}$$

Here, M_{ij} are the minors of the matrix H_{ij}. As

$$\frac{d}{dt}(\frac{\partial X_i(t,\alpha)}{\partial \alpha_j}) = \frac{\partial}{\partial \alpha_j}(u_i(t,X(t,\alpha))) = \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k}\frac{\partial X_k}{\partial \alpha_j},$$

we get

$$\frac{dJ}{dt} = \sum_{i,j,k=1}^{n} (-1)^{i+j} M_{ij} \frac{\partial u_i}{\partial x_k}\frac{\partial X_k}{\partial \alpha_j} = \sum_{i,k=1}^{n} \frac{\partial u_i}{\partial x_k} J\delta_{ik} = J(\nabla \cdot u). \tag{1.5}$$

Preservation of the volume means that $J \equiv 1$, which is, thus, equivalent to the incompressibility condition:

$$\nabla \cdot u = 0. \tag{1.6}$$

Here, we use the notation

$$\nabla \cdot u = \text{div} u = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k}.$$

More generally, if the density is not constant, mass conservation would require that for any initial volume V_0 we would have (recall that $\rho(t,x)$ is the fluid density)

$$\frac{d}{dt}\int_{V(t)} \rho(t,x)dx = 0. \tag{1.7}$$

Writing

$$\int_{V(t)} \rho(t,x)dx = \int_{V_0} \rho(t,X(t,\alpha))J(t,\alpha)d\alpha, \tag{1.8}$$

we see that mass conservation is equivalent to the condition

$$\frac{d}{dt}(\rho(t,X(t,\alpha))J(t,\alpha)) = 0. \tag{1.9}$$

Using expression (1.5) leads to

$$\frac{\partial \rho}{\partial t}J + (u \cdot \nabla\rho)J + \rho(\nabla \cdot u)J = 0. \tag{1.10}$$

Dividing by J we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \tag{1.11}$$

We note briefly some basic properties of (1.11). First, the total mass over the whole space is conserved:

$$\int_{\mathbb{R}^n} \rho(t,x)dx = \int_{\mathbb{R}^n} \rho(0,x)dx. \tag{1.12}$$

This follows both from (1.11) after integration (assuming an appropriate decay at infinity), and from our derivation of the continuity equation. If (1.11) is posed in a bounded domain Ω then, in order to ensure mass preservation, one may assume that the flow does not penetrate the boundary $\partial\Omega$:

$$u \cdot \nu = 0 \text{ on } \partial\Omega. \tag{1.13}$$

Here, ν is the outward normal to $\partial\Omega$. Under this condition, we have

$$\int_\Omega \rho(t,x)dx = \int_\Omega \rho(0,x)dx. \tag{1.14}$$

This may be verified directly from (1.11) but it also follows from our derivation of the continuity equation since (1.13) implies that Ω is an invariant region for the flow u.

Furthermore, (1.11) preserves the positivity of the solution: if $\rho(0,x) \geq 0$ then $\rho(t,x) \geq 0$ for all $t > 0$ and x.

Newton's second law in an inviscid fluid

The continuity equation for the evolution of the density $\rho(t,x)$ should be supplemented by an evolution equation for the fluid velocity $u(t,x)$. This will come from Newton's second law of motion. Consider a fluid volume V. If the fluid is inviscid, so that there is no "internal friction" in the fluid, the only force acting on this volume is due to the pressure:

$$F = -\int_{\partial V} p\nu dS = -\int_V \nabla p dx, \tag{1.15}$$

where ∂V is the boundary of V, and ν is the outside normal to ∂V. Taking V to be an infinitesimal volume at a point $X(t)$, which moves with the fluid, Newton's second law of motion leads to the balance

$$\rho(t,X(t))\ddot{X}(t) = -\nabla p(t,X(t)). \tag{1.16}$$

We may compute $\ddot{X}(t)$:

$$\begin{aligned}\ddot{X}_j(t) &= \frac{d}{dt}(u_j(t,X(t)) = \frac{\partial u_j(t,X(t))}{\partial t} + \sum_k \dot{X}_k(t)\frac{\partial u_j(t,X(t))}{\partial x_k} \\ &= \frac{\partial u_j(t,X(t))}{\partial t} + u(t,X(t))\cdot\nabla u_j(t,X(t)).\end{aligned} \tag{1.17}$$

Therefore, we have the following equation of motion:

$$\rho\Big(\frac{\partial u}{\partial t} + u\cdot\nabla u\Big) + \nabla p = 0. \tag{1.18}$$

Equations (1.11) and (1.18) do not form a closed system of equations by themselves – they involve $n+1$ equations for $n+2$ unknowns (the density $\rho(t,x)$, the pressure $p(t,x)$ and the fluid velocity $u(t,x)$). The missing equation should provide the connection between the density and the pressure, and this comes from the physics of the problem. In gas dynamics, it often takes the form of a constitutive relation $p = F(\rho)$, where $F(\rho)$ is a given function, such as $F(\rho) = C\rho^\gamma$ with some constant $\gamma > 0$. Then, the full system becomes

$$\begin{aligned}&\rho_t + \nabla\cdot(\rho u) = 0 \\ &u_t + u\cdot\nabla u + \frac{1}{\rho}\nabla p = 0, \\ &p = F(\rho).\end{aligned} \tag{1.19}$$

The pressure may also depend on the temperature, and then the evolution of the local temperature has to be included as well but we will not discuss this at the moment.

The linearized equations

The simplest solution of (1.19) is the constant density and pressure, zero fluid velocity state:

$$\rho = \rho_0,\ p = p_0 = F(\rho_0) \text{ and } u = 0. \tag{1.20}$$

Let us consider a small perturbation around this state:

$$\begin{aligned}
&\rho = \rho_0 + \varepsilon\eta + O(\varepsilon^2),\\
&p = p_0 + \varepsilon F'(\rho_0)\eta + O(\varepsilon^2)\\
&u = \varepsilon v + O(\varepsilon^2).
\end{aligned} \tag{1.21}$$

Inserting these expansions into (1.19) gives, in the (leading) order $O(\varepsilon)$:

$$\begin{aligned}
&\eta_t + \rho_0 \nabla \cdot v = 0\\
&v_t + \frac{F'(\rho_0)}{\rho_0}\nabla\eta = 0.
\end{aligned} \tag{1.22}$$

It is common to write this system in terms of v and the pressure perturbation $\tilde{p} = F'(\rho_0)\eta$. After dropping the tilde it becomes the linearized acoustic system

$$\kappa_0 p_t + \nabla \cdot v = 0 \tag{1.23}$$

$$\rho_0 v_t + \nabla p = 0. \tag{1.24}$$

Here, $\kappa_0 = 1/(F'(\rho_0)\rho_0)$ is the compressibility constant. Differentiating (1.23) in time and using (1.24) leads to the wave equation for pressure:

$$\frac{1}{c_0^2} p_{tt} - \Delta p = 0, \tag{1.25}$$

with the sound speed

$$c_0 = \frac{1}{\sqrt{\rho_0\kappa_0}} = \sqrt{F'(\rho_0)}. \tag{1.26}$$

The linearized acoustics is what governs most of the "real-world" applications at "bearable" sound levels but we will not pursue this direction at the moment, and rather focus on incompressible fluids.

Euler's equations in incompressible fluids

A common approximation in the fluid dynamics is to assume that the fluid is incompressible, that is, its density is constant: $\rho(t, x) = \rho_0$. Using this condition in (1.11), leads to another form of the incompressibility condition:

$$\nabla \cdot u = 0, \tag{1.27}$$

that we have already seen before in (1.6) as the volume preservation condition for the flow.

Equations (1.18) and (1.27) together form Euler's equations for an incompressible fluid:

$$\begin{aligned}&\frac{\partial u}{\partial t}+u\cdot\nabla u+\frac{1}{\rho_0}\nabla p=0,\\&\nabla\cdot u=0.\end{aligned} \tag{1.28}$$

In order to find the pressure, we may take the divergence of the first equation above, leading to the Poisson equation for the pressure in terms of the velocity field:

$$\Delta p=-\rho_0\nabla\cdot(u\cdot\nabla u)=-\rho_0\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(u_k\frac{\partial u_j}{\partial x_k}\Big)=-\rho_0\sum_{i,j=1}^{n}\frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}. \tag{1.29}$$

We used the incompressibility condition in the last equality above. Equations (1.28)-(1.29) together may be thought of as a closed system of equations for the velocity $u(t,x)$ alone. The Poisson equation for the pressure means that it is a non-local function of the velocity, hence the Euler equations are a non-local system of equations for the fluid velocity – the pressure field at a given point depends on the velocity distribution in the whole space.

When the problem is posed in a bounded domain, we need to prescribe the boundary conditions for the fluid velocity and pressure. If the physical domain Ω is fixed and the fluid does not penetrate through its boundary, a natural physical condition for the fluid velocity is that the normal component of the velocity vanishes at the boundary:

$$\nu\cdot u=0 \text{ on } \partial\Omega, \tag{1.30}$$

where ν is the outward normal to the boundary. It follows that

$$\nu\cdot\frac{\partial u}{\partial t}=0 \text{ on } \partial\Omega, \tag{1.31}$$

thus the pressure satisfies the Neumann boundary conditions

$$\frac{\partial p}{\partial\nu}=-\rho_0\nu\cdot(u\cdot\nabla u) \text{ on } \partial\Omega. \tag{1.32}$$

Often, as a simplification we will consider the Euler equations either in the whole space, with the decaying boundary conditions at infinity, or with the periodic boundary conditions on a two- or three-dimensional torus, as the boundaries bring extra (and very interesting) difficulties into an already difficult problem.

The viscous stress and the Navier-Stokes equations

The previous discussion did not take into account the viscosity of a fluid, which comes from the forces that resist the shearing motions because of the microscopic friction. The forces normal to a given area element are associated to the pressure (which we did take into account), while those acting in the plane of the area element are associated to the shear stress. In order to derive the fluid motion equations, as a generalization of the force on a volume element V coming from the pressure field:

$$F=-\int_{\partial V}p\nu dS=-\int_V\nabla p dx, \tag{1.33}$$

we may write, for the force that acts on an infinitesimal surface area dS of a volume element V:

$$dF_j = \sum_{k=1}^{n} \nu_k \tau_{kj} dS, \tag{1.34}$$

where ν is the outward normal to dS, and τ is the total stress tensor (that includes both the pressure and the shear stress). Integrating this expression leads to the total force acting on the volume V:

$$F_j = \sum_{k=1}^{n} \int_{\partial V} \nu_k \tau_{kj} dS = \sum_{k=1}^{n} \int_V \frac{\partial \tau_{kj}}{\partial x_k} dx. \tag{1.35}$$

We will use the notation $\nabla \cdot \tau$ for the vector with the components

$$(\nabla \cdot \tau)_j = \sum_{k=1}^{n} \frac{\partial \tau_{kj}}{\partial x_k}, \tag{1.36}$$

as well as denote

$$(\nu \cdot \tau)_j = \sum_{k=1}^{n} \nu_k \frac{\partial \tau_{kj}}{\partial x_k}, \tag{1.37}$$

Let us assume for the moment that the fluid is in equilibrium, and let f be the internal forces and τ the stress tensor, and V an arbitrary volume element. Then the balance of forces says that

$$\int_V f dx + \int_V \nabla \cdot \tau dx = 0, \tag{1.38}$$

which means that

$$f + \nabla \cdot \tau = 0. \tag{1.39}$$

The total angular momentum of the force should also vanish, meaning that (in three dimensions)

$$\int_V (f \times x) dx + \int_{\partial V} (\nu \cdot \tau) \times x dS = 0, \tag{1.40}$$

for each volume element V. The surface integral above can be re-written as[1]

$$\int_{\partial V} \varepsilon_{ijk} \nu_l \tau_{lj} x_k dS = \int_V \varepsilon_{ijk} \frac{\partial}{\partial x_l} (\tau_{lj} x_k) dx = \int_V \varepsilon_{ijk} \Big(\frac{\partial \tau_{lj}}{\partial x_l} x_k + \tau_{kj} \Big) dx, \quad \text{for each } i = 1, 2, 3. \tag{1.41}$$

Here, ε_{ink} is the totally anti-symmetric tensor: $(v \times w)_i = \varepsilon_{ijk} v_j w_k$, and $\varepsilon_{ijk} = 0$ if any pair of the indices i, j, k coincide, while if all i, j, k are different, then $\varepsilon_{ijk} = (-1)^{p+1}$, where $p = 1$ if (ijk) is an even permutation, and $p = 0$ if it is odd. Using (1.39) in (1.41), we get

$$\int_{\partial V} \varepsilon_{ijk} \nu_l \tau_{lj} x_k dS = \int_V \varepsilon_{ijk} \Big(- f_j x_k + \tau_{kj} \Big) dx, \quad \text{for each } i = 1, 2, 3. \tag{1.42}$$

Returning to (1.40), and combing it with (1.42), we obtain

$$0 = \int_V \varepsilon_{ijk} f_j x_k dx + \int_V \varepsilon_{ijk} \Big(- f_j x_k + \tau_{kj} \Big) dx = \int_V \varepsilon_{ijk} \tau_{kj} dx, \text{ for each } i = 1, 2, 3. \tag{1.43}$$

[1] From now we will use the convention that the repeated indices are summed unless specified otherwise.

As a consequence,

$$\varepsilon_{ijk}\tau_{jk} = 0, \text{ for each } i = 1, 2, 3, \tag{1.44}$$

which means that the tensor τ_{ij} has to be symmetric.

Exercise. Modify the above computation to show that the stress tensor is symmetric even if the fluid is not in an equilibrium.

We may now go back to the derivation of the Euler equations and proceed as before, the difference being that the force term in the Newton second law is not $-\nabla p$ but $\nabla \cdot \tau$. This will lead to the equation of motion

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \frac{1}{\rho} \nabla \cdot \tau. \tag{1.45}$$

As for the Euler equations, the evolution equation for the fluid velocity needs to be supplemented by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \tag{1.46}$$

Previously, we needed also to prescribe the equation of state – the relation between the pressure and the density. Now, we need to postulate, or derive, an expression for the stress tensor. We will decompose it as

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}. \tag{1.47}$$

The first term comes from the pressure – it leads to a force acting on a surface element in the direction normal to the surface element. The second term comes from the shear stress, and comes from the friction inside the fluid. It is natural to assume that it depends locally on ∇u – if the flow is uniform there is no shearing force. In order to understand this dependence, recall that, given a flow

$$\frac{dX}{dt} = u(t, X(t)), \quad X(0) = \alpha, \tag{1.48}$$

the deformation tensor $J_{ij} = \partial X_i / \partial \alpha_j$ obeys

$$\frac{dJ_{ij}}{dt} = \frac{\partial u_i}{\partial x_m} J_{mj}, \quad J_{ij}(0) = \delta_{ij}. \tag{1.49}$$

Therefore, the skew-symmetric part of the matrix ∇u (locally in time and space) leads to a rigid-body rotation and does not contribute to the shearing force. Hence, it is natural to assume that the shear stress σ_{ij} depends only on the symmetric part of ∇u:

$$D_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{1.50}$$

In a Newtonian fluid, the shear stress depends linearly on the deformation tensor D_{ij}: $\sigma = L(D)$ for some linear map between symmetric matrices. The map L should not depend on the point x and it should be isotropic: for each rotation matrix Q we should have

$$L(QDQ^*) = QL(D)Q^*. \tag{1.51}$$

Exercise. Show that the above conditions imply that the map L has to have the form

$$[L(D)]_{ij} = 2\mu D_{ij} + \lambda\delta_{ij}\mathrm{Tr}(D), \tag{1.52}$$

with some constants λ and μ. These constants are called the Lamé parameters in the context of the elasticity theory.

For an incompressible fluid, we have

$$\mathrm{Tr}D = \nabla \cdot u = 0, \tag{1.53}$$

hence the stress tensor has a simpler form

$$\sigma_{ij} = 2\mu D_{ij}. \tag{1.54}$$

We will make an additional assumption that μ and λ are constants that do not depend on other physical parameters such as temperature, density or pressure. Then the force term in (1.45) can be written as

$$\begin{aligned}[\nabla \cdot \tau]_k = \frac{\partial \tau_{jk}}{\partial x_j} = \frac{\partial}{\partial x_j}\Big[- p\delta_{jk} + \mu(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}) + \lambda(\nabla \cdot u)\delta_{jk}\Big] \\ = -\frac{\partial p}{\partial x_k} + \mu\Delta u_k + (\mu + \lambda)\frac{\partial}{\partial x_k}(\nabla \cdot u).\end{aligned} \tag{1.55}$$

This leads to the Navier-Stokes equations of compressible fluid dynamics

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho}\nabla p = \frac{\mu}{\rho}\Delta u + \frac{(\mu + \lambda)}{\rho}\nabla(\nabla \cdot u) \tag{1.56}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{1.57}$$

$$p = F(\rho). \tag{1.58}$$

As with the Euler equations, the equation of state may also involve the temperature, and then the evolution equation for the temperature should also be prescribed.

The incompressibility constraint (constant density) simplifies the system (1.56)-(1.58) to the system of incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho_0}\nabla p = \frac{\mu}{\rho_0}\Delta u \tag{1.59}$$

$$\nabla \cdot u = 0. \tag{1.60}$$

Note that Euler's equations are formally recovered from the Navier-Stokes equation by setting the viscosity $\mu = 0$.

Remark. From now on, unless specified otherwise, we will consider only the incompressible Euler and Navier-Stokes equations.

Two-dimensional flows

We will sometimes consider the two-dimensional version of the Navier-Stokes equations, which has exactly the same form as the three-dimensional equations (1.59)-(1.60) but with the fluid velocity that has only two components: $u = (u_1, u_2)$, and, in addition, the problem is posed for $x \in \mathbb{R}^2$. These can be interpreted as the solutions of the three-dimensional Navier-Stokes system of a special form $u = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$ with the pressure $p = p(x_1, x_2)$ – that is, they are independent of x_3 and the third component of the fluid velocity vanishes. It is straightforward to check that, indeed, they satisfy (1.59)-(1.60) provided that $\tilde{u} = (u_1, u_2)$ satisfies

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} + \frac{1}{\rho_0} \nabla p = \frac{\mu}{\rho_0} \Delta \tilde{u} \tag{1.61}$$

$$\nabla \cdot \tilde{u} = 0, \tag{1.62}$$

posed in $\mathbb{R}^2$ and not $\mathbb{R}^3$.

2 The vorticity evolution

The vorticity

An important role in the theory of fluids is played by the fluid vorticity. It is defined in terms of the fluid velocity $u(t, x)$ as a vector

$$\omega = \text{curl} u = \nabla \times u, \quad \omega_i = \varepsilon_{ijk} \partial_j u_k, \tag{2.1}$$

in three dimensions, and as a scalar

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \tag{2.2}$$

in two dimensions. The two-dimensional vorticity can be understood as the x_3-component of the three-dimensional vorticity of the flow $(u_1(x_1, x_2), u_2(x_1, x_2), 0)$ – the other two components of the vorticity vanish for such flows.

The vorticity vector field in three dimensions is always divergence free:

$$\nabla \cdot \omega = \varepsilon_{ijk} \partial_i \partial_j u_k = 0. \tag{2.3}$$

Vorticity conservation in two dimensions

Let us now compute the evolution equation for the vorticity in two and three dimensions. In the two-dimensional case, we start with the Navier-Stokes equations (we will set the density $\rho_0 = 1$ for simplicity from now on, unless specified otherwise)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u, \tag{2.4}$$

and compute

$$
\begin{aligned}
\frac{\partial \omega}{\partial t} &= \frac{\partial}{\partial x_1}\Big(\nu\Delta u_2 - \frac{\partial p}{\partial x_2} - u_1\frac{\partial u_2}{\partial x_1} - u_2\frac{\partial u_2}{\partial x_2}\Big) - \frac{\partial}{\partial x_2}\Big(\nu\Delta u_1 - \frac{\partial p}{\partial x_1} - u_1\frac{\partial u_1}{\partial x_1} - u_2\frac{\partial u_1}{\partial x_2}\Big) \\
&= \nu\Delta\omega - \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} - u_1\frac{\partial^2 u_2}{\partial {x_1}^2} - \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} - u_2\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_1} + u_1\frac{\partial^2 u_1}{\partial x_1\partial x_2} \qquad (2.5) \\
&+\frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} + u_2\frac{\partial^2 u_1}{\partial {x_2}^2} = \nu\Delta\omega - u_1\frac{\partial}{\partial x_1}(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) - u_2\frac{\partial}{\partial x_2}(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) = \Delta\omega - u\cdot\nabla\omega.
\end{aligned}
$$

In the last step, we computed that

$$
\begin{aligned}
&-\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial x_1}\Big(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\Big) + \frac{\partial u_2}{\partial x_2}\Big(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\Big) \\
&= -\omega\nabla\cdot u = 0. \qquad (2.6)
\end{aligned}
$$

The "miracle" is that in two dimensions the term we have calculated in (2.6), and which in three dimensions will contribute to the vorticity growth, cancels out completely because of the incompressibility condition. Thus, in two dimensions, the vorticity satisfies an advection-diffusion equation

$$
\frac{\partial \omega}{\partial t} + u\cdot\nabla\omega = \nu\Delta\omega. \qquad (2.7)
$$

This is very remarkable as (2.7) obeys the maximum principle: with appropriate decay conditions at infinity (if (2.7) is posed in the whole space $\mathbb{R}^2$), we can immediately conclude that

$$
\|\omega(t,\cdot)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}, \qquad (2.8)
$$

where $\omega_0(x) = \omega(0,x)$ is the initial data for the vorticity. Furthermore, in an inviscid fluid, when $\nu = 0$ the vorticity is simply advected along the flow lines; solution of

$$
\frac{\partial \omega}{\partial t} + u\cdot\nabla\omega = 0 \qquad (2.9)
$$

is simply

$$
\omega(t,x) = \omega_0(t, A(t,x)), \qquad (2.10)
$$

where $A(t,x)$ is the "back-to-labels" map for (1.1). This will help us later to prove the regularity of the solutions of the Euler and Navier-Stokes equations in two dimensions (though it will not imply the regularity immediately).

Note also that the pressure term is nowhere to be seen in (2.7). Thus, in order to close the problem, we only need to supplement the evolution equation (2.7) for vorticity by an expression for the fluid velocity in terms of vorticity. To this end, observe, that, as $u(t,x)$ is divergence free, and the problem is posed in all of $\mathbb{R}^2$, there exists a function $\psi(t,x)$, called the stream function, so that $u(t,x)$ has the form

$$
u(t,x) = \nabla^\perp\psi(t,x) = (-\psi_{x_2}(t,x), \psi_{x_1}(t,x)). \qquad (2.11)
$$

To see this, note that, because of the divergence-free condition for $u(t,x)$, the flow

$$
v(t,x) = (u_2, -u_1), \qquad (2.12)
$$

satisfies

$$\frac{\partial v_1}{\partial x_2} = \frac{\partial v_2}{\partial x_1}, \tag{2.13}$$

hence there exists a function $\psi(t,x)$ so that $v(t,x) = \nabla\psi(t,x)$, which is equivalent to (2.11).

The vorticity can be expressed in terms of the stream function as

$$\Delta\psi = \omega, \tag{2.14}$$

or, more explicitly,

$$\psi(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|)\omega(t,y)dy. \tag{2.15}$$

Differentiating (2.15) formally, we obtain an expression for the fluid velocity in terms of its vorticity

$$u(t,x) = \int_{\mathbb{R}^2} K_2(x-y)\omega(t,y)dy, \tag{2.16}$$

with the vector-valued integral kernel

$$K_2(x) = \frac{1}{2\pi}\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big). \tag{2.17}$$

Thus, the Navier-Stokes equations in two dimensions can be formulated purely in terms of vorticity as the advection-diffusion equation for the scalar vorticity

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \nu\Delta\omega, \tag{2.18}$$

with the velocity $u(t,x)$ given in terms of $\omega(t,x)$ by (2.16). The function $K_2(x)$ is singular, homogeneous of degree (-1) in x. Thus, it is not obvious that (2.17) gives a sufficiently regular velocity field $u(t,x)$ for the coupled problem to have a smooth solution even if the initial data $\omega_0(x) = \omega(0,x)$ is smooth and rapidly decaying at infinity. However, the "$1/x$" singularity in two dimensions is sufficiently mild: writing (2.16) in the polar coordinates gives (with $x^\perp = (-x_2, x_1)$)

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}(-\sin\phi,\cos\phi)\omega(x_1-r\cos\phi, x_2-r\sin\phi)d\phi dr, \tag{2.19}$$

There is no longer a singularity in (2.19), and the expression for the velocity "makes sense".

The system (2.16), (2.17), (2.18) is an example of an active scalar – the vorticity $\omega(t,x)$ is a solution of an advection-diffusion equation with the velocity coupled to the advected scalar itself.

Vorticity evolution in three dimensions

The situation in three dimensions is very different. In order to compute the evolution equation for the vorticity vector, first, note that the advection term in the Navier-Stokes equations can be written as

$$(u\cdot\nabla u)_i = u_j\frac{\partial u_i}{\partial x_j} = u_j\Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big) + u_j\frac{\partial u_j}{\partial x_i}, \tag{2.20}$$

and that

$$(\omega \times u)_i = \varepsilon_{ijk}\omega_j u_k = \varepsilon_{ijk}\varepsilon_{jmn}(\partial_m u_n)u_k = (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn})(\partial_m u_n)u_k$$
$$= (\partial_k u_i)u_k - (\partial_i u_k)u_k. \tag{2.21}$$

We used above the identity

$$\varepsilon_{jik}\varepsilon_{jmn} = \delta_{im}\delta_{kn} - \delta_{in}\delta_{km} \tag{2.22}$$

and anti-symmetry of ε_{ijk}. We see that

$$u \cdot \nabla u = \omega \times u + \nabla\Big(\frac{|u|^2}{2}\Big). \tag{2.23}$$

Therefore, the Navier-Stokes equations with a force f can be written as

$$u_t + \omega \times u + \nabla\Big(\frac{|u|^2}{2} + p\Big) = \nu\Delta u + f. \tag{2.24}$$

The formula

$$\mathrm{curl}(a \times b) = -a \cdot \nabla b + b \cdot \nabla a + a(\nabla \cdot b) - b(\nabla \cdot a) \tag{2.25}$$

helps us to take the curl of (2.24), leading to the vorticity equation:

$$\omega_t + u \cdot \nabla\omega = \nu\Delta\omega + V(t,x)\omega, \tag{2.26}$$

with

$$V(t,x)\omega = \omega \cdot \nabla u, \quad V_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{2.27}$$

We can decompose the matrix V into its symmetric and anti-symmetric parts:

$$V = D + \Omega, \quad D = \frac{1}{2}(V + V^T), \quad \Omega = \frac{1}{2}(V - V^T), \tag{2.28}$$

and observe that, for any $h \in \mathbb{R}^3$

$$\Omega_{ij}h_j = \frac{1}{2}[\partial_j u_i - \partial_i u_j]h_j = \frac{1}{2}\partial_m u_k[\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}]h_j = \frac{1}{2}\varepsilon_{lij}\varepsilon_{lkm}(\partial_m u_k)h_j$$
$$= -\frac{1}{2}\varepsilon_{lij}\varepsilon_{lmk}(\partial_m u_k)h_j = -\frac{1}{2}\varepsilon_{lij}\omega_l h_j = \frac{1}{2}\varepsilon_{ilj}\omega_l h_j = \frac{1}{2}[\omega \times h]_i, \tag{2.29}$$

that is,

$$\Omega h = \frac{1}{2}\omega \times h. \tag{2.30}$$

The matrix Ω has an explicit form

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{2.31}$$

As a consequence, we have $\Omega\omega = 0$, thus $V\omega = D\omega$, and the vorticity equation has the form

$$\omega_t + u \cdot \nabla\omega = \nu\Delta\omega + D(t,x)\omega, \tag{2.32}$$

with

$$D_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{2.33}$$

The term $D\omega$ in the vorticity equation is known as the vortex stretching term, and it is maybe the main reason why the solutions of the three- dimensional Navier-Stokes equations exhibit such rich behavior and complexity. As we have done in two dimensions, it is possible to express the velocity $u(t,x)$ in terms of the vorticity – this relation is known as the Biot-Savart law, leading to the "pure vorticity" formulation of the Navier-Stokes equations, but we will postpone this computation until slightly later.

The evolution of the matrix D itself is obtained by differentiating the Navier-Stokes equations to get an evolution equation for the matrix V:

$$\frac{\partial}{\partial t}\Big(\frac{\partial u_i}{\partial x_j}\Big) + u_m \frac{\partial}{\partial x_m}\Big(\frac{\partial u_i}{\partial x_j}\Big) + \frac{\partial u_m}{\partial x_j}\frac{\partial u_i}{\partial x_m} + \frac{\partial^2 p}{\partial x_i \partial x_j} = \nu\Delta\frac{\partial u_i}{\partial x_j}, \tag{2.34}$$

which, in the matrix form is

$$\frac{\partial V}{\partial t} + u\cdot\nabla V + V^2 + H = \nu\Delta V, \tag{2.35}$$

where H is the Hessian of the pressure. Taking the symmetric part gives

$$\frac{\partial D}{\partial t} + u\cdot\nabla D + D^2 + \Omega^2 + H = \nu\Delta D. \tag{2.36}$$

An analogy to the Burgers' equation

The vorticity equation (2.32) has a quadratic term in ω in the right side. Such quadratic nonlinearities may potentially lead to a blow up. This is easily seen on the simple ODE example

$$\dot z = z^2, \quad z(0) = z_0. \tag{2.37}$$

Its explicit solution is

$$z(t) = \frac{z_0}{1 - z_0 t}. \tag{2.38}$$

If $z_0 > 0$, the solution becomes infinite at the time

$$t_c = \frac{1}{z_0}. \tag{2.39}$$

At a slightly more sophisticated level, we can look at the familiar Burgers' equation on the line:

$$u_t + uu_x = 0, \quad u(0,x) = u_0(x). \tag{2.40}$$

which is sIts solutions develop a finite time singularity if the initial condition $u_0(x)$ is decreasing on some interval. Such discontinuities are known as shocks. In order to make a connection to the vorticity equation, set set $\omega = -u_x$, this function satisfies

$$\omega_t + u\omega_x = \omega^2, \quad \omega(0,x) = \omega_0(x) = -u_0'(x). \tag{2.41}$$

This equation is analogous to the vorticity equation except the nonlinearity has a different form: $D(\omega)\omega$ is replaced by ω^2. As in the case of the quadratic ODE (2.37), the function $\omega(t,x)$ becomes infinite in a finite time if there are points where $\omega_0(x) > 0$. One should mention that there are two regularizations of the inviscid Burgers' equation (2.40): first, adding a diffusive (dissipative) term gives the viscous Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \quad u(0,x) = u_0(x), \tag{2.42}$$

which has global in time smooth solutions if $u_0(x)$ is smooth. A natural question which we may revisit later is why is the u_{xx} term sufficiently regularizing? More precisely, one may consider equations of the form

$$u_t + uu_x = Au, \quad u(0,x) = u_0(x), \tag{2.43}$$

where A is a linear dissipative operator in the sense that

$$(Au,u) = \int_{\mathbb{R}} (Au(x))u(x)dx \le 0. \tag{2.44}$$

If A commutes with differentiation, the "vorticity" equation will have the form

$$\omega_t + u\omega_x = A\omega + \omega^2, \quad \omega(0,x) = \omega_0(x) = -u_0'(x). \tag{2.45}$$

Then, the dissipative effect of $A\omega$ will compete with the growth caused by ω^2 in the right side. The issue of when the dissipation will win is rather delicate – we will revisit it later if we have time.

There is a different approach to the blow up in the Burgers' equation that illustrates a general strategy of trying to control integral functionals of the solution rather than solutions themselves. Let us consider, for simplicity, the solution of the Burgers' equation on the line with a periodic initial condition $u_0(x)$:

$$u_0(x + 2\pi) = u_0(x).$$

Then the solution of

$$u_t + uu_x = 0, \quad u(0,x) = u_0(x) \tag{2.46}$$

will stay periodic for all $t > 0$ (as long as it exists):

$$u(t, x + 2\pi) = u(t,x). \tag{2.47}$$

If, in addition, the initial data is odd: $u_0(-x) = -u_0(x)$, then solution remains odd as well: $u(t,x) = -u(t,x)$. This means that, as long as the solution remains smooth, the functional

$$L(t) = \int_{\pi}^{\pi} \frac{u(t,x)}{x} dx \tag{2.48}$$

is well-defined and finite – the function $u(t,x)$ vanishes at $x = 0$. Differentiating $L(t)$ in time gives

$$\frac{dL(t)}{dt} = \int_{\pi}^{\pi} \frac{u_t(t,x)}{x} dx = -\int_{-\pi}^{\pi} \frac{1}{x} uu_x dx = -\frac{1}{2}\int_{-\pi}^{\pi} \frac{u^2(t,x)}{x^2} dx. \tag{2.49}$$

The Cauchy-Schwartz inequality implies that

$$L^2(t) = \left(\int_{-\pi}^{\pi} \frac{u(t,x)}{x} dx\right)^2 \leq 2\pi \int_{-\pi}^{\pi} \frac{u^2(t,x)}{x^2} dx. \tag{2.50}$$

Hence, the function $L(t)$ satisfies a differential inequality

$$\frac{dL}{dt} \leq -\frac{1}{4\pi} L^2(t). \tag{2.51}$$

Integrating this inequality in time gives

$$\frac{1}{L_0} - \frac{1}{L(t)} \leq -\frac{t}{4\pi}. \tag{2.52}$$

Hence, we have

$$L(t) \leq \frac{4\pi L_0}{4\pi + L_0 t}. \tag{2.53}$$

We conclude that if $L_0 < 0$ then $L(t) = -\infty$ at some time $t < -4\pi/L_0$, thus solution may not remain smooth past this time. The condition that $L_0 < 0$ distinguishes between the initial data that "look like" $u_0(x) = \sin x$ and like $u_0(x) = -\sin x$. The latter is decreasing at $x = 0$, hence the shock is expected to form there, thus it is reasonable to expect that $L(t)$, which has x in the denominator in the integrand, will blow-up. On the other hand, the former is increasing at $x = 0$, thus the shock would not form there, and $L(t)$ should not capture the singularity formation. A different functional should be considered to capture the blow-up.

Another very interesting regularization of the inviscid Burgers' equation is via dispersion:

$$u_t + u u_x = \mu u_{xxx}, \quad u(0,x) = u_0(x). \tag{2.54}$$

This is the Kortweg-de Vries equation which describes a regime of the shallow water waves. Its mathematics is incredibly rich and is connected by now with nearly every area of mathematics. If we have time, we will go back to it as well.

Flows with spatially homogenous vorticity

As an example, we consider flows that have a spatially uniform vorticity $\omega(t)$. Let us choose a symmetric matrix $D(t)$ with $\mathrm{Tr}D(t) = 0$, and a vector-valued function $\omega(t) \neq 0$ such that

$$\frac{d\omega}{dt} = D(t)\omega(t), \quad \omega(0) = \omega_0. \tag{2.55}$$

We also define an anti-symmetric matrix $\Omega(t)$, given by (2.31), so that

$$\Omega(t)h = \frac{1}{2}\omega(t) \times h, \text{ for any } h \in \mathbb{R}^3, \quad \Omega_{ij} = \varepsilon_{imj}\omega_m. \tag{2.56}$$

A direct computation, using the symmetry of D, the assumption $\mathrm{Tr}D = 0$, and (2.31), gives

$$\dot{\Omega} + D\Omega + \Omega D = 0. \tag{2.57}$$

The observation is that the flow

$$u(t,x) = \frac{1}{2}\omega(t) \times x + D(t)x \tag{2.58}$$

gives an exact solution of the three-dimensional Euler and Navier-Stokes equations, with the vorticity $\text{curl}v = \omega$. Indeed, first, as the trace of $D(t)$ vanishes, both components in (2.58) are divergence-free:

$$\nabla \cdot u = \partial_j(\varepsilon_{jkl}\omega_k x_l) + \partial_j(D_{jk}x_k) = \varepsilon_{jkl}\omega_k\delta_{jl} + D_{jk}\delta_{jk} = 0. \tag{2.59}$$

Moreover, the second term in (2.58) is the gradient of the function $(1/2)(D(t)x \cdot x)$, hence its vorticity vanishes, while identity (2.25) means that

$$\text{curl}u = \frac{1}{2}\text{curl}(\omega(t) \times x) = -\frac{1}{2}\omega \cdot \nabla x + \frac{1}{2}\omega(\nabla \cdot x) = -\frac{1}{2}\omega + \frac{3}{2}\omega = \omega. \tag{2.60}$$

Next, we compute

$$u_t = \frac{1}{2}\dot{\omega} \times x + \dot{D}x, \tag{2.61}$$

and

$$\partial_j u_k = \frac{1}{2}\partial_j(\varepsilon_{kmn}\omega_m x_n) + \partial_j(D_{km}x_m) = \frac{1}{2}\varepsilon_{kmj}\omega_m + D_{kj}, \tag{2.62}$$

so that

$$u \cdot \nabla u_k = u_j\partial_j u_k = \frac{1}{2}\varepsilon_{kmj}u_j\omega_m + u_j D_{kj} = \frac{1}{2}\omega \times u + Du. \tag{2.63}$$

Putting these equations together and using (2.56) leads to

$$\begin{aligned} u_t + u \cdot \nabla u &= \frac{1}{2}\dot{\omega} \times x + \dot{D}x + \frac{1}{2}\omega \times u + Du = \frac{1}{2}\dot{\omega} \times x + \dot{D}x \\ &+ \frac{1}{2}\omega \times \Big(\frac{1}{2}\omega \times x + Dx\Big) + D\Big(\frac{1}{2}\omega \times x + Dx\Big) \\ &= (\dot{D} + \dot{\Omega} + \Omega^2 + D^2 + D\Omega + \Omega D)x = (\dot{D} + \Omega^2 + D^2)x = -\nabla p(t,x) \end{aligned} \tag{2.64}$$

We have used (2.57) in the next to last equality above. The pressure is given explicitly by

$$p(t,x) = -\frac{1}{2}\Big(\frac{\partial D}{\partial t} + D^2 + \Omega^2\Big)x \cdot x. \tag{2.65}$$

We conclude that, given any symmetric trace-less matrix $D(t)$, we may construct a solution of the Euler equations as above.

Example 1. A jet flow. As the first example of using the above construction, we may take $\omega_0 = 0$, so that $\omega(t) = 0$ and $D(t) = \text{diag}(-\gamma_1, -\gamma_2, \gamma_1 + \gamma_2)$ with $\gamma_1, \gamma_2 > 0$. The flow is

$$u(t,x) = (-\gamma_1 x_1, -\gamma_2 x_2, (\gamma_1 + \gamma_2)x_3). \tag{2.66}$$

The particle trajectories are

$$X(t,\alpha) = (e^{-\gamma_1 t}\alpha_1, e^{-\gamma_2 t}\alpha_2, e^{(\gamma_1+\gamma_2)t}\alpha_3), \tag{2.67}$$

and have the form of a jet, going toward the x_3-axis, and up along this line for $x_3 > 0$, and down this direction for $x_3 < 0$.

Example 2. A strain flow. Consider $D = \text{diag}(-\gamma, \gamma, 0)$ with $\gamma > 0$, and, once again, vorticity $\omega = 0$, so that

$$u(t, x) = (-\gamma x_1, \gamma x_2, 0). \tag{2.68}$$

Then the particle trajectories are

$$X(t, \alpha) = (e^{-\gamma t}\alpha_1, e^{\gamma t}\alpha_2, \alpha_3). \tag{2.69}$$

The particle trajectories stay in a fixed plane orthogonal to the x_3-axis and are stretched in this plane: nearby two particles starting near the x_1-axis with $\alpha_2 > 0$ and $\alpha_2 < 0$ will separate exponentially fast in time.

Shear layer solutions

Here, we will generalize the second example above: we will be looking at flows of the form generalizing (2.68):

$$u(t, x) = (-\gamma x_1, \gamma x_2, w(t, x_1)), \tag{2.70}$$

that is, the third flow component depends only on x_1 and t. Such flows satisfy the Navier-Stokes equations with the pressure $p(t, x) = \gamma(x_1^2 + x_2^2)/2$, provided that the vertical component of the flow w satisfies a linear advection-diffusion equation

$$\frac{\partial w}{\partial t} - \gamma x_1 \frac{\partial w}{\partial x_1} = \nu \frac{\partial^2 w}{\partial {x_1}^2}. \tag{2.71}$$

The vorticity is given by

$$\omega(t, x) = (0, -\frac{\partial w}{\partial x_1}, 0), \tag{2.72}$$

and its second component $\tilde{\omega} = -w_{x_1}$ satisfies (after dropping the tilde)

$$\frac{\partial \omega}{\partial t} - \gamma x_1 \frac{\partial \omega}{\partial x_1} = \nu \frac{\partial^2 \omega}{\partial {x_1}^2} + \gamma\omega. \tag{2.73}$$

Here, we see clearly the three competing effects in the vorticity evolution: the diffusive (dissipative) term $\nu\omega_{x_1 x_1}$, the convective term $-\gamma x_1 \omega_{x_1}$ and the vorticity growth term $\gamma\omega$. It is instructive to look at the three effects in this very simple setting.

First, let us note that when $\gamma > 0$, the vorticity equation (2.73) admits steady solutions:

$$-\gamma x_1 \bar{\omega}' = \nu\bar{\omega}'' + \gamma\bar{\omega}. \tag{2.74}$$

Indeed, setting $y = \lambda x_1$ leads to

$$-\gamma y \bar{\omega}_y = \lambda^2 \nu \bar{\omega}_{yy} + \gamma\bar{\omega}, \tag{2.75}$$

thus, choosing $\lambda = \sqrt{\gamma/\nu}$, we arrive at

$$-y\bar{\omega}_y = \bar{\omega}_{yy} + \bar{\omega}. \tag{2.76}$$

This equation has an explicit steady solution

$$\bar{\omega}(y) = e^{-y^2}/2, \tag{2.77}$$

hence a steady solution of (2.74) is

$$\bar{\omega}(x_1) = e^{-\gamma x_1^2/(2\nu)}. \tag{2.78}$$

Such solutions do not exist when $\gamma = 0$ – they are sustained by the stretch, and are localized in a layer of the width $O(\sqrt{\nu/\gamma})$ around the plane $\{x_1 = 0\}$. They may also not exist at zero viscosity: if $\gamma = 0$ then (2.74) has no non-trivial bounded steady solutions – thus, they are a result of a balance between the stretch and the friction.

Equation (2.73) can be solved explicitly. Fitst, writing

$$\omega(t, x) = e^{\gamma t} z(t, x_1) \tag{2.79}$$

gives

$$\frac{\partial z}{\partial t} - \gamma x_1 \frac{\partial z}{\partial x_1} = \nu \frac{\partial^2 z}{\partial {x_1}^2}. \tag{2.80}$$

Next, making a change of variables:

$$z(t, x) = \eta(\tau(t), e^{\gamma t} x_1) \tag{2.81}$$

with the function $\tau(t)$ to be determined, leads to

$$\dot{\tau}\frac{\partial \eta}{\partial \tau} + \gamma e^{\gamma t} x_1 \frac{\partial \eta}{\partial \xi} - \gamma x_1 e^{\gamma t} \frac{\partial \eta}{\partial \xi} = \nu e^{2\gamma t} \frac{\partial^2 \eta}{\partial \xi^2}. \tag{2.82}$$

Taking

$$\dot{\tau} = \nu e^{2\gamma t}, \tag{2.83}$$

or

$$\tau(t) = \frac{\nu}{2\gamma}\Big(e^{2\gamma t} - 1\Big), \tag{2.84}$$

leads to the standard heat equation

$$\frac{\partial \eta}{\partial \tau} = \frac{\partial^2 \eta}{\partial \xi^2}, \quad \tau > 0, \ \xi \in \mathbb{R}, \tag{2.85}$$

with the initial condition $\eta(0, \xi) = \omega_0(\xi)$. Therefore, the vorticity is

$$\omega(t, x_1) = e^{\gamma t} \int G\Big(\frac{\nu}{2\gamma}(e^{2\gamma t} - 1), e^{\gamma t} x_1 - y\Big)\omega_0(y)dy, \tag{2.86}$$

where $G(t, x_1)$ is the standard heat kernel:

$$G(t, x_1) = \frac{1}{\sqrt{4\pi t}} e^{-|x_1|^2/(4t)}. \tag{2.87}$$

Let us look at the long time behavior of vorticity:

$$\omega(t,x_1)=e^{\gamma t}\Big(\frac{4\pi\nu}{2\gamma}(e^{2\gamma t}-1)\Big)^{-1/2}\int\exp\Big\{-\frac{|e^{\gamma t}x_1-y|^2}{\frac{4\nu}{2\gamma}(e^{2\gamma t}-1)}\Big\}\omega_0(y)dy \tag{2.88}$$
$$\to\bar\omega(x)=\Big(\frac{\gamma}{2\pi\nu}\Big)^{1/2}e^{-\gamma|x_1|^2/(2\nu)}\int\omega_0(y)dy,$$

provided that the initial vorticity $\omega_0\in L^1(\mathbb{R})$. Thus, the vorticity is localized as $t\to+\infty$ around $x_1=0$, in a layer of the width $O(\sqrt{\nu/\gamma})$, and its long time limit is a multiple of the steady solution (2.78).

The Biot-Savart law

We now return to the vorticity equation in three dimensions

$$\omega_t+u\cdot\nabla\omega=\nu\Delta\omega+\omega\cdot\nabla u. \tag{2.89}$$

Our goal is to derive an expression for the velocity u in terms of the vorticity ω, so as to formulate the Euler and Navier-Stokes equations purely in terms of vorticity. In two dimensions, this was done using the stream function, solution of

$$\Delta\psi=\omega, \tag{2.90}$$

with u given by

$$u=\nabla^\perp\psi=(-\psi_{x_2},\psi_{x_1}), \tag{2.91}$$

or, equivalently,

$$u(t,x)=\int_{\mathbb{R}^2}K_2(x-y)\omega(y)dy, \tag{2.92}$$

with the vector-valued integral kernel

$$K_2(x)=\frac{1}{2\pi}\Big(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\Big). \tag{2.93}$$

In three dimensions, given a divergence-free vector field $\omega(x)$ we need to find a divergence-free vector field $u(t,x)$ so that

$$\nabla\times u=\omega,\quad\nabla\cdot u=0. \tag{2.94}$$

Attempting the same strategy as in two dimensions, we define the stream vector ψ via

$$\Delta\psi=\omega, \tag{2.95}$$

and

$$u(x)=-\nabla\times\psi(x). \tag{2.96}$$

Note that $\nabla\cdot\psi$ satisfies

$$\Delta(\nabla\cdot\psi)=0, \tag{2.97}$$

hence, if we assume that $\nabla\cdot\psi$ is bounded, then $\nabla\cdot\psi=0$, and ψ is also divergence-free. The flow u defined by (2.96) is divergence-free: $\nabla\cdot u=0$, and

$$[\nabla\times u]_i=-\varepsilon_{ijk}\partial_j\varepsilon_{kmn}\partial_m\psi_n=-\varepsilon_{kij}\varepsilon_{kmn}\partial_j\partial_m\psi_n=-(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\partial_j\partial_m\psi_n$$
$$=-\partial_i\partial_j\psi_j+\Delta\psi_i, \tag{2.98}$$

that is,

$$\nabla\times u=-\nabla(\nabla\cdot\psi)+\Delta\psi=\omega. \tag{2.99}$$

We have an explicit expression for the stream-vector $\psi(x)$ as the solution of the Poisson equation (2.95):

$$\psi(x)=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\omega(y)dy. \tag{2.100}$$

The velocity is then given by

$$u_i(x)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\varepsilon_{ijk}\partial_j\Big(\frac{1}{|x-y|}\Big)\omega_k(y)dy=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\varepsilon_{ijk}\frac{x_j-y_j}{|x-y|^3}\omega_k(y)dy, \tag{2.101}$$

so that

$$u(x)=\frac{1}{4\pi}\int_{\mathbb{R}^3}K(x-y)\times\omega(y)dy, \tag{2.102}$$

with

$$K(x)=-\frac{1}{4\pi}\frac{x}{|x|^3}. \tag{2.103}$$

As in the two-dimensional case, the integral operator defining $u(x)$ in terms of the vorticity $\omega(x)$ is not "really singular" – the singularity of the $1/|x|^2$ type is cancelled in three dimensions by the Jacobian if we pass to the spherical coordinates. However, unlike in two dimensions, the vorticity equation involves not only $u(x)$ but also the gradient ∇u. Formally differentiating (2.102) leads to (this identity is not quite correct because of the singularity of the integrals involved)

$$\nabla u(x)\text{''}=\text{''}\int_{\mathbb{R}^3}\nabla K(x-y)\times\omega(y)dy. \tag{2.104}$$

The integral kernel $\nabla K(x)$ in (2.104) has the singularity of the type $x/|x|^4$, which can not be simply cancelled by the Jacobian if we pass to the spherical coordinates. Integral operators with a singularity of this type are known as singular integral operators, and we will deal with them in some detail later, leaving for now the vorticity equation on a formal level.

3 The conserved quantities

We will now discuss the physical quantities conserved by the Euler and Navier-Stokes equations. They are important both from the physical and mathematical points of view – a system that possesses sufficiently regular integrals of motion will not have irregular solutions if the initial data is smooth. As we will see, the integrals of motion for the fluid equations are often insufficient to deduce the existence and regularity of solutions.

Kelvin's theorem

Consider a smooth, oriented, closed curve C_0, and let $C(t)$ be its image under a flow $u(t,x)$:

$$C(t) = \{X(t,\alpha): \ \alpha \in C_0\}, \tag{3.1}$$

with

$$\frac{dX}{dt} = u(t,X), \quad X(0,\alpha) = \alpha. \tag{3.2}$$

The circulation around $C(t)$ is

$$\Gamma_{C(t)} = \oint_{C(t)} u(t,x)\cdot d\ell, \tag{3.3}$$

where $d\ell$ is the length element along $\Gamma(t)$. Let us parametrize the initial and evolved curves as

$$C_0 = \{\gamma(s),\ 0 \le s \le 1\}, \quad C(t) = \{X(t,\gamma(s)),\ 0 \le s \le 1\}, \tag{3.4}$$

then the length element along the evolved curve has the components (prime denotes the derivative with respect to the parametrization parameter s)

$$d\ell_j = \frac{\partial X_j}{\partial \gamma_k}\gamma_k' ds, \tag{3.5}$$

or $C'(t,s) = H(t, X(t,\gamma(s))\gamma'(s)$, with the matrix

$$H_{ij}(t, X(t,\alpha)) = \frac{\partial X_i(t,\alpha)}{\partial \alpha_j}, \tag{3.6}$$

which satisfies (1.3)

$$\frac{dH}{dt} = (\nabla u)H. \tag{3.7}$$

Now, we may compute

$$\begin{aligned}
&\frac{d}{dt}\oint_{C(t)} u(t,x)\cdot d\ell = \frac{d}{dt}\int_0^1 u(t,X(t,\gamma_0(s))\cdot(H\gamma')ds = \int_0^1 [(\dot{u}\cdot H\gamma') + (u\cdot \dot{H}\gamma')]ds \\
&= \int_0^1 [(u_t + u\cdot\nabla u)\cdot H\gamma') + (u\cdot(\nabla u H)\gamma')]ds \\
&= \oint_{C(t)} (u_t + u\cdot\nabla u)\cdot d\ell + \oint_{C(t)} (\nabla u)^t u\cdot d\ell
\end{aligned} \tag{3.8}$$

If u satisfies the Euler equations, we have for the first term in the last line above:

$$\oint_{C(t)} (u_t + u\cdot\nabla u)\cdot d\ell = -\oint \nabla p\cdot d\ell = 0. \tag{3.9}$$

The second term can be written as

$$\oint_{C(t)} (\nabla u)^t) u\cdot d\ell = \oint_{C(t)} \frac{\partial u_k}{\partial x_j} u_k d\ell_j = \oint_{C(t)} \nabla\Big(\frac{|u|^2}{2}\Big)\cdot d\ell = 0. \tag{3.10}$$

We see that

$$\frac{d}{dt}\oint_{C(t)} u(t,x)\cdot d\ell = 0. \tag{3.11}$$

This is Kelvin's theorem for the Euler equations: the circulation of the flow along a curve that evolves with the flow is preserved in time.

Conservation of the total velocity and vorticity

If u is a divergence-free velocity field, and q is a scalar function, and both of them decay sufficiently fast at infinity, we have

$$\int_{\mathbb{R}^n}(u\cdot\nabla\phi)dx = -\int(\nabla\cdot u)\phi dx = 0. \tag{3.12}$$

Therefore, integrating either the Euler or the Navier-Stokes equations with solutions that decay rapidly at infinity, we conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^n} u dx = 0, \tag{3.13}$$

both in two and three dimensions. The same identity implies that in two dimensions the total vorticity is preserved: integrating (2.18), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2}\omega dx = -\nu\int_{\mathbb{R}^2}\Delta\omega dx - \int_{\mathbb{R}^2}(u\cdot\nabla\omega)dx = 0. \tag{3.14}$$

However, in that case we know more: any regular solution of (2.18) can be decomposed as $\omega = \omega^+(t,x) - \omega^-(t,x)$, where $\omega^\pm$ are the solutions of (2.18) with the initial data $\omega_0^\pm(x)$, respectively. It follows that

$$\int_{\mathbb{R}^2}|\omega|dx \le \int_{\mathbb{R}^2}\omega^+(t,x)dx + \int_{\mathbb{R}^2}\omega^-(t,x)dx = \int_{\mathbb{R}^2}|\omega_0|dx, \tag{3.15}$$

that is, not only the integral of the vorticity is preserved but its L^1-norm does not grow in two dimensions. In addition, for the solutions of the Euler equations, vorticity satisfies the advection equation

$$\omega_t + u\cdot\nabla\omega = 0. \tag{3.16}$$

Therefore, not only the integral of the vorticity but all L^p-norms of ω are preserved, with any $1 \le p \le \infty$:

$$\int_{\mathbb{R}^2}|\omega(t,x)|^p dx = \int_{\mathbb{R}^2}|\omega_0(x)|^p dx. \tag{3.17}$$

In three dimensions, the vorticity vector satisfies (2.27). Integrating this equation leads to

$$\frac{d}{dt}\int_{\mathbb{R}^3}\omega_i dx = \int_{\mathbb{R}^3}(\omega\cdot\nabla u_i)dx = 0, \tag{3.18}$$

since $\omega(t,x)$ is also a divergence-free field. Thus, the total integral of the vorticity is preserved also in three dimensions. However, conservation of the L^p-norms does not follow, and vorticity may grow.

Evolution of energy, dissipation and enstrophy

The kinetic energy of the fluid is

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |u(t,x)|^2 dx. \tag{3.19}$$

Differentiating in time gives

$$\begin{aligned} \frac{dE}{dt} &= \int_{\mathbb{R}^n} (u \cdot u_t) dx = \int_{\mathbb{R}^n} (-u_j u_k \frac{\partial u_j}{\partial x_k} - u \cdot \nabla p + \nu u_j \Delta u_j) dx \\ &= -\int_{\mathbb{R}^n} (u \cdot \nabla\Big(\frac{|u|^2}{2} + p\Big) - \nu \int_{\mathbb{R}^n} |\nabla u|^2 dx = -\nu \int_{\mathbb{R}^n} |\nabla u|^2 dx. \end{aligned} \tag{3.20}$$

Therefore, the energy of the solutions of the Euler equations ($\nu = 0$) is preserved in time:

$$E(t) = E(0), \tag{3.21}$$

while the energy of the solutions of the Navier-Stokes equations is dissipating:

$$\frac{dE}{dt} = -\nu \mathcal{D}(t), \tag{3.22}$$

where $\mathcal{D}(t)$ is the enstrophy

$$\mathcal{D}(t) = \int_{\mathbb{R}^n} |\nabla u|^2 dx. \tag{3.23}$$

The enstrophy can be expressed purely in terms of vorticity using the identity

$$|\omega|^2 = \varepsilon_{ijk}\varepsilon_{imn}(\partial_j u_k)(\partial_m u_n) = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})(\partial_j u_k)(\partial_m u_n) = |\nabla u|^2 - (\partial_j u_k)(\partial_k u_j). \tag{3.24}$$

Note that

$$\int_{\mathbb{R}^n} (\partial_j u_k)(\partial_k u_j) dx = -\int_{\mathbb{R}^n} u_k (\partial_k \partial_j u_j) dx = 0 \tag{3.25}$$

We used the incompressibility condition on u in the last step. This implies that the enstrophy for a divergence-free flow is

$$\mathcal{D}(t) = \int_{\mathbb{R}^n} |\omega|^2 dx. \tag{3.26}$$

Therefore, large vorticity leads to increased energy dissipation – this, however, does not lead to regularity.

Conservation of helicity

The helicity of a flow is

$$H = \int_{\mathbb{R}^3} (u \cdot \omega) dx. \tag{3.27}$$

This definition is non-trivial only in three dimensions, as in two dimensions we have, for any incompressible flow,

$$\begin{aligned} \int_{\mathbb{R}^2} u_1 \omega dx &= \int_{\mathbb{R}^2} u_1 \Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big) dx = -\int_{\mathbb{R}^2} u_2 \Big(\frac{\partial u_1}{\partial x_1} + \frac{1}{2}\frac{\partial (u_1^2)}{\partial x_2}\Big) dx \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \frac{\partial}{\partial x_2}(u_2^2 - u_1^2) dx = 0. \end{aligned} \tag{3.28}$$

In three dimensions, however, helicity is a non-trivial quantity, and, for the solutions of the Euler equations, we may compute

$$\frac{dH}{dt} = \int_{\mathbb{R}^3} (u_t \cdot \omega + u \cdot \omega_t) dx. \tag{3.29}$$

We have

$$u_t \cdot \omega + (u \cdot \nabla u) \cdot \omega + \omega \cdot \nabla p = 0, \tag{3.30}$$

and

$$u \cdot \omega_t + (u \cdot \nabla \omega) \cdot u = u \cdot (\omega \cdot \nabla u). \tag{3.31}$$

The last term in (3.30) integrates to zero since $\nabla \cdot \omega = 0$:

$$\int_{\mathbb{R}^3} (\omega \cdot \nabla p) dx = 0. \tag{3.32}$$

The other terms lead to

$$\frac{dH}{dt} = -\int_{\mathbb{R}^3} (u_k(\partial_k u_j)\omega_j + u_k u_j \partial_j \omega_k - u_j \omega_k \partial_k u_j) dx = 0, \tag{3.33}$$

as the first and the second terms in the rights side vanish since $\nabla \cdot u = 0$ while the last vanishes because $\nabla \cdot \omega = 0$. Thus, helicity is preserved for the solutions of the Euler equations. In particular, the velocity field and the vorticity can not "too aligned" in any growth or blow-up scenario.

4 The Constantin-Lax-Majda toy model

The formulation of the model

In order to appreciate the difficulties of the problem of the regularity for the solutions of the Euler and the Navier-Stokes equations, and in particular, focus on the effect vortex stretching term, we consider here a toy model studied by Constantin, Lax and Majda in 1985. The vortex stretching term in the three-dimensional vorticity equation for the Euler equation

$$\omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u, \tag{4.1}$$

has the form (2.104) – once again, it should not be taken to literally because of the singularities in the integrals,

$$\nabla u(x)\text{"} = \text{"} \int_{\mathbb{R}^3} \nabla K(x-y) \times \omega(y) dy, \tag{4.2}$$

with

$$K(x) = -\frac{1}{4\pi} \frac{x}{|x|^3}. \tag{4.3}$$

The Constantin-Lax-Majda model aims to imitate three important properties of the right side in the vorticity equation (4.1): first, it is quadratic in ω, second, its integral vanishes:

$$\int_{\mathbb{R}^3} \omega \cdot \nabla u \; dx = 0. \tag{4.4}$$

The third feature is that the kernel $\nabla K(x)$ has the singularity of the type $x/|x|^4$, which is of the kind $x/|x|^{n+1}$ in n dimensions. Constantin, Lax and Majda considered a one-dimensional model, with an analogous singularity

$$\frac{\partial \omega(t,x)}{\partial t} = H[\omega]\omega, \quad x \in \mathbb{R}, \tag{4.5}$$

with the initial condition $\omega(0,x) = \omega_0(x)$. Here, $H(\omega)$ is the Hilbert transform, a singular integral operator in one dimension:

$$H[\omega](x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{\omega(y)}{x-y} dy. \tag{4.6}$$

The principal value above is understood as

$$H[\omega](x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} \frac{\omega(x-y)}{y} dy = \frac{1}{\pi} \int_{|y|>1} \frac{\omega(x-y)}{y} dy + \frac{1}{\pi} \int_{-1}^{1} \frac{\omega(x-y)-\omega(x)}{y} dy. \tag{4.7}$$

The singularity $1/x$ in the kernel of the one-dimensional Hilbert transform is analogous to the singularity $x/|x|^4$ in three dimensions that appears in the kernel ∇K in (4.2): both are odd, and their size is $1/|x|^n$.

The toyest model of all

Before proceeding with the analysis of the Constantin-Lax-Majda model, let us pause and see what would happen if we would consider the simplest model that would preserve only the quadratic nature of the nonlinearity in the vorticity equation:

$$\frac{d\omega(t,x)}{dt} = \omega^2(t,x), \quad \omega(0,x) = \omega_0(x), \quad x \in \mathbb{R}. \tag{4.8}$$

Its explicit solution is

$$\omega(t,x) = \frac{\omega_0(x)}{1 - t\omega_0(x)}. \tag{4.9}$$

If there exist $x \in \mathbb{R}$ so that $\omega_0(x) > 0$, this solution exists until the denominator vanishes, that is, until the time

$$T_c = \inf \Big[\frac{1}{\omega_0(x)} : \ \omega_0(x) > 0 \Big]. \tag{4.10}$$

Let us assume that the function $\omega_0(x)$ attains its maximum at $x = x_m$. The function $\omega(t,x)$ at the time $t = T_c$ has an asymptotic expansion near the point $x = x_m$:

$$\omega(T_c,x) = \frac{\omega_0(x)}{1 - T_c\omega_0(x)} \approx \frac{\omega_0(x_m)}{-(T_c/2)\omega_0''(x_m)(x-x_m)^2}. \tag{4.11}$$

Thus, the function $\omega(t,x)$ blows up at the point x_m and the blow-up profile is $O(x-x_m)^{-2}$. As a consequence, all L^p-norms of $\omega(t,x)$ blow up as well:

$$\int_{\mathbb{R}} |\omega(t,x)|^p dx \to +\infty \text{ as } t \uparrow T_c, \tag{4.12}$$

for all $p \geq 1$. Moreover, if we define the "velocity" as

$$v(t,x) = \int_{-\infty} \omega(t,y)dy, \tag{4.13}$$

then $v(t,x)$ also blows-up at the time T_c and its blow-up profile is $O(x-x_m)^{-1}$. Therefore, the L^p-norm of the velocity blows up as well:

$$\int_{\mathbb{R}} |v(t,x)|^p dx \to +\infty \text{ as } t \uparrow T_c, \tag{4.14}$$

for all $p \geq 1$. In particular, the kinetic energy blows up as well:

$$\int_{\mathbb{R}} |v(t,x)|^2 dx \to +\infty \text{ as } t \uparrow T_c. \tag{4.15}$$

This is in contrast to the energy conservation in the true Euler equations. Thus, the toy model (4.8) can not be even "toyishly" correct.

The Hilbert transform

In order to understand the Constantin-Lax-Majda model, let us first recall some basic properties of the Hilbert transform and its alternative definition in terms of complex analysis. Given a Schwartz class function $f(x) \in \mathcal{S}(\mathbb{R})$ define a function

$$u(x,y) = \int_{\mathbb{R}} e^{-2\pi y|\xi|} \hat{f}(\xi) e^{2\pi i x\xi} d\xi, \quad y \geq 0, \quad x \in \mathbb{R}.$$

Here, the Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx, \quad f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi. \tag{4.16}$$

The function $u(x,y)$ is harmonic in the upper half plane:

$$\Delta_{x,y} u = 0 \text{ in } \mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty),$$

and satisfies the boundary condition on the line $y = 0$:

$$u(x,0) = f(x), \quad x \in \mathbb{R}.$$

We can write $u(x,y)$ as a convolution

$$u(x,y) = P_y \star f = \int P_y(x-x')f(x')dx',$$

with

$$\hat{P}_y(\xi) = e^{-2\pi y|\xi|},$$

and

$$P_y(x) = \int_{-\infty}^{\infty} e^{-2\pi y|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{2\pi(y-ix)} + \frac{1}{2\pi(y+ix)} = \frac{y}{\pi(x^2+y^2)}.$$

Next, set $z = x + iy$ and write

$$u(z) = \int_{\mathbb{R}} e^{-2\pi y|\xi|} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = \int_0^\infty \hat{f}(\xi) e^{2\pi i z\xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z}\xi} d\xi.$$

Consider the function $v(z)$ given by

$$iv(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z\xi} d\xi - \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z}\xi} d\xi.$$

As the function

$$u(z) + iv(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z\xi} d\xi$$

is analytic in the upper half-plane $\{\mathrm{Im} z > 0\}$, the function v is the harmonic conjugate of u. It can be written as

$$v(z) = \int_{\mathbb{R}} (-i\mathrm{sgn}(\xi)) e^{-2\pi y|\xi|} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = Q_y \star f,$$

with

$$\hat{Q}_y(\xi) = -i\mathrm{sgn}(\xi) e^{-2\pi y|\xi|}, \tag{4.17}$$

and

$$Q_y(x) = -i \int_{-\infty}^\infty \mathrm{sgn}(\xi) e^{-2\pi y|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi} \frac{x}{x^2 + y^2}.$$

The Poisson kernel and its conjugate are related by

$$P_y(x) + iQ_y(x) = \frac{i}{\pi(x + iy)} = \frac{1}{i\pi z},$$

which is analytic in $\{\mathrm{Im} z \geq 0\}$.

In order to consider the limit of Q_y as $y \to 0$, we relate it to the principal value of $1/x$ defined as in (4.7): it is an element of the space $\mathcal{S}'(\mathbb{R})$ of the Schwartz distributions, defined by

$$\mathrm{P.V.}\frac{1}{x}(\phi) = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\phi(x)}{x} dx = \int_{|x|<1} \frac{\phi(x) - \phi(0)}{x} dx + \int_{|x|>1} \frac{\phi(x)}{x} dx, \quad \phi \in \mathcal{S}(\mathbb{R}), \tag{4.18}$$

which is well-defined for $\phi \in \mathcal{S}(\mathbb{R})$. The conjugate Poisson kernel Q_y and the principal value of $1/x$ are related as follows.

Proposition 4.1 *Let* $Q_y = \dfrac{1}{\pi} \dfrac{x}{x^2 + y^2}$, *then for any function* $\phi \in \mathcal{S}(\mathbb{R})$

$$\frac{1}{\pi} P.V.\frac{1}{x}(\phi) = \lim_{y \to 0} \int_{\mathbb{R}} Q_y(x) \phi(x) dx.$$

Proof. Let

$$\psi_y(x) = \frac{1}{x}\chi_{y<|x|}(x)$$

so that

$$\text{P.V.}\frac{1}{x}(\phi) = \lim_{y\to 0}\int_{\mathbb{R}}\psi_y(x)\phi(x)dx.$$

Note, however, that

$$\begin{aligned}
&\int(\pi Q_y(x) - \psi_y(x))\phi(x)dx = \int_{\mathbb{R}}\frac{x\phi(x)}{x^2+y^2}dx - \int_{|x|>y}\frac{\phi(x)}{x}dx \\
&= \int_{|x|<y}\frac{x\phi(x)}{x^2+y^2}dx + \int_{|x|>y}\left[\frac{x}{x^2+y^2} - \frac{1}{x}\right]\phi(x)dx \qquad (4.19)\\
&= \int_{|x|<1}\frac{x\phi(xy)}{x^2+1}dx - \int_{|x|>y}\frac{y^2\phi(x)}{x(x^2+y^2)}dx = \int_{|x|<1}\frac{x\phi(xy)}{x^2+1}dx - \int_{|x|>1}\frac{\phi(xy)}{x(x^2+1)}dx.
\end{aligned}$$

The dominated convergence theorem implies that both integrals on the utmost right side above tend to zero as $y \to 0$. □

It is important to note that the computation in (4.19) worked only because the kernel $1/x$ is odd – this produces the cancellation that saves the day. This would not happen, for instance, for a kernel behaving as $1/|x|$ near $x = 0$.

Thus, the Hilbert transform defined as

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy. \qquad (4.20)$$

can be also written as

$$Hf(x) = \lim_{y\to 0} Q_y \star f(x). \qquad (4.21)$$

In other words, we take the function $f(x)$, extend it as a harmonic function $u(x,y)$ to the upper half-plane, and find the conjugate harmonic function $v(x,y)$. Then, $Hf(x) = v(x,0)$, the restriction of $v(x,y)$ to the real axis. It follows from (4.17) that

$$\widehat{Hf}(\xi) = \lim_{t\downarrow 0}\hat{Q}_t(\xi)\hat{f}(\xi) = -i\text{sgn}(\xi)\hat{f}(\xi). \qquad (4.22)$$

Therefore, the Hilbert transform may be extended to an isometry $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, with

$$\|Hf\|_{L^2} = \|f\|_{L^2},\ \ H(Hf) = -f, \qquad (4.23)$$

and

$$\int(Hf)(x)g(x)dx = -\int f(x)(Hg)(x)dx. \qquad (4.24)$$

Back to the Constantin-Lax-Majda model

Let us now return to the CLM model

$$\omega_t = H[\omega]\omega, \quad \omega(0,x) = \omega_0(x). \qquad (4.25)$$

The term $H[\omega]\omega$ in the right side of (4.25) is similar to the vorticity stretching term $D\omega$ in the true three-dimensional vorticity equation in the three aspects we have discussed above, below (4.3). It is quadratic in ω, it follows from (4.24) that the operator H is skew-symmetric:

$$\int_{\mathbb{R}} H[\omega](x)\omega(x)dx = 0, \tag{4.26}$$

so the right side of (4.25) integrates to zero, as in (4.4), and the kernel $1/x$ has the correct singularity – it is odd and of the size $1/|x|^n$ (where n is the dimension). It follows from (4.26) that the integral of the solution of the toy model (4.25) is preserved:

$$\frac{d}{dt}\int_{\mathbb{R}} \omega(t,x)dx = 0. \tag{4.27}$$

Let us now use the "complex analysis" definition of $\psi = H[\phi]$, and set $u(x,y)$ and $v(x,y)$ so that the function $f = u + iv$ is analytic in $\{y > 0\}$, and $u(x,0) = \phi(x)$, $v(x,0) = \psi(x)$. As we may write

$$-if^2 = 2uv + i(v^2 - u^2), \tag{4.28}$$

it follows that the harmonic conjugate of uv is $(v^2 - u^2)/2$. Restricting this identity to the real line gives

$$H(\phi H[\phi]) = \frac{1}{2}(H[\phi])^2 - \frac{1}{2}\phi^2. \tag{4.29}$$

Applying the Hilbert transform to the toy vorticity equation gives then

$$\frac{d}{dt}H[\omega] = \frac{1}{2}H[\omega]^2 - \frac{\omega^2}{2}. \tag{4.30}$$

Therefore, the function

$$w(t,x) = H[\omega](t,x) + i\omega(t,x) \tag{4.31}$$

satisfies the simple quadratic ODE

$$\frac{dw}{dt} = \frac{1}{2}(H[\omega])^2 - \frac{1}{2}\omega^2 + iH[\omega]\omega = \frac{1}{2}w^2. \tag{4.32}$$

Hence, the function $w(t,x)$ is given explicitly by

$$w(t,x) = \frac{w_0(x)}{1 - \frac{1}{2}tw_0(x)}. \tag{4.33}$$

Taking the imaginary part of (4.33) gives an explicit formula for the solution of the toy vorticity equation:

$$\begin{aligned}\omega(t,x) &= \mathrm{Im}\frac{w_0(x)}{1-\frac{1}{2}tw_0(x)} = \mathrm{Im}\frac{2(H[\omega_0](x)+i\omega_0(x))}{2-t(H[\omega_0](x)+i\omega_0(x))}\\ &= \mathrm{Im}\frac{2(H[\omega_0](x)+i\omega_0(x))(2-tH[\omega_0](x)+it\omega_0(x))}{(2-tH[\omega_0](x))^2+t^2(\omega_0(x))^2}\\ &= 2\frac{t\omega_0(x)H[\omega_0](x)+\omega_0(x)(2-tH[\omega_0](x))}{(2-tH[\omega_0](x))^2+t^2(\omega_0(x))^2} = \frac{4\omega_0(x)}{(2-tH[\omega_0](x))^2+t^2(\omega_0(x))^2}.\end{aligned} \tag{4.34}$$

The explicit formula

$$\omega(t,x) = \frac{4\omega_0(x)}{(2 - tH[\omega_0](x))^2 + t^2(\omega_0(x))^2}. \tag{4.35}$$

gives an explicit criterion for the solution of the vorticity to exist for all times $t > 0$. Namely, the solution $\omega(t,x)$ exists and remains smooth provided that there does not exist a point $x \in \mathbb{R}$ so that $\omega_0(x) = 0$ and $H[\omega_0](x) > 0$. The explicit breakdown time for a smooth solution is

$$T_c = \inf \Big\{ \frac{2}{H[\omega_0](x)} : \quad \omega_0(x) = 0, \; H[\omega_0](x) > 0 \Big\}. \tag{4.36}$$

As an example, consider $\omega_0(x) = \cos x$, so that $H[\omega_0](x) = \sin x$, and

$$\omega(t,x) = \frac{4\cos x}{(2 - t\sin x)^2 + t^2\cos^2 x} = \frac{4\cos x}{4 + t^2 - 4t\sin x}. \tag{4.37}$$

The breakdown time $T_c = 2$, and the corresponding "toy velocity" is

$$v(t,x) = \int_0^x \omega(t,y)dy = \frac{1}{t}\log(1 + \frac{t^2}{4} - t\sin x). \tag{4.38}$$

Therefore,

$$\int_{-\pi}^{\pi} |\omega(t,x)|^p dx \to +\infty \tag{4.39}$$

as $t \uparrow T_c$, for any $1 \le p < \infty$. On the other hand, the L^p-norms of the velocity stay finite:

$$\int_{-\pi}^{\pi} |v(t,x)|^p dx \to M_p < +\infty, \tag{4.40}$$

for all $1 \le p < +\infty$, as $t \to\uparrow T_c$. In particular, the kinetic energy does not blow-up at the time T_c:

$$\int_{-\pi}^{\pi} |v(t,x)|^2 dx \to M_2 < +\infty, \tag{4.41}$$

This is in contrast to what happens in the "most toyest" model (4.8), where, the kinetic energy blows up at the blow-up time. Thus, while the Constantin-Lax-Majda model does not necessarily capture the physics of the Euler equations, it provides a "reasonable" one-dimensional playground.

5 The weak solutions of the Navier-Stokes equations

We will now start looking at the existence and regularity of the solutions of the Navier-Stokes equations. In order to focus on the less technical points, we will consider the periodic solutions of the Navier-Stokes equations:

$$\begin{aligned} &u_t + u \cdot \nabla u - \nu\Delta u + \nabla p = f(x), \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0. \end{aligned} \tag{5.1}$$

Here, f is the forcing term, and $u_0(x)$ is the initial data. We assume both to be 1-periodic in all directions: $f(x+e_j) = f(x)$, $u_0(x+e_j) = u_0(x)$, with $j = 1,2$ in $\mathbb{R}^2$ and $j = 1,2,3$ in $\mathbb{R}^3$. We will look for periodic solutions of (5.1) in $\mathbb{R}^n$, $n = 2,3$. As the integral of u is conserved if $f = 0$, we may (and will) assume without loss of generality that

$$\langle u \rangle = \int_{\mathbb{T}^n} u(t,x)dx = 0. \tag{5.2}$$

Here, $\mathbb{T}^n = [0,1]^n$ is the unit period. When $f \neq 0$, (5.2) holds, provided that $\langle f \rangle = 0$. Otherwise, we have a separate equation for $\langle u \rangle$:

$$\frac{d\langle u \rangle}{dt} = \langle f \rangle. \tag{5.3}$$

Thus, we may assume without loss of generality that $\langle f \rangle = 0$, and (5.2) holds.

The two and three dimensional cases are very different. In two dimensions, we will be able to show existence of regular solutions for all $t > 0$, provided that the forcing $f(x)$ and the initial condition $u_0(x)$ are sufficiently regular. On the other hand, in three dimensions, we will only be able to show that there exists a time T_c that depends on the force f and the initial condition u_0 so that the solution of the Navier-Stokes equations remains regular until the time T_c. However, if the initial data and the forcing are sufficiently small (in a sense to be made precise later), then solutions of the Navier-Stokes equations remain regular for all times $t > 0$. This will be shown using the dominance of diffusion over the nonlinearity for small data.

The weak solutions

The distinction between two and three dimensions is less dramatic if we talk about weak solutions. As is usual in the theory of weak solutions of partial differential equations, the definition of a weak solution of the Navier-Stokes equations (5.1) comes from multiplying it by a smooth test function and integrating by parts. First, we note that any test vector field ψ can be decomposed as a sum of a gradient field and a divergence-free field:

$$\psi(x) = \phi(x) + \nabla\eta(x), \tag{5.4}$$

with $\nabla \cdot \phi(x) = 0$. This is known as the Hodge decomposition. In the periodic case it is quite explicit: write $\psi(x)$ in terms of the Fourier transform

$$\psi(x) = \sum_{k\in\mathbb{Z}^n} \psi_k e^{2\pi i k\cdot x}, \tag{5.5}$$

and set

$$\eta(x) = \sum_{k\in\mathbb{Z}^n, k\neq 0} \frac{(\psi_k \cdot k)}{2\pi i|k|^2} e^{2\pi i k\cdot x}, \quad \nabla\eta(x) = \sum_{k\in\mathbb{Z}^n, k\neq 0} \frac{(\psi_k \cdot k)}{|k|^2} k e^{2\pi i k\cdot x}. \tag{5.6}$$

Then, the Fourier coefficients of the difference

$$\phi(x) = \psi(x) - \nabla\eta(x) = \sum_{k\in\mathbb{Z}^n, k\neq 0} \Big(\psi_k - \frac{(\psi_k \cdot k)}{|k|^2} k\Big) e^{2\pi i k\cdot x} \tag{5.7}$$

are

$$\phi(x) = \sum_{k \in \mathbb{Z}^n} \phi_k e^{2\pi i k \cdot x}, \quad \phi_k = \psi_k - \frac{(\psi_k \cdot k)}{|k|^2} k. \tag{5.8}$$

They satisfy

$$\phi_k \cdot k = 0, \tag{5.9}$$

thus the vector field $\phi(x)$ is divergence-free:

$$\nabla \cdot \phi(x) = 0. \tag{5.10}$$

Let now $u(t,x)$ be a smooth solution of the Navier-Stokes equations

$$u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + g, \tag{5.11}$$

$$\nabla \cdot u = 0. \tag{5.12}$$

We will also decompose the forcing term

$$g = f + \nabla \zeta \text{ with } \nabla \cdot f = 0. \tag{5.13}$$

The first observation is that if we multiply (5.11) by $\nabla \eta(x)$ and integrate, then we simply get the Poisson equation for the pressure. Indeed, if w is a smooth periodic vector field, and $\nabla \cdot w = 0$, then

$$\int_{\mathbb{T}^n} w(x) \cdot \nabla \eta(x) dx = - \int_{\mathbb{T}^n} \eta(x) (\nabla \cdot w)(x) dx = 0. \tag{5.14}$$

It follows that

$$\int_{\mathbb{T}^n} (u_t \cdot \nabla \eta) dx = \int_{\mathbb{T}^n} (\Delta u \cdot \nabla \eta) dx = 0. \tag{5.15}$$

For the pressure we have:

$$\int_{\mathbb{T}^n} (\nabla p \cdot \nabla \eta) dx = - \int_{\mathbb{T}^n} p \Delta \eta dx, \tag{5.16}$$

while for the nonlinear term we get, after an integration by parts, using the divergence-free condition on u:

$$\int_{\mathbb{T}^n} ((u \cdot \nabla u) \cdot \nabla \eta) dx = \int_{\mathbb{T}^n} u_j (\partial_j u_k) \partial_k \eta dx = - \int_{\mathbb{T}^n} u_j u_k (\partial_j \partial_k \eta) dx. \tag{5.17}$$

We deduce that, for any test function $\eta(x)$, we have

$$\int_{\mathbb{T}^n} (p \Delta \eta + u_j u_k (\partial_j \partial_k \eta) dx = \int_{\mathbb{T}^n} g \cdot \nabla \eta = \int_{\mathbb{T}^n} \nabla \zeta \cdot \nabla \eta. \tag{5.18}$$

This is the weak form of the Poisson equation

$$-\Delta p = (\partial_j u_k)(\partial_k u_j) - \Delta \zeta. \tag{5.19}$$

On the other hand, when we multiply (5.11) by a divergence-free smooth vector field $w(x)$, the pressure term disappears:

$$\int_{\mathbb{T}^n} (w \cdot \nabla p) dx = 0, \tag{5.20}$$

and the nonlinear term may be written as

$$\int_{\mathbb{T}^n}((u\cdot\nabla u)\cdot w)dx=\int_{\mathbb{T}^n}u_j(\partial_j u_k)w_k dx=-\int_{\mathbb{T}^n}u_j u_k\partial_j w_k dx. \tag{5.21}$$

Thus, if w is a $C^\infty(\mathbb{T}^n)$ periodic divergence-free field, integration by parts gives

$$\int_{\mathbb{T}^n}[u_t\cdot w-u_j u_k\partial_j w_k]dx=\nu\int_{\mathbb{T}^n}(u\cdot\Delta w)dx+\int_{\mathbb{T}^n}(f\cdot w)dx. \tag{5.22}$$

For now, we say that $u(t,x)$ is a weak solution of the Navier-Stokes equations if (5.22) holds for all periodic smooth divergence-free vector fields $w(x)$. A little later, we will make this notion more precise, setting up the proper spaces in which the weak solutions live, and relaxing the C^∞ assumption on the test function.

The Galerkin approximation

In order to construct the weak solutions, we will consider the Galerkin approximation of the Navier-Stokes equations. In the periodic case, this is equivalent to the projection of the equations on the divergence-free Fourier modes with $|k|\le m$, where $m>0$ is fixed. That is, given a vector-field

$$\psi(x)=\sum_{k\in\mathbb{Z}^n}a_k e^{2\pi ik\cdot x}, \tag{5.23}$$

we set

$$\psi^{(m)}(x)=P_m\psi(x)=\sum_{|k|\le m}\Big(a_k-\frac{(a_k\cdot k)}{|k|^2}k\Big)e^{2\pi ik\cdot x}, \tag{5.24}$$

so that, in particular,

$$\nabla\cdot\psi^{(m)}=0. \tag{5.25}$$

Note that if ψ is a divergence-free vector field then $\psi^{(m)}$ is simply the projection on the Fourier modes with $|k|\le m$.

The Galerkin approximation of the Navier-Stokes equations

$$u_t+u\cdot\nabla u+\nabla p=\nu\Delta u+f, \tag{5.26}$$

with $u(0,x)=u_0(x)$, and a divergence-free force f: $\nabla\cdot f=0$, is the system

$$\frac{\partial u^{(m)}}{\partial t}+P_m(u^{(m)}\cdot\nabla u^{(m)})=\nu\Delta u^{(m)}+f^{(m)},\quad u^{(m)}(0)=u_0^{(m)}. \tag{5.27}$$

This is a finite-dimensional constant coefficients system of quadratic ODE's for the Fourier coefficients u_m of the function $u(x)$ with $|k|\le m$. If the function f is time-independent, this system is autonomous. The goal is obtain bounds on the solution $u^{(m)}$ of the Galerkin system that would allow us to pass to the limit $m\to+\infty$, leading to a weak solution of the Navier-Stokes equations.

A bound on the energy for the Galerkin solutions

We fix an arbitrary time $T > 0$ throughout the analysis of the Galerkin system. As (5.27) is a system of constant coefficient non-linear ODEs for the coefficients u_k, $|k| \leq m$, it has a solution for a sufficiently small time $t > 0$ (which a priori may depend on the initial data $u_0^{(m)}$, as well as on m). However, unlike partial differential equations, such ODEs may lose solutions only via the blow-up of the energy

$$\|u^{(m)}\|_2^2 = \sum_{|k|\leq m} |u_k|^2, \tag{5.28}$$

and that, as we will now show, can not happen in a finite time. Indeed, we have

$$\int_{\mathbb{T}^n} (P_m(u^{(m)} \cdot \nabla u^{(m)}) \cdot u^{(m)}) dx = \int_{\mathbb{T}^n} ((u^{(m)} \cdot \nabla u^{(m)}) \cdot u^{(m)}) dx = 0. \tag{5.29}$$

We used the definition of the projection P_m in the first identity, and the incompressibility of $u^{(m)}$ in the second. Therefore, multiplying (5.27) by $u^{(m)}$ and integrating, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n} |u^{(m)}|^2 dx = -\nu \int_{\mathbb{T}^n} |\nabla u^{(m)}|^2 dx + \int_{\mathbb{T}^n} (f^{(m)} \cdot u^{(m)}) dx. \tag{5.30}$$

We will now use the Poincaré inequality

$$4\pi^2 \int_{\mathbb{T}^n} |\phi|^2 dx \leq \int_{\mathbb{T}^n} |\nabla \phi|^2, \tag{5.31}$$

that holds for all mean-zero periodic functions ϕ. With its help, identity (5.30) implies that $E(t) = \|u^{(m)}\|_2^2$ satisfies

$$\begin{aligned} \frac{1}{2}\frac{dE}{dt} &\leq -4\pi^2\nu E(t) + \|f^{(m)}\|_2\sqrt{E(t)} \leq -4\pi^2\nu E(t) + 2\pi^2\nu E(t) + \frac{1}{8\pi^2\nu}\|f\|_2^2 \\ &\leq -2\pi^2\nu E(t) + \frac{1}{8\pi^2\nu}\|f\|_2^2. \end{aligned} \tag{5.32}$$

Therefore, we have the inequality

$$\frac{d}{dt}\Big(E(t)e^{4\pi^2\nu t}\Big) \leq \frac{1}{4\pi^2\nu}\|f\|_2^2 e^{4\pi^2\nu t}. \tag{5.33}$$

Integrating in time leads to an estimate

$$E(t) \leq E(0)e^{-4\pi^2\nu t} + \frac{1}{4\pi^2\nu}\int_0^t e^{-4\pi^2\nu(t-s)}\|f(s)\|_2^2 ds. \tag{5.34}$$

The estimate (5.34) relies on the finiteness of the L^2-norm of the forcing f. Another way to estimate the right side in (5.30), relying only on the finiteness of a weaker norm of f, is to use the inequality

$$\Big|\int_{\mathbb{T}^n} (f \cdot g) dx\Big| = \Big|\sum_{k\in\mathbb{Z}^n} f_k g_k\Big| \leq \Big(\sum_{k\in\mathbb{Z}^n} 4\pi^2 k^2 |f_k|^2\Big)^{1/2} \Big(\sum_{k\in\mathbb{Z}^n} \frac{|g_k|^2}{4\pi^2 k^2}\Big)^{1/2} = \|\nabla f\|_2 \|g\|_{H^{-1}}, \tag{5.35}$$

with the H^{-1}-norm defined as in the above inequality. Using this inequality in (5.30) gives

$$\frac{1}{2}\frac{dE}{dt} \le -\nu\|\nabla u^{(m)}\|_2^2 + \|\nabla u^{(m)}\|_2\|f\|_{H^{-1}} \le -\nu\|\nabla u^{(m)}\|_2^2 + \frac{\nu}{2}\|\nabla u^{(m)}\|_2^2 + \frac{1}{2\nu}\|f\|_{H^{-1}}^2$$
$$= -\frac{\nu}{2}\|\nabla u^{(m)}\|_2^2 + \frac{1}{2\nu}\|f\|_{H^{-1}}^2. \tag{5.36}$$

Now, we use the Poincaré inequality to obtain:

$$\frac{dE}{dt} \le -C_1\nu E + \frac{C_2}{\nu}\|f\|_{H^{-1}}^2, \tag{5.37}$$

with universal constants C_1 and C_2. Integrating this differential inequality in time leads to another estimate for $E(t)$, which involves only $\|f\|_{H^{-1}}$ and not $\|f\|_2$:

$$E(t) \le E(0)e^{-C_1\nu t} + \frac{C_2'}{\nu}\int_0^t e^{-C_1\nu(t-s)}\|f(s)\|_{H^{-1}}^2 ds. \tag{5.38}$$

An enstrophy bound

The same argument provides a bound on the enstrophy $D(t) = \|\nabla u(t)\|_2^2$. Indeed, integrating inequality (5.36) in time leads to

$$\frac{1}{2}\|u^{(m)}(T)\|_2^2 + \frac{\nu}{2}\int_0^T\int_{\mathbb{T}^n}|\nabla u^{(m)}(s,x)|^2 dx ds \le \frac{1}{2}\|u_0^{(m)}\|_2^2 + \frac{1}{2\nu}\int_0^T \|f^{(m)}(s)\|_{H^{-1}}^2 ds. \tag{5.39}$$

The function spaces and an executive summary

Now, we need to introduce certain spaces. We denote by H the space of all mean-zero vector-valued functions u in the space $[L^2(\mathbb{T}^n)]^n$, with zero divergence (in the sense of distributions):

$$H = \{u \in L^2(\mathbb{T}^n) :\ \nabla\cdot u = 0,\ \langle u\rangle = 0\}, \tag{5.40}$$

with the inner product

$$(f,g) = \int_{\mathbb{T}^n}(f\cdot g)dx. \tag{5.41}$$

In other words, a vector field $u \in H$ if its Fourier coefficients in the expansion

$$u(x) = \sum_{k\in\mathbb{Z}^n} u_k e^{2\pi i k\cdot x} \tag{5.42}$$

satisfy $u_0 = 0$, $k\cdot u_k = 0$ for all $k \in \mathbb{Z}^n_* = \mathbb{Z}^n \setminus \{0\}$, and

$$\|u\|_H^2 = \sum_{k\in\mathbb{Z}^n_*}|u_k|^2 < +\infty. \tag{5.43}$$

We also denote by V the space of divergence-free functions in the Sobolev space $H^1(\mathbb{T}^n)$:

$$V = \{u \in H^1(\mathbb{T}^n) :\ \nabla\cdot u = 0,\ \langle u\rangle = 0\}, \tag{5.44}$$

with the inner product

$$\langle f, g\rangle = \int_\Omega (\frac{\partial u}{\partial x_i}\cdot\frac{\partial g}{\partial x_i})dx, \tag{5.45}$$

for two vector-valued functions f and g. That is, $u\in V$ if its Fourier coefficients satisfy $u_0=0$, as well as $k\cdot u_k=0$ for all k, and

$$\|u\|_V^2 = \sum_{k\in\mathbb{Z}_*^n}|k|^2|u_k|^2 < +\infty. \tag{5.46}$$

The dual space to V consists of all distributions with the Fourier coefficients that satisfy

$$\|u\|_{V'}^2 = \sum_{k\in\mathbb{Z}_*^n}\frac{|u_k|^2}{|k|^2} < +\infty,\ u_0=0 \text{ and } k\cdot u_k = 0. \tag{5.47}$$

We will occasionally use the Sobolev spaces H^s, $s\in\mathbb{R}$, of divergence-free functions: we say that $u\in H^s(\mathbb{T}^n)$ if its Fourier coefficients u_k satisfy

$$u_0=0,\ k\cdot u_k=0 \text{ and } \|u\|_{H^s} = \left(\sum_{k\in\mathbb{Z}_*^n}|k|^{2s}|u_k|^2\right)^{1/2} < +\infty. \tag{5.48}$$

We have, with this notation $V=H^1$ and $V'=H^{-1}$. The spaces $L^2(0,T;H)$ and $L^2(0,T;V)$ have the respective norms

$$\|u\|_{L^2(0,T;H)}^2 = \int_0^T \|u(t)\|_H^2 dt,\quad \|u\|_{L^2(0,T;V)}^2 = \int_0^T \|u(t)\|_V^2 dt. \tag{5.49}$$

Summarizing our analysis of the Galerkin system so far, and rephrasing the results in terms of the spaces H, V and V', we have proved the following.

Proposition 5.1 *Assume that $f\in L^\infty(0,T;H)$. Then, the Galerkin system (5.27) has a unique solution $u^{(m)}\in L^2(0,T;V)\cap L^\infty(0,T;H)$. More precisely, there exist two universal constants $C_1>0$ and $C_2>0$ so that*

$$\|u^{(m)}(t)\|_H^2 \le \|u_0\|_H^2 e^{-4\pi^2\nu t} + \frac{1}{4\pi^2\nu}\int_0^t e^{-4\pi^2\nu(t-s)}\|f(s)\|_H^2 ds, \tag{5.50}$$

$$\|u^{(m)}(t)\|_H^2 \le \|u_0\|_H^2 e^{-C_1\nu t} + \frac{C_2}{\nu}\int_0^t e^{-C_1\nu(t-s)}\|f(s)\|_{V'}^2 ds \tag{5.51}$$

$$\nu\int_0^T \|u^{(m)}(s)\|_V^2 ds \le \|u_0\|_2^2 + \frac{1}{2\nu}\int_0^T \|f(s)\|_{V'}^2 ds. \tag{5.52}$$

The Galerkin approximation: bounds on the time derivative

The next step is obtain bounds on the time derivative of $u^{(m)}$. They will be needed in the passage to the limit $m\to+\infty$, to ensure that the limit is weakly continuous in time. Let us write the Galerkin approximation of the Navier-Stokes equaitons as

$$\frac{\partial u^{(m)}}{\partial t} = \nu\Delta u^{(m)} - P_m(u^{(m)}\cdot\nabla u^{(m)}) + f^{(m)},\quad u^{(m)}(0) = u_0^{(m)}. \tag{5.53}$$

We will aim to obtain the following bounds on $u_t^{(m)}$. The estimates are slightly different in two and three dimensions.

Proposition 5.2 *Assume that $f \in L^2(0,T;V')$. There exists a constant C which depends on the norm $\|u_0\|_H$ of the initial data u_0, the $L^2(0,T;V')$-norm of the forcing f, and the viscosity ν but not on m so that the solution of the Galerkin system (5.27) in dimension $n = 3$ satisfies the estimate*

$$\int_0^T \Big\|\frac{\partial u^{(m)}}{\partial t}(t)\Big\|_{V'}^{4/3} \leq C. \tag{5.54}$$

and in dimension $n = 2$ it satisfies

$$\int_0^T \Big\|\frac{\partial u^{(m)}}{\partial t}(t)\Big\|_{V'}^{2} \leq C. \tag{5.55}$$

For the proof, we will estimate individually each of the terms in the right side of (5.53). As we assume that $f \in L^2(0,T;V')$, the forcing term in is not be a problem either in dimension two or three. The Laplacian term in (5.53) is also bounded in $L^2(0,T;V')$, as follows from (5.52): the Fourier coefficients of Δu are $|k|^2 u_k$, hence

$$\|\Delta u\|_{V'}^2 = \sum_{k\in\mathbb{Z}^n} \frac{|k|^4}{|k|^2}|u_k|^2 = \|u\|_V^2, \tag{5.56}$$

thus

$$\int_0^T \|\Delta u^{(m)}(s)\|_{V'}^2 ds = \int_0^T \|u^{(m)}(s)\|_V^2 ds \leq \frac{1}{\nu}\|u_0^{(m)}\|_2^2 + \frac{1}{2\nu^2}\int_0^T \|f^{(m)}(s)\|_{V'}^2 ds. \tag{5.57}$$

The nonlinear term will require the most effort. We will establish the following bounds.

Lemma 5.3 *There exists a constant C that so that in two dimensions we have, for any function $u \in V$:*

$$\|(u\cdot\nabla u)\|_{V'} \leq C\|u\|_H\|u\|_V, \quad n = 2, \tag{5.58}$$

and in three dimensions we have

$$\|(u\cdot\nabla u)\|_{V'} \leq C\|u\|_H^{1/2}\|u\|_V^{3/2}, \quad n = 3. \tag{5.59}$$

Together with the uniform energy bound (5.51) and the enstrophy bound (5.52), this implies the conclusion of Proposition 5.2. Indeed, in dimension $n = 2$, (5.58) gives

$$\int_0^T \|P_m(u\cdot\nabla u)(s)\|_{V'}^2 ds \leq \int_0^T \|(u\cdot\nabla u)(s)\|_{V'}^2 ds \leq (\sup_{0\leq t\leq T}\|u(t)\|_H^2)\int_0^T \|u(s)\|_V^2 ds \leq C,$$

and in dimension $n = 3$, (5.59) leads to

$$\int_0^T \|P_m(u\cdot\nabla u)(s)\|_{V'}^{4/3} ds \leq \int_0^T \|(u\cdot\nabla u)(s)\|_{V'}^2 ds \leq (\sup_{0\leq t\leq T}\|u(t)\|_H^{2/3})\int_0^T \|u(s)\|_V^2 ds \leq C.$$

Thus the proof of Proposition 5.2 is reduced to proving Lemma 5.3.

The proof of Lemma 5.3: bounds on the nonlinear term

Note that

$$\|(u\cdot\nabla u)\|_{V'} = \|(-\Delta)^{-1/2}(u\cdot\nabla u)\|_H. \tag{5.60}$$

The operator $(-\Delta)^{-1/2}$ is defined via its action on the Fourier coefficients of a mean-zero function $u(x)$:

$$(-\Delta)^{-1/2}u(x) = \sum_{k\in\mathbb{Z}^n}\frac{u_k}{|k|}e^{2\pi ik\cdot x}. \tag{5.61}$$

This operator commutes with the projection P_m, as, in particular, it preserves the incompressibility of u. Continuing (5.60) leads to

$$\|(u\cdot\nabla u)\|_{V'} = \|(-\Delta)^{-1/2}(u\cdot\nabla u)\|_H = \|(-\Delta)^{-1/2}(u\cdot\nabla u)\|_H. \tag{5.62}$$

Hence, Lemma 5.3 can be restated as follows.

Lemma 5.4 *Let $u\in V$, then in three dimensions we have the estimate*

$$\|(-\Delta)^{-1/2}(u\cdot\nabla u)\|_H \le C\|u\|_H^{1/2}\|u\|_V^{3/2}, \tag{5.63}$$

while in two dimensions we have

$$\|(-\Delta)^{-1/2}(u\cdot\nabla u)\|_H \le C\|u\|_H\|u\|_V, \tag{5.64}$$

Proof. In this proof, we will use interchangeably the notation $\|u\|_{H^1}$ and $\|u\|_V$. Take an arbitrary $u\in H$ and $w\in H$ and write, for the inner product in H:

$$((-\Delta)^{-1/2}(u\cdot\nabla u),w) = ((u\cdot\nabla u),(-\Delta)^{-1/2}w). \tag{5.65}$$

In three dimensions, we will show

Lemma 5.5 *In dimension $n=3$, for any $u,v,w\in V$ we have*

$$|((u\cdot\nabla v),w)| \le C\|u\|_{H^{1/2}}\|v\|_{H^1}\|w\|_{H^1}. \tag{5.66}$$

Applying this estimate in (5.65) gives

$$|((-\Delta)^{-1/2}(u\cdot\nabla u),w)| = |((u\cdot\nabla u),(-\Delta)^{-1/2}w)| \le C\|u\|_{H^{1/2}}\|u\|_{H^1}\|(-\Delta)^{-1/2}w)\|_{H^1}. \tag{5.67}$$

As

$$\|(-\Delta)^{-1/2}w)\|_{H^1} = \|w\|_H, \tag{5.68}$$

and

$$\|u\|_{H^{1/2}}^2 = \sum_{k\in\mathbb{Z}^n}|k||u_k|^2 \le \left(\sum_{k\in\mathbb{Z}^n}|k|^2|u_k|^2\right)^{1/2}\left(\sum_{k\in\mathbb{Z}^n}|u_k|^2\right)^{1/2} = \|u\|_H\|u\|_V, \tag{5.69}$$

we deduce from (5.65) that in three dimensions we have

$$|((-\Delta)^{-1/2}(u\cdot\nabla u),w)| \le C\|u\|_H^{1/2}\|u\|_V^{3/2}\|w\|_H. \tag{5.70}$$

As this estimate holds for all $w\in H$, (5.63) follows.

In two dimensions, we will show

Lemma 5.6 *In dimension $n = 2$, we have*

$$|((u \cdot \nabla v), u)| \leq C\|u\|_2\|u\|_{H^1}\|v\|_{H^1}. \tag{5.71}$$

To see that this implies (5.64), we write, using incompressibility of u:

$$((-\Delta)^{-1/2}(u \cdot \nabla u), w) = ((u \cdot \nabla u), (-\Delta)^{-1/2}w) = -((u \cdot \nabla(-\Delta)^{-1/2}w), u). \tag{5.72}$$

Applying estimate (5.71) in (5.72) gives

$$\begin{aligned} &|((-\Delta)^{-1/2}(u \cdot \nabla u), w)| = |((u \cdot \nabla(-\Delta)^{-1/2}w), u)| \\ &\leq C\|u\|_2\|(-\Delta)^{-1/2}w\|_{H^1}\|u\|_{H^1} = C\|u\|_2\|u\|_{H^1}\|w\|_H. \end{aligned} \tag{5.73}$$

As this holds for any $w \in H$, we conclude that (5.64) holds in two dimensions.

Thus, we only need to verify (5.66) in three dimensions and (5.71) in two dimensions to finish the proof of Lemma 5.4.

Proof of Lemma 5.5. In three dimensions, we use Hölder's inequality to get

$$\begin{aligned} |((u \cdot \nabla v), w)| &\leq \int_{\mathbb{T}^3} |u_j(\partial_j v_k)w_k|dx \leq \|u\|_{L^3(\mathbb{T}^3)}\|\nabla v\|_{L^2(\mathbb{T}^3)}\|w\|_{L^6(\mathbb{T}^3)} \\ &= \|u\|_{L^3(\mathbb{T}^3)}\|v\|_{H^1(\mathbb{T}^3)}\|w\|_{L^6(\mathbb{T}^3)}. \end{aligned} \tag{5.74}$$

The Sobolev inequality says that, for $m < n/2$,

$$\|f\|_{L^q(\mathbb{T}^n)} \leq C\|f\|_{H^m(\mathbb{T}^n)}, \tag{5.75}$$

as long as

$$\frac{1}{q} \geq \frac{1}{2} - \frac{m}{n}. \tag{5.76}$$

Therefore, we have in three dimensions

$$\|u\|_{L^3(\mathbb{T}^3)} \leq C\|u\|_{H^{1/2}}, \tag{5.77}$$

and

$$\|w\|_{L^6(\mathbb{T}^3)} \leq C\|w\|_{H^1(\mathbb{T}^3)}. \tag{5.78}$$

It follows then from (5.74) that

$$|((u \cdot \nabla v), w)| \leq \|u\|_{L^3(\mathbb{T}^3)}\|v\|_{H^1(\mathbb{T}^3)}\|w\|_{L^6(\mathbb{T}^3)} \leq C\|u\|_{H^{1/2}(\mathbb{T}^3)}\|v\|_{H^1(\mathbb{T}^3)}\|w\|_{H^1(\mathbb{T}^3)}, \tag{5.79}$$

which is (5.66).

Proof of Lemma 5.6. In two dimensions, we proceed similarly: Hölder's inequality implies

$$|((u \cdot \nabla v), w)| \leq \|u\|_{L^4(\mathbb{T}^2)}\|w\|_{L^4(\mathbb{T}^2)}\|v\|_{H^1(\mathbb{T}^2)}. \tag{5.80}$$

The Sobolev inequality (5.75) in two dimensions implies that

$$\|f\|_{L^4(\mathbb{T}^2)} \leq C\|f\|_{H^{1/2}(\mathbb{T}^2)}. \tag{5.81}$$

Using this in (5.80) leads to

$$|((u \cdot \nabla v), w)| \leq \|u\|_{L^4(\mathbb{T}^2)} \|w\|_{L^4(\mathbb{T}^2)} \|v\|_{H^1(\mathbb{T}^2)} \leq C \|u\|_{H^{1/2}(\mathbb{T}^2)} \|w\|_{H^{1/2}(\mathbb{T}^2)} \|v\|_{H^1(\mathbb{T}^2)}. \quad (5.82)$$

As

$$\|u\|_{H^{1/2}}^2 \leq \|u\|_2 \|u\|_{H^1}, \quad (5.83)$$

we obtain

$$|((u \cdot \nabla v), w)| \leq C(\|u\|_{L^2} \|u\|_{H^1} \|w\|_{L^2} \|w\|_{H^1})^{1/2} \|v\|_{H^1(\mathbb{T}^2)}, \quad (5.84)$$

hence

$$|((u \cdot \nabla v), u)| \leq C \|u\|_{L^2} \|u\|_{H^1} \|v\|_{H^1(\mathbb{T}^2)}, \quad (5.85)$$

which is (5.71). This finishes the proof of Lemma 5.4. □

A compactness theorem

We have deduced above uniform in m a priori bounds on the solution $u^{(m)}$ of the Galerkin system

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)} \cdot \nabla u^{(m)}) = \nu \Delta u^{(m)} + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}. \quad (5.86)$$

The next step is to use these uniform bounds to show that the sequence $u^{(m)}$ has a (strongly) convergent subsequence in $L^2(0, T; H)$. As we will see, the limit of this subsequence will be a weak solution of the Navier-Stokes equations. We will use the following result.

Proposition 5.7 *Let u_m be a sequence of functions satisfying*

$$\|u_m(t)\|_H \leq C, \quad (5.87)$$

for all $0 \leq t \leq T$,

$$\int_0^T \|u_m(s)\|_V^2 ds \leq C, \text{ for all } m = 1, 2, \ldots \quad (5.88)$$

and

$$\int_0^T \left\| \frac{\partial u^{(m)}}{\partial t}(t) \right\|_{V'}^p \leq C, \text{ for all } m = 1, 2, \ldots, \quad (5.89)$$

with some $C > 0$ and $p > 1$. Then there exists a subsequence u_{m_j} of u_m which converges strongly in $L^2(0, T; H)$ to a function $u \in L^2(0, T; V)$.

Proof. The uniform bound (5.88) implies that there exists a subsequence u_{m_j} which converges weakly in $L^2(0, T; V)$ to a function $u \in L^2(0, T; V)$, which also obeys the bound (5.88). In addition, using the diagonal argument, we may ensure that the sequence of time derivatives $u_t^{(m)}$ converges weakly to the derivative u_t in $L^p(0, T; V')$. Thus, the estimate (5.89) also holds for the function u. The difference

$$w_j = u_{m_j} - u$$

converges weakly to zero in $L^2(0,T;V)$, and (5.87)-(5.89) hold for w_j as well. Our goal is to prove that the convergence of w_j to zero is strong in $L^2(0,T;H)$. Note that for any $f \in V$

$$\|f\|_H \le (\|f\|_V \|f\|_{V'})^{1/2}, \tag{5.90}$$

hence, for any $\delta > 0$ we have

$$\|f\|_H^2 \le \delta \|f\|_V^2 + \frac{1}{\delta}\|f\|_{V'}^2. \tag{5.91}$$

The uniform bound (5.88) for the functions w_j and (5.91) imply

$$\int_0^T \|w_j\|_H^2 dt \le C\delta + \frac{1}{\delta}\int_0^T \|w_j\|_{V'}^2 dt. \tag{5.92}$$

Our goal is to estimate the second term in (5.92), and show that it goes to zero as $j \to +\infty$, with $\delta > 0$ fixed. Note that

$$\|w_j(t)\|_{V'} \le \|w_j(t)\|_H \le C. \tag{5.93}$$

Thus, the Lebesgue dominated convergence theorem shows that it suffices to show that

$$\|w_j\|_{V'} \to 0 \text{ pointwise in } t \in [0,T]. \tag{5.94}$$

To this end, given a time $\varepsilon > 0$ and $\varepsilon \le t \le T$, let us write

$$w_j(t,x) = w_j(s,x) + \int_s^t \frac{\partial w_j(\tau,x)}{\partial \tau} d\tau, \tag{5.95}$$

and average this identity over $s \in [t-\varepsilon, t]$:

$$\begin{aligned} w_j(t,x) &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^t w_j(s,x)ds + \frac{1}{\varepsilon}\int_{t-\varepsilon}^t ds \int_s^t \frac{\partial w_j(\tau,x)}{\partial \tau} d\tau \\ &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^t w_j(s,x)ds + \frac{1}{\varepsilon}\int_{t-\varepsilon}^t (\tau - t + \varepsilon)\frac{\partial w_j(\tau,x)}{\partial \tau} d\tau. \end{aligned} \tag{5.96}$$

In order to bound the first term, note that for any $0 \le a \le b \le T$ the integral

$$I_j(x) = \int_a^b w_j(t,x)dt \tag{5.97}$$

converges weakly to zero in V. Indeed, for any $v \in V'$, the function $\chi_{[a,b]}(t)v(x)$ is an element of $L^2(0,T;V')$, and $w_j \to 0$ weakly in $L^2(0,T;V)$, thus we have

$$\int_{\mathbb{T}^n} I_j(x)v(x)dx = \int_0^T \int_{\mathbb{T}^n} w_j(t,x)\chi_{[a,b]}(t)v(x)dxdt \to 0 \text{ as } j \to \infty. \tag{5.98}$$

As V is compactly embedded into H, weak convergence in V implies strong convergence in H: the sequence I_j converges strongly to zero in H. Thus, it also converges strongly to zero in V'. In particular, given any $\varepsilon > 0$ and $\delta > 0$, for all j sufficiently large we have

$$\frac{1}{\varepsilon}\left\|\int_{t-\varepsilon}^t w_j(s,x)ds\right\|_{V'} < \delta \text{ for } j \ge J(\varepsilon,\delta,t), \tag{5.99}$$

giving a pointwise in time estimate for the first term in (5.96). For the second term in (5.96), we may use the Minkowski inequality, followed by Hölder's inequality, with $1/q + 1/p = 1$:

$$\frac{1}{\varepsilon}\left\|\int_{t-\varepsilon}^{t}(\tau - t + \varepsilon)\frac{\partial w_j(\tau,x)}{\partial \tau}d\tau\right\|_{V'} \leq \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}(\tau - t + \varepsilon)\left\|\frac{\partial w_j(\tau,x)}{\partial \tau}\right\|_{V'} d\tau \quad (5.100)$$
$$\leq \frac{1}{\varepsilon}\left(\int_{t-\varepsilon}^{t}(\tau - t + \varepsilon)^q d\tau\right)^{1/q}\left(\int_{t-\varepsilon}^{t}\left\|\frac{\partial w_j(\tau,x)}{\partial \tau}\right\|_{V'}^p d\tau\right)^{1/p}$$
$$\leq C\varepsilon^{1/q}\left(\int_0^T\left\|\frac{\partial w_j(\tau,x)}{\partial \tau}\right\|_{V'}^p d\tau\right)^{1/p} \leq C\varepsilon^{1/q},$$

for all $j \geq 1$. It follows from the above analysis that, given any $\varepsilon > 0$ and $\delta > 0$, we may find $J(\varepsilon, \delta, t)$ so that

$$\|w_j(t)\|_{V'} \leq \delta + C\varepsilon^{1/q}, \text{ for all } j \geq J(\varepsilon, \delta, t). \quad (5.101)$$

In other words, we have shown that

$$\|w_j(t)\|_{V'} \to 0 \text{ as } j \to \infty, \text{ pointwise in } t \in [0,T]. \quad (5.102)$$

As we have explained above, we may use the Lebesgue dominated convergence theorem to conclude from (5.92) that the sequence w_j converges strongly to zero in $L^2(0,T;H)$. This finishes the proof of Proposition 5.7. □

The weak solutions as limits of the Galerkin solutions

We will now construct the weak solutions of the Navier-Stokes equations as a limit of the solutions $u^{(m)}$ of the Galerkin system as $m \to \infty$. In particular, the definition of the weak solution we will adopt is motivated by the estimates on $u^{(m)}$ we have obtained above. We say that $u \in C_w(0,T;H)$ if the function $\psi(t) = (u(t), h)$ is continuous for all $h \in H$.

Definition 5.8 *A function u is a weak solution of the (periodic) Navier-Stokes equations if*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap C_w(0,T;H) \text{ and } \frac{\partial u}{\partial t} \in L^1_{loc}(0,T;V'), \quad (5.103)$$

and, for any $v \in V$ we have

$$\int_{\mathbb{T}^n} u(t,x)\cdot v(x)dx + \nu\int_0^t\int_{\mathbb{T}^n}\nabla u\cdot\nabla v dxds + \int_0^t\int_{\mathbb{T}^n}((u\cdot\nabla u)\cdot v)dxds$$
$$= \int_{\mathbb{T}^n} u_0(x)\cdot v(x)dx + \int_0^t\int f\cdot v dxds, \text{ for all } v \in V \text{ and } 0 \leq t \leq T. \quad (5.104)$$

Let us check that each term in (5.104) makes sense if u satisfies (5.103), and $v \in V$. The first term is finite since $u \in L^\infty(0,T;H)$. The second is finite since $u \in L^2(0,T;V)$. The last term in the left side is finite because of the estimate (5.66):

$$|((u\cdot\nabla u), v)| \leq C\|u\|_{H^{1/2}}\|u\|_{H^1}\|v\|_{H^1} \leq C\|u\|_H^{1/2}\|u\|_V^{3/2}\|v\|_V, \quad (5.105)$$

as $\|u\|_H$ is uniformly bounded in t, and $u \in L^2(0,T;V)$. Finally, the right side in (5.104) is finite provided that $f \in L^2(0,T;V')$ and $u_0 \in H$. The following theorem, due to Leray, is one of the most classical results in the mathematical theory of the Navier-Stokes equations (we state here its simpler version for the periodic case).

Theorem 5.9 *Given $u_0 \in H$ and $f \in L^2(0,T;V')$, there exists a weak solution of the Navier-Stokes equations*

$$\begin{aligned} &u_t + u\cdot\nabla u + \nabla p = \nu\Delta u + f, \quad t>0,\ x\in\mathbb{T}^n, \\ &\nabla\cdot u = 0, \\ &u(0,x) = u_0(x). \end{aligned} \tag{5.106}$$

In additioon, this weak solution satisfies the energy inequality

$$\frac{1}{2}\int_{\mathbb{T}^n}|u(t,x)|^2dx + \nu\int_0^t\int_{\mathbb{T}^n}|\nabla u(s,x)|^2dxds \le \frac{1}{2}\int_{\mathbb{T}^n}|u_0(x)|^2dx + \int_0^t\int_{\mathbb{T}^n} f(s,x)\cdot u(s,x)dxds. \tag{5.107}$$

Moreover, we have

$$\frac{\partial u}{\partial t} \in L^{4/3}(0,T;V') \text{ in dimension } n=3, \tag{5.108}$$

and

$$\frac{\partial u}{\partial t} \in L^{2}(0,T;V') \text{ in dimension } n=2. \tag{5.109}$$

Proof. Let $u^{(m)}$ be the solutions of the Galerkin system (5.27):

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)}\cdot\nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}. \tag{5.110}$$

The estimates we have obtained in the previous section imply that, after extracting a subsequence, $u^{(m)}$ converge strongly in $L^2(0,T;H)$ and weakly in $L^2(0,T;V)$ to some u. Moreover, the functions $u^{(m)}$ satisfy a uniform continuity in time bound in V':

$$u^{(m)}(t) - u^{(m)}(s) = \int_s^t \frac{\partial u^{(m)}}{\partial\tau}d\tau, \tag{5.111}$$

thus

$$\begin{aligned} \|u^{(m)}(t) - u^{(m)}(s)\|_{V'} &\le \int_s^t \left\|\frac{\partial u^{(m)}}{\partial\tau}\right\|_{V'}d\tau \le (t-s)^{1/q}\left(\int_s^t\left\|\frac{\partial u^{(m)}}{\partial\tau}\right\|_{V'}^p d\tau\right)^{1/p} \\ &\le (t-s)^{1/q}\left(\int_0^T\left\|\frac{\partial u^{(m)}}{\partial\tau}\right\|_{V'}^p d\tau\right)^{1/p} \le C(t-s)^{1/q}, \end{aligned} \tag{5.112}$$

with $p=q=2$ in dimension $n=2$, and $p=4/3$, $q=4$ in dimension $n=3$. Thus, u obeys the same estimate, and $u\in C(0,T;V')$. We also know that

$$\frac{\partial u^{(m)}}{\partial t} \to \frac{\partial u}{\partial t},$$

weakly in $L^{4/3}(0,T;V')$ in three dimensions, and weakly in $L^2(0,T;V')$ in two dimensions.

Given any $v\in V$ we multiply the Galerkin system (5.110) by v and integrate:

$$\begin{aligned} &\int_{\mathbb{T}^n} u^{(m)}(t,x)v(x)dx + \int_0^t\int_{\mathbb{T}^n}(u^{(m)}\cdot\nabla u^{(m)})\cdot(P_m v)dxds \\ &= -\nu\int_0^t\int_{\mathbb{T}^n}\nabla u^{(m)}\cdot\nabla v dxds + \int_{\mathbb{T}^n} u_0^{(m)}(x)v(x)dx + \int_0^t\int_{\mathbb{T}^n} fvdxds. \end{aligned} \tag{5.113}$$

We pass now to the limit in this identity, looking at each term individually. The first term in the right side is easy:

$$\int_0^t \int_{\mathbb{T}^n} \nabla u^{(m)} \cdot \nabla v dx ds \to \int_0^t \int_{\mathbb{T}^n} \nabla u \cdot \nabla v dx ds, \tag{5.114}$$

because $u^{(m)}$ converges weakly to u in $L^2(0,T;V)$. Next, we look at the nonlinear term: set

$$A_m = \int_0^t \int_{\mathbb{T}^n} (u^{(m)} \cdot \nabla u^{(m)}) \cdot (P_m v) dx ds - \int_0^t \int_{\mathbb{T}^n} (u \cdot \nabla u) \cdot v dx ds. \tag{5.115}$$

Let us recall (5.66):

$$|((u \cdot \nabla v), w)| \leq C\|u\|_{H^{1/2}} \|v\|_{H^1} \|w\|_{H^1}. \tag{5.116}$$

This inequality holds both in two and three dimensions and implies that

$$\left| \int_0^t \int_{\mathbb{T}^n} (u \cdot \nabla u) \cdot (P_m v - v) dx ds \right| \leq \left(\int_0^t \|u(s)\|_V^2 ds \right) \|P_m v - v\|_V \leq C\|P_m v - v\|_V \to 0, \tag{5.117}$$

as $m \to \infty$. Hence, A_m has the same limit as $m \to \infty$ as

$$A'_m = \int_{t_0}^t \int_{\mathbb{T}^n} (u^{(m)} \cdot \nabla u^{(m)} - u \cdot \nabla u) \cdot (P_m v) dx ds = B_1 + B_2, \tag{5.118}$$

where $B_{1,2}$ correspond to the decomposition

$$\begin{aligned} u^{(m)} \cdot \nabla u^{(m)} - u \cdot \nabla u &= u^{(m)} \cdot \nabla u^{(m)} - u^{(m)} \cdot \nabla u + u^{(m)} \cdot \nabla u - u \cdot \nabla u \\ &= u^{(m)} \cdot (\nabla u^{(m)} - \nabla u) + (u^{(m)} - u) \cdot \nabla u. \end{aligned} \tag{5.119}$$

To estimate B_1, we write

$$B_1 = \int_0^t \int_{\mathbb{T}^n} (u^{(m)} \cdot (\nabla u^{(m)} - \nabla u)) \cdot (P_m v) dx ds = -\int_0^t \int_{\mathbb{T}^n} (u^{(m)} \cdot \nabla P_m v) \cdot (u^{(m)} - u) dx ds. \tag{5.120}$$

The same proof as for (5.66) shows that

$$|(u \cdot \nabla v), w)| \leq \|u\|_V \|v\|_V \|w\|_{H^{1/2}}. \tag{5.121}$$

Using this in (5.120) gives

$$\begin{aligned} |B_1| &\leq \int_{t_0}^t \|u^{(m)}(s)\|_V \|v\|_V \|u^{(m)}(s) - u(s)\|_{H^{1/2}} ds \qquad (5.122) \\ &\leq \|v\|_V \left(\int_0^t \|u^{(m)}(s)\|_V^2 ds \right)^{1/2} \left(\int_0^t \|u^{(m)}(s) - u(s)\|_{H^{1/2}}^2 ds \right)^{1/2} \\ &\leq C\|v\|_V \left(\int_0^t \|u^{(m)}(s) - u(s)\|_V^2 ds \right)^{1/4} \left(\int_0^t \|u^{(m)}(s) - u(s)\|_H^2 ds \right)^{1/4} \\ &\leq C\|u^{(m)} - u\|_{L^2(0,T;H)} \to 0, \quad \text{as } m \to \infty, \end{aligned}$$

as $u^{(m)}$ converges to u strongly in $L^2(0,T;H)$. As for B_2, we write

$$|B_2| = \left|\int_0^t \int_{\mathbb{T}^n} ((u^{(m)} - u)\cdot\nabla u)\cdot(P_m v)dxds\right| \leq \int_0^t \|u^{(m)}(s) - u(s)\|_{H^{1/2}}\|u(s)\|_V\|v\|_V ds$$
$$\leq \|v\|_V\|u\|_{L^2(0,T;V)}\|u^{(m)}(s) - u(s)\|_{L^2(0,T;H^{1/2})} \to 0, \tag{5.123}$$

for the same reason as in (5.122).

In order to pass to the limit in the two terms in (5.113) that do not involve the time integration, we first note that $u_0^{(m)}$ converges strongly in H to u_0. Furthermore, as $u^{(m)}$ converges weakly to u in $L^2(0,T;V)$, we may extract a subsequence so that $u^{(m)}(t)$ converges weakly in V to $u(t)$ (pointwise in t), except for $t \in E$, where E is an exceptional set of times in $[0,T]$ of measure zero. Weak convergence in V implies that $u^{(m)}(t)$ converges strongly to $u(t)$ in H for $t \notin E$. Hence, taking $t \notin E$ and passing to the limit $m \to \infty$ in (5.113) we arrive at

$$\int_{\mathbb{T}^n} u(t,x)v(x)dx = \int_{\mathbb{T}^n} u_0(x)v(x)dx - \int_0^t \int_{\mathbb{T}^n} (u\cdot\nabla u)\cdot vdxds$$
$$-\nu \int_0^t \int_{\mathbb{T}^n} \nabla u\cdot\nabla vdxds + \int_0^t \int_{\mathbb{T}^n} fvdxds. \tag{5.124}$$

Given the a priori bounds on u, the right side of (5.124) is a continuous function of t, defined for all $t \in [0,T]$, not just $t \in E$. In addition, we know that $u(t)$ is continuous in $C_w(0,T;V')$, and coincides with the aforementioned right side of (5.124) for $t \notin E$. This continuity implies that $u(t)$ coincides with the right side of (5.124), which means that it satisfies (5.124) for all $t \in [0,T]$, giving us a weak solution of the Navier-Stokes equations.

The fact that $u \in C_w(0,T;H)$, and not just $u \in C(0,T;V')$ follows from (5.124), the density of V in H and the uniform in t bound on $\|u(t)\|_H$.

To obtain the energy inequality, we start with the identity

$$\frac{1}{2}\|u^{(m)}(t)\|_H^2 + \nu \int_0^t \|u^{(m)}(s)\|_V^2 ds = \frac{1}{2}\|u_0^{(m)}\|_H^2 + \int_0^t \int_{\mathbb{T}^n} f\cdot u^{(m)}dxds. \tag{5.125}$$

The right side converges, as $m \to \infty$, to

$$\frac{1}{2}\|u_0\|_H^2 + \nu \int_{t_0}^t \int_{\mathbb{T}^n} f\cdot udxds. \tag{5.126}$$

In the left side, we may use the Fatou lemma to conclude that, as $u^{(m)}(t)$ converges weakly in H to $u(t)$ for all $t \in [0,T]$, we have

$$\frac{1}{2}\|u(t)\|_H^2 + \nu \int_0^t \|u(s)\|_V^2 ds \leq \frac{1}{2}\|u_0\|_H^2 + \int_0^t \int_{\mathbb{T}^n} f\cdot udxds. \tag{5.127}$$

This completes the proof. □

Uniqueness of the weak solutions in two dimensions

One of the main issues with weak solutions in general in nonlinear partial differential equations is the issue of uniqueness – it is often much easier to show that they exist than to prove their uniqueness. Uniqueness of a weak solution hints that it is a "correct" solution. The problem of the uniqueness of the weak solutions for the Navier-Stokes equations in three dimensions is still open. In two dimensions, we know that the weak solutions of

$$\begin{aligned} &u_t + u\cdot\nabla u + \nabla p = \nu\Delta u, \quad t>0,\ x\in\mathbb{T}^2, \\ &\nabla\cdot u = 0, \\ &u(0,x) = u_0(x). \end{aligned} \tag{5.128}$$

are unique.

Theorem 5.10 *Let $f\in L^2(0,T;V')$ and $u_0\in H$. If u_1 and u_2 are two weak solutions of (5.128) which both lie in $L^2(0,T;V)\cap L^\infty(0,T;H)\cap C_w(0,T;H)$, then $u_1=u_2$.*

Proof. First, we note that if u is a weak solution of (5.128) in $L^2(0,T;V)\cap L^\infty(0,T;H)$ then $u_t\in L^2(0,T;V')$. Indeed, for any $v\in V$ we have then

$$(u_t,v) = -\nu(\nabla u,\nabla v) - (u\cdot\nabla u, v). \tag{5.129}$$

As

$$\|\Delta u\|_{V'}^2 = \|u\|_V^2, \tag{5.130}$$

the first term in the right side of (5.129) is bounded in $L^2(0,T;V')$. For the second term, we use the bound

$$|(u\cdot\nabla u),v| \le C\|u\|_H\|u\|_V\|v\|_V, \tag{5.131}$$

which holds in two dimensions and implies that

$$\|u\cdot\nabla u\|_{V'} \le C\|u\|_H\|u\|_V \tag{5.132}$$

and thus

$$\int_0^T \|u\cdot\nabla u(s)\|_{V'}^2 ds \le \big(\sup_{t\in[0,T]}\|u\|_H^2\big)\int_0^T \|u\|_V^2 ds = \|u\|_{L^\infty(0,T;H)}\|u\|^2_{L^2(0,T;V)}. \tag{5.133}$$

Thus, we know that $u_t\in L^2(0,T;V')$. Let us denote $w=u_1-u_2$. This function satisfies

$$\begin{aligned} &w_t + u_1\cdot\nabla w + w\cdot\nabla u_2 + \nabla p' = \nu\Delta w, \quad t>0,\ x\in\mathbb{T}^2, \\ &\nabla\cdot w = 0, \\ &w(0,x) = 0, \end{aligned} \tag{5.134}$$

with $p'=p_1-p_2$, and we know that $w_t\in L^2(0,T;V')$.

Multiplying by w and integrating over the torus gives

$$\int_{\mathbb{T}^2} w_t\cdot w + \nu\int_{\mathbb{T}^2}|\nabla w|^2 dx + \int_{\mathbb{T}^2} w_k(\partial_j u_{2,m})w_m dx = 0. \tag{5.135}$$

As $w_t \in V'$ for a.e. t, and $w \in V$ for a.e. $t \in [0,T]$, identity (5.135) holds for a.e. $t \in [0,T]$. We have, as in (5.131):

$$|(w \cdot \nabla u_2, w)| \leq C\|w\|_H \|u_2\|_V \|w\|_V. \tag{5.136}$$

As $w \in L^\infty(0,T;H)$ and $u_2, w \in L^2(0,T,H)$, we conclude from (5.135) and (5.136) that

$$\int_0^T |(w_t(t), w(t))| dt < +\infty.$$

Now, (5.135) implies that

$$\frac{d}{dt}\|w\|_H^2 \leq C\|w\|_H\|u_2\|_V\|w\|_V - \nu\|w\|_V^2 \leq \frac{C}{\nu}\|u_2\|_V^2\|w\|_H^2. \tag{5.137}$$

As

$$\int_0^T \|u_2\|_V^2 dt < +\infty,$$

Gronwall's inequality implies that

$$\|w(t)\|_H^2 \leq \|w(0)\|_H^2 \exp\Big\{\int_0^t \|u_2(s)\|_V^2 ds\Big\} = 0, \tag{5.138}$$

since $w(0) = 0$. This finishes the proof. □

6 Strong solutions in two and three dimensions

Uniqueness of strong solutions in three dimensions

We say that u is a strong solution of the Navier-Stokes equations (in either two or three dimensions) if u is a weak solution, and, in addition, $u \in C_w(0,T;V)$, and the following bounds hold:

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^n} |\nabla u(t,x)|^2 dx < +\infty, \tag{6.1}$$

and

$$\int_0^T \int_{\mathbb{T}^n} |\Delta u(t,x)|^2 dx dt < +\infty. \tag{6.2}$$

The motivation for this definition comes from two properties that we will prove: first, unlike for the weak solutions, one can show that strong solutions are unique in three dimensions (existence of strong solutions in three dimensions is an important open problem). Second, as we will show, the conditions in the definition of the strong solutions are sufficient to show that they are actually infinitely differentiable if the initial data u_0 and the forcing f are.

First, we prove their uniqueness in three dimensions.

Theorem 6.1 *Let $u_{1,2}$ be two solutions of the Navier-Stokes equations on $\mathbb{T}^3$ with the initial data $u_0 \in H$ and $f \in L^2(0,T;H)$. If both $u_{1,2}$ satisfy (6.1) and (6.2), and they lie in $C_w(0,T;V)$ then $u_1 = u_2$.*

Proof. We argue as in the proof of uniqueness of the weak solutions in two dimensions. Let $w = u_1 - u_2$, so that

$$(\frac{\partial w}{\partial t}, w) + \nu\|w\|_V^2 + (w \cdot \nabla u_2, w) = 0, \tag{6.3}$$

as in (5.135). We now use the estimate

$$|((w \cdot \nabla u, w)| \leq C\|w\|_{L^2}\|w\|_{H^1}\|u\|_{H^1}^{1/2}\|\Delta u\|_2^{1/2}. \tag{6.4}$$

It is obtained as follows: recall that in three dimensions we have

$$\|w\|_{L^3(\mathbb{T}^3)} \leq C\|w\|_{H^{1/2}}, \tag{6.5}$$

thus

$$\begin{aligned} |((w \cdot \nabla u, w)| \leq \int_{\mathbb{T}^3} |w||\nabla u||w|dx \leq \|w\|_{L^3}\|\nabla u\|_{L^3}\|w\|_{L^3} \leq C\|w\|_{H^{1/2}}^2\|\nabla u\|_{H^{1/2}} \quad (6.6)\\ \leq C\|w\|_{L^2}\|w\|_{H^1}\|u\|_{H^1}^{1/2}\|\Delta u\|_{L^2}^{1/2}, \end{aligned}$$

which is (6.4). Using the bound (6.4) in (6.3) leads to

$$\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2) + \nu\|w\|_{H^1}^2 \leq \frac{C}{\nu}\|w\|_{L^2}^2\|u\|_{H^1}\|\Delta u\|_2 + \nu\|w\|_{H^1}^2. \tag{6.7}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2) \leq \frac{C}{\nu}\|u\|_{H^1}\|\Delta u\|_2\|w\|_{L^2}^2. \tag{6.8}$$

Now, Grownwall's inequality implies that $w(t) = 0$ provided that $w(0) = 0$, and

$$\int_0^t \|u\|_{H^1}\|\Delta u\|_2 ds < +\infty, \tag{6.9}$$

which is a consequence of (6.1)-(6.2). □

Construction of the strong solutions in two dimensions

We now use the Galerkin system in two dimensions to show existence of global in time strong solutions of the Navier-Stokes equations in two dimensions. Once again, we restrict ourselves to the simpler case of the two-dimensional torus $\mathbb{T}^2$. As we did to show the existence of weak solutions, we will use the Galerkin system

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)} \cdot \nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}, \tag{6.10}$$

and then pass to the limit $m \to +\infty$. As we have already shown the uniqueness of the weak solutions in the two-dimensional case, this will show that weak solutions are actually strong in two dimensions.

Galerkin solutions are often not large

The first step is to show that solutions of the Galerkin system are "often not large" – this will be made precise soon. The second stepp will be to show that if solutions are not too large often then they can never be large. Taking the inner product with $u^{(m)}$ we obtain the familiar identity

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|_H^2 + \nu\|\nabla u^{(m)}\|_H^2 = (f, u^{(m)}). \tag{6.11}$$

We may use the Poincaré inequality

$$\int_{\mathbb{T}^2} |u(x)|^2 dx = \sum_{k\in\mathbb{Z}^n} |u_k|^2 \le \sum_{k\in\mathbb{Z}^n} |k|^2|u_k|^2 = \frac{1}{4\pi^2}\int_{\mathbb{T}^n} |\nabla u|^2 dx, \tag{6.12}$$

to conclude from (6.11) that

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|_H^2 + \nu\|\nabla u^{(m)}\|_H^2 \le \frac{1}{2\cdot 4\pi^2\nu}\|f\|_H^2 + \frac{4\pi^2\nu}{2}\|u^{(m)})\|_H^2 \le \frac{1}{8\pi^2\nu}\|f\|_H^2 + \frac{\nu}{2}\|\nabla u^{(m)})\|_H^2. \tag{6.13}$$

We deduce the bounds we have seen before: there exist two explicit constants $C_{1,2} > 0$, so that

$$\nu\int_0^t \|\nabla u^{(m)}\|_V^2 ds \le \|u_0\|_H^2 + \frac{C_1}{\nu}\int_0^t \|f\|_H^2 ds, \tag{6.14}$$

and

$$\|u^{(m)}(t)\|_H^2 \le \|u_0\|_H^2 e^{-C_2\nu t} + \frac{C_1}{\nu}\int_0^t e^{-C_2\nu(t-s)}\|f\|_H^2 ds. \tag{6.15}$$

In particular, if $f \in L^\infty(0,T;H)$, then

$$\|u^{(m)}(t)\|_H^2 \le \|u_0\|_H^2 e^{-C_2\nu t} + \frac{C_1}{\nu^2}\|f\|_\infty^2, \tag{6.16}$$

with

$$\|f\|_\infty = \sup_{t>0}\|f(t)\|_H. \tag{6.17}$$

Our next goal is to get uniform in time bounds on $\|u^{(m)}(t)\|_V$ – this is not something we have done in the construction of the weak solutions, because such bound holds only in two dimensions, and not in three, while the weak solutions can be constructed both in two and three dimensions. The first step in that direction is to show that this norm can not be large for too long a time.

Proposition 6.2 *Let $u^{(m)}(t)$ be the solution for the Galerkin system with $f \in L^\infty(0,+\infty;H)$ and $u_0 \in H$, in either two or three dimensions. Then in every time interval of length $\tau > 0$ there exists a time t_0 so that*

$$\|u^{(m)}(t_0)\|_V^2 \le \frac{2}{\tau\nu}\left(\|u_0\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty(\frac{1}{\nu}+\tau)\right). \tag{6.18}$$

Proof. Inequality (6.15) implies that

$$\nu \int_0^t \|\nabla u^{(m)}\|_V^2 ds \le \|u_0\|_H^2 + \frac{C_1 t}{\nu}\|f\|_\infty^2, \tag{6.19}$$

and (6.15) that

$$\|u^{(m)}(t)\|_H^2 \le \|u_0\|_H^2 + \frac{C_1}{\nu^2}\|f\|_\infty^2. \tag{6.20}$$

Let us also integrate (6.13) between the times t and $t+\tau$, leading to

$$\nu \int_t^{t+\tau} \|u(s)\|_V^2 ds \le \|u(t)\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty \tau \le \|u_0\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty (\frac{1}{\nu} + \tau). \tag{6.21}$$

The right side above does not depend on the time t. Therefore, on any time interval $[t, t+\tau]$ we may estimate the Lebesgue measure of the set of times when $\|u(s)\|_V$ is large:

$$\left|\{s: \ s \in [t, t+\tau] \text{ s.t. } \|u^{(m)}(s)\|_V \ge \rho\}\right| \le \frac{1}{\nu\rho^2}\left(\|u_0\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty(\frac{1}{\nu}+\tau)\right). \tag{6.22}$$

In particular, taking

$$\rho_0 = \left[\frac{2}{\tau\nu}\left(\|u_0\|_H^2 + \frac{C_1}{\nu}\|f\|_\infty(\frac{1}{\nu}+\tau)\right)\right]^{1/2},$$

we arrive at the conclusion of Proposition 6.2. □

Galerkin solutions are never large

Next, we will get rid of the "sometimes not large" restriction in Proposition 6.2, showing that in two dimensions Galerkin solutions are never large in V. We will prove the following estimate for the solutions of the Galerkin system

$$\frac{\partial u^{(m)}}{\partial t} + P_m(u^{(m)} \cdot \nabla u^{(m)}) = \nu\Delta u^{(m)} + f^{(m)}, \quad u^{(m)}(0) = u_0^{(m)}. \tag{6.23}$$

Proposition 6.3 *Let $u^{(m)}$ be the solution of the Galerkin system (6.23) with the initial data $u_0 \in H$ and $f \in L^\infty(0,T;H)$. There exists a constant α that depends on ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m so that $u^{(m)}$ satisfies the bounds*

$$\|u^{(m)}(t)\|_V \le \alpha \text{ for all } t \ge 1, \tag{6.24}$$

and

$$\|u^{(m)}(t)\|_V \le \frac{\alpha}{t} \text{ for all } 0 < t < 1. \tag{6.25}$$

In addition, if $u_0 \in V$ then there exists a constant α_1 which depends ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m so that

$$\|u^{(m)}(t)\|_V \le \alpha_1 \text{ for all } 0 < t < 1. \tag{6.26}$$

Proof. The idea is to use Proposition 6.2 – we know that for any time $t > 1$ there is a time $t_0 \in [t-1, t]$ so that the norm $\|u(t_0)\|_V \le \alpha$, with the constant α which depends only on ν, $\|u_0\|_H$ and $\|f\|_\infty$. The additional ingredient in this proof will be a control of the growth of $\|u\|_V$ on the time intervals of length 1.

We multiply (6.23) by Δu and integrate. The first term gives

$$\int_{\mathbb{T}^2} u_t^{(m)} \cdot \Delta u^{(m)} dx = -\int_{\mathbb{T}^2} \nabla u_t^{(m)} \cdot \nabla u^{(m)} dx = -\frac{1}{2}\frac{d}{dt}\|\nabla u^{(m)}(t)\|_H^2, \tag{6.27}$$

so that the overall balance is

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^{(m)}(t)\|_H^2 + \nu\|\Delta u^{(m)}\|_H^2 - ((u^{(m)} \cdot \nabla u^{(m)}), \Delta u^{(m)}) = -(f, \Delta u^{(m)}). \tag{6.28}$$

For the nonlinear term, we will use the inequality

$$|((u \cdot \nabla u), \Delta u)| \le \|u\|_H^{1/2}\|u\|_V\|\Delta u\|_H^{3/2}, \tag{6.29}$$

which holds in two dimensions. The proof is similar to that of (5.71): we write

$$|((u \cdot \nabla v), w)| \le \int_{\mathbb{T}^n} |(u_j \partial_j v_k) w_k| dx \le \|u \cdot \nabla v\|_{L^2}\|w\|_{L^2} \le \|u\|_{L^4}\|\nabla v\|_{L^4}\|w\|_{L^2}. \tag{6.30}$$

The Sobolev inequality

$$\|f\|_{L^q(\mathbb{T}^n)} \le C\|f\|_{H^m(\mathbb{T}^n)}, \qquad \frac{1}{q} \ge \frac{1}{2} - \frac{m}{n} \tag{6.31}$$

implies that in two dimensions we have

$$\|f\|_{L^4(\mathbb{T}^2)} \le C\|f\|_{H^{1/2}(\mathbb{T}^2)}. \tag{6.32}$$

Using this in (6.30) leads to

$$\begin{aligned} &|((u \cdot \nabla u), \Delta u)| \le \|u\|_{H^{1/2}}\|u\|_{H^{3/2}}\|\Delta u\|_{L^2} \le \|u\|_H^{1/2}\|u\|_V^{1/2}\|u\|_V^{1/2}\|\Delta u\|_H^{1/2}\|\Delta u\|_H \\ &= \|u\|_H^{1/2}\|u\|_V\|\Delta u\|_H^{3/2}, \end{aligned} \tag{6.33}$$

which is (6.29). It follows that the nonlinear term can be estimated, using the inequality

$$ab \le \frac{\nu}{4}a^{4/3} + \frac{C}{\nu^3}b^4$$

as

$$|((u \cdot \nabla u), \Delta u)| \le \frac{\nu}{4}\|\Delta u\|_H^2 + \frac{C}{\nu^3}\|u\|_H^2\|u\|_V^4. \tag{6.34}$$

Returning to (6.28), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u^{(m)}(t)\|_H^2 + \nu\|\Delta u^{(m)}\|_H^2 \le |((u^{(m)} \cdot \nabla u^{(m)}), \Delta u^{(m)})| + \|f\|_\infty\|\Delta u^{(m)}\|_H \\ &\le \frac{\nu}{4}\|\Delta u^{(m)}\|_H^2 + \frac{C}{\nu^3}\|u^{(m)}\|_H^2\|u^{(m)}\|_V^4 + \frac{\nu}{4}\|\Delta u^{(m)}\|_H^2 + \frac{C}{\nu}\|f\|_\infty^2. \end{aligned} \tag{6.35}$$

We conclude that

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2 + \frac{\nu}{2}\|\Delta u^{(m)}\|_H^2 \le \frac{C}{\nu^3}\|u^{(m)}\|_H^2\|u^{(m)}\|_V^4 + \frac{C}{\nu}\|f\|_\infty^2. \tag{6.36}$$

Let us set

$$G(t_0;t) = \frac{2C}{\nu^3}\int_{t_0}^t \|u(s)\|_H^2\|u(s)\|_V^2 ds, \tag{6.37}$$

then (6.36) implies, for any $t \ge t_0$:

$$\frac{d}{dt}\Big(\|u^{(m)}\|_V^2 \exp\{-G(t_0;t)\}\Big) \le \frac{C}{\nu}\|f\|_\infty^2 \exp\{-G(t_0;t)\}. \tag{6.38}$$

Integrating between t_0 and t gives

$$\begin{aligned}
\|u^{(m)}(t)\|_V^2 &\le \|u^{(m)}(t_0)\|_V^2 \exp\{G(t_0;t)\} + \frac{C}{\nu}\|f\|_\infty^2 \exp\{G(t_0;t)\}\int_{t_0}^t \exp\{-G(t_0;s)\}ds \\
&\le \|u^{(m)}(t_0)\|_V^2 \exp\{G(t_0;t)\} + \frac{C}{\nu}\|f\|_\infty^2 \int_{t_0}^t \exp\{G(s;t)\}ds \\
&\le \|u^{(m)}(t_0)\|_V^2 \exp\{G(t_0;t)\} + \frac{C}{\nu}\|f\|_\infty^2 (t-t_0)\exp\{G(t_0;t)\}. \qquad (6.39)
\end{aligned}$$

Now we will use the "sometimes small" result in Proposition 6.2. Given $\tau > 0$ and $t > \tau$ we may find $t_0 \in [t-\tau, t]$ such that

$$\|u(t_0)\|_V \le \alpha(1 + \frac{1}{\tau}), \tag{6.40}$$

with the constant $\alpha > 0$ that only depends on ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m or $\|u_0\|_V$. We may also use (6.21) to estimate $G(t_0;t)$:

$$G(t_0;t) \le \alpha(1+\tau). \tag{6.41}$$

Using this in (6.39) shows that for all $t > \tau$ we have

$$\begin{aligned}
&\|u^{(m)}(t)\|_V^2 \le \|u^{(m)}(t_0)\|_V^2 \exp\{G(t_0;t)\} + \frac{C}{\nu}\|f\|_\infty^2 (t-t_0)\exp\{G(t_0;t)\} \qquad (6.42) \\
&\le \alpha(1+\frac{1}{\tau})e^{\alpha(1+\tau)} + \alpha\tau e^{\alpha(1+\tau)}.
\end{aligned}$$

This bound is uniform in $t > \tau$. Hence, if we fix $\tau = 1$, we get a uniform in m estimate for $\|u^{(m)}(t)\|_V$ for all $t > 1$, giving the bound (6.24).

In order to deal with times $t < 1$, we will use (6.42) on the time intervals $t \in [1/2^{k+1}, 1/2^k]$ with $\tau = 1/2^{k+1}$. The point is that for such times t and τ are comparable: $\tau \le t \le 2\tau$. Therefore, for $t < 1$ we have an estimate

$$t\|u^{(m)}(t)\|_V^2 \le \alpha, \tag{6.43}$$

with the constant α that only depends on ν, $\|u_0\|_H$ and $\|f\|_\infty$ but not on m or $\|u_0\|_V$, which is (6.25).

Finally, if we allow the dependence on the norm $\|u_0\|_V$, then for times $t < 1$ we may simply use the first line in (6.42) with $t_0 = 0$, together with the estimate

$$G(t_0 = 0, t = 1) \le 2\alpha, \tag{6.44}$$

which follows from (6.41). This gives (6.26) and finishes the proof of Proposition 6.3. □

The strong solutions in two dimensions

The above bounds on the solutions $u^{(m)}$ of the Galerkin system (6.23) allow us to pass to the limit $m \to \infty$ to construct solutions of the Navier-Stokes equations on a two-dimensional torus

$$\begin{aligned} & u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \quad t > 0, \quad x \in \mathbb{T}^2, \\ & \nabla \cdot u = 0, \\ & u(0,x) = u_0(x). \end{aligned} \tag{6.45}$$

Theorem 6.4 *Assume that $T > 0$, $u_0 \in H$ and $f \in L^\infty(0,T;H)$. Then there exists a constant $C > 0$ which depends only on ν, $\|u_0\|_H$ and $\|f\|_\infty$, and a solution of the Navier-Stokes equation (6.45) which satisfies the bounds*

$$\|u(t)\|_H \leq C, \tag{6.46}$$

$$\|u(t)\|_V \leq C \text{ for } t \geq 1, \text{ and } \|u(t)\| \leq \frac{C}{t} \text{ for } 0 < t < 1, \tag{6.47}$$

$$\int_0^T \|u(t)\|_V^2 dt \leq C. \tag{6.48}$$

In addition, for any $s > 0$ there exists C_s so that

$$\int_s^T \|\Delta u(t)\|_H^2 dt \leq C_s T. \tag{6.49}$$

Moreover, if $u_0 \in V$ then there exists a constant $C > 0$ which depends only on ν, $\|u_0\|_V$ and $\|f\|_\infty$ so that

$$\|u(t)\|_V \leq C \text{ for all } t \geq 0, \tag{6.50}$$

and

$$\int_0^T \|\Delta u(t)\|_H^2 dt \leq CT. \tag{6.51}$$

These bounds are inherited from the solutions of the Galerkin system, we leave the details of this passage to the reader, as they are very close to what was done in the corresponding passage in the construction of the weak solutions. We only mention that the $L^2(0,T;H)$ estimate for Δu follows from (6.36). Note that we do not yet claim that if u_0 is an infinitely differentiable function, then the solution $u(t,x)$ is also smooth but only that u is a strong solution in the sense that the aforementioned bounds on $u(t,x)$ hold. We will improve them soon, assuming that u_0 is smooth.

Strong solutions in three dimensions: small data

While existence of global in time strong solutions in three dimensions is not known, strong solutions do exist if the initial data and the forcing are small.

Theorem 6.5 *Let $u_0 \in V$ and $f \in L^2(0,T;H)$. There exists a constant $C > 0$ which depend only on ν, so that if*

$$\|u_0\|_H + \int_0^T \|f(t)\|_H^2 dt \le C, \tag{6.52}$$

then the Navier-Stokes equations

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \quad t > 0, \quad x \in \mathbb{T}^3, \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.53}$$

have a strong solution on the time interval $[0,T]$ that satisfies

$$\|u(t)\|_V^2 + \int_0^T \|\Delta u(t)\|_H^2 dt \le \frac{1}{C}, \tag{6.54}$$

for all $0 \le t \le T$.

Note that if $f = 0$ then solutions exist for all $t > 0$ if the initial condition is small: $\|u_0\|_H \le C$. The proof of this theorem, once again, relies on the estimates for the Galerkin solutions

$$u_t^{(m)} + P_m(u^{(m)} \cdot \nabla u^{(m)}) = \nu \Delta u^{(m)}, \quad u^{(m)}(0,x) = u_0^{(m)}(x), \quad t > 0, \quad x \in \mathbb{T}^3. \tag{6.55}$$

Taking the inner product with $\Delta u^{(m)}$, as we did in the two-dimensional case, we obtain, as in(6.28):

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2 + \nu\|\Delta u^{(m)}\|_H^2 - (u^{(m)} \cdot \nabla u^{(m)}, \Delta u^{(m)}) = -(f, \Delta u^{(m)}). \tag{6.56}$$

In three dimensions, we may not use the two-dimensional estimate (6.29) for the nonlinear term. Instead, we will bound it as

$$|(u \cdot \nabla u, \Delta u)| \le C\|u\|_V^{3/2}\|\Delta u\|_H^{3/2} \le \frac{C}{\nu^3}\|u\|_V^6 + \frac{\nu}{4}\|\Delta u\|_H^2. \tag{6.57}$$

This comes from the estimate

$$|(u \cdot \nabla u, \Delta u)| \le C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\Delta u\|_{L^2}. \tag{6.58}$$

The Sobolev inequality implies that in three dimensions we have

$$\|u\|_{L^3} \le C\|u\|_{H^{1/2}}, \quad \|v\|_{L^6} \le C\|v\|_{H^1}. \tag{6.59}$$

Using this in (6.58) gives

$$|(u \cdot \nabla u, \Delta u)| \le C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\Delta u\|_{L^2} \le C\|u\|_{H^1}\|\nabla u\|_{H^{1/2}}\|\Delta u\|_{L^2} \le C\|u\|_{H^1}^{3/2}\|\Delta u\|_{L^2}^{3/2}, \tag{6.60}$$

which is (6.57). We will estimate the forcing term in (6.56) as

$$|(f, \Delta u)| \le \frac{4}{\nu}\|f\|_H^2 + \frac{\nu}{4}\|\Delta u\|_H^2. \tag{6.61}$$

Altogether, with the above estimates, (6.56) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2+\nu\|\Delta u^{(m)}\|_H^2 &\le (u^{(m)}\cdot\nabla u^{(m)},\Delta u^{(m)})-(f,\Delta u^{(m)}) \\ &\le \frac{C}{\nu^3}\|u^{(m)}\|_V^6+\frac{\nu}{4}\|\Delta u^{(m)}\|_H^2+\frac{C}{\nu}\|f\|_H^2+\frac{\nu}{4}\|\Delta u^{(m)}\|_H^2.\end{aligned} \tag{6.62}$$

This gives

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2\le\frac{C}{\nu^3}\|u^{(m)}\|_V^6-\frac{\nu}{2}\|\Delta u^{(m)}\|_H^2+\frac{C}{\nu}\|f\|_H^2\le\frac{C}{\nu^3}\|u^{(m)}\|_V^6-\frac{\nu}{2}\|u^{(m)}\|_V^2+\frac{C}{\nu}\|f\|_H^2. \tag{6.63}$$

Therefore, the function $y(t)=\|u^{(m)}(t)\|_V^2$ satisfies a differential inequality

$$\frac{dy}{dt}\le\frac{C}{\nu^3}y^3-\nu y+\frac{C}{\nu}\|f\|_H^2. \tag{6.64}$$

Hence, as long as

$$y(s)\le\frac{\nu^2}{\sqrt{C}},\ \text{for all } 0<s<t, \tag{6.65}$$

we have

$$\frac{dy}{dt}\le\frac{C}{\nu}\|f\|_H^2, \tag{6.66}$$

and

$$y(t)\le y(0)+\frac{C}{\nu}\int_0^t\|f(s)\|_H^2ds. \tag{6.67}$$

It follows that if

$$\|u_0\|_V^2+\frac{C}{\nu}\int_0^\infty\|f(s)\|_H^2ds\le\frac{\nu^2}{\sqrt{C}}, \tag{6.68}$$

with a universal constant $C>0$, then

$$\|u^{(m)}(t)\|_V^2\le\frac{\nu^2}{\sqrt{C}}, \tag{6.69}$$

for all $t>0$. This is the part of the bound (6.54) on $\|u^{(m)}\|_V$. In order to get the bound on $\Delta u^{(m)}$ in $L^2(0,T;H)$, we go back to (6.62):

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2+\frac{\nu}{2}\|\Delta u^{(m)}\|_H^2\le\frac{C}{\nu^3}\|u^{(m)}\|_V^6+\frac{C}{\nu}\|f\|_H^2\le C\nu\|u^{(m)}\|_V^2+\frac{C}{\nu}\|f\|_H^2, \tag{6.70}$$

leading to

$$\frac{\nu}{2}\int_0^T\|\Delta u^{(m)}(t)\|_H^2dt\le\|u_0^{(m)}\|_V^2+C\nu\int_0^T\|u^{(m)}(t)\|_V^2dt+\frac{C}{\nu}\int_0^T\|f(t)\|_H^2dt. \tag{6.71}$$

As we also have

$$\nu\int_0^T\|u^{(m)}(t)\|_V^2dt\le\|u_0\|_H^2+\frac{C}{\nu}\int_0^T\|f(t)\|_H^2dt, \tag{6.72}$$

we deduce that under the assumptions (6.52) we have

$$\int_0^T\|\Delta u^{(m)}(t)\|_H^2dt\le C. \tag{6.73}$$

Passing to the limit $m\to\infty$ we construct a solution of the Navier-Stokes equations $u(t)$ that satisfies the same estimates (6.54). Uniqueness of the strong solution finishes the proof.

Strong solutions in three dimensions: short times

Next, we show that strong solutions of the Navier-Stokes exist for a sufficiently short time even if the data is not small.

Theorem 6.6 *Let $u_0 \in V$ and $f \in L^2(0,T;H)$. There exists a constant $C_0 > 0$ which depends on ν and $\|u_0\|_V$, so that if*

$$T_0 + \int_0^{T_0} \|f(t)\|_H^2 dt \le C_0, \tag{6.74}$$

then the Navier-Stokes equations

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \quad t > 0, \quad x \in \mathbb{T}^3, \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.75}$$

have a strong solution on the time interval $[0, T_0]$ that satisfies

$$\|u(t)\|_V^2 \le C_0^{-1}, \tag{6.76}$$

for all $0 \le t \le T_0$.

For the proof, we recall (6.70):

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|_V^2 + \frac{\nu}{2}\|\Delta u^{(m)}\|_H^2 \le \frac{C}{\nu^3}\|u^{(m)}\|_V^6 + \frac{C}{\nu}\|f\|_H^2, \tag{6.77}$$

which, in particular, implies that the function $y(t) = \|u^{(m)}(t)\|_V^2$ satisfies a differential inequality

$$\dot{y}(t) \le C y(t)^3 + C\|f\|_H^2, \tag{6.78}$$

with the constant C that depends on ν. Dividing by $(1+y)^3$ we get

$$\frac{\dot{y}}{(1+y)^3} \le \frac{Cy^3 + C\|f\|_H^2}{(1+y)^3} \le C + C\|f\|_H^2, \tag{6.79}$$

Integrating in time leads to

$$\frac{1}{(1+y_0)^2} - \frac{1}{(1+y(t))^2} \le Ct + C\int_0^t \|f(s)\|_H^2 ds. \tag{6.80}$$

Therefore, as long as the time t is such that (6.80) holds, or, rather, as long as T_0 satisfies

$$CT_0 + C\int_0^{T_0} \|f(s)\|_H^2 ds \le \frac{1}{2(1+\|u_0\|_V^2)^2} \le \frac{1}{2(1+y_0)^2}, \tag{6.81}$$

we have, for all $0 \le t \le T_0$:

$$\frac{1}{(1+y(t))^2} \ge \frac{1}{2(1+y_0)^2} \ge \frac{1}{2(1+\|u_0\|_V^2)^2}. \tag{6.82}$$

Therefore, as long as the time t is sufficiently small, so that (6.80) holds, we have

$$\|u^{(m)}(t)\|_V^2 \le 2(1 + \|u_0\|_V^2). \tag{6.83}$$

As usual, this uniform bound on the Galerkin approximations $u^{(m)}(t)$ implies that, passing to the limit $m \to +\infty$, we construct a strong solution of the Navier-Stokes equations for times $0 \le t \le T_0$.

In general, for an arbitrary $m > 1$ $\Box$

Strong solutions are smooth if the data are smooth

We now show that if the initial condition u_0 and the forcing f are smooth, then the strong solution of the Navier-Stokes equations (if it exists) is also infinitely differentiable. We consider only the three-dimensional case but the analysis applies essentially verbatim to the two-dimensional case as well.

Theorem 6.7 *Let $u(t,x)$ be the strong solution of the Navier-Stokes equations*

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \quad 0 < t \le T, \quad x \in \mathbb{T}^3, \\ &\nabla \cdot u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{6.84}$$

in the sense that there exists $C > 0$ so that

$$\sup_{0 \le t \le T} \|u(t)\|_V \le C, \quad \int_0^T \|\Delta u(s)\|_H^2 ds \le C. \tag{6.85}$$

Assume that $u_0 \in C^\infty(\mathbb{T}^3)$ and $f \in C^\infty(0,T;\mathbb{T}^3)$, then $u \in C^\infty(0,T;\mathbb{T}^3)$.

The strategy of the proof will be to estimate $\|\Delta^m u(t)\|_H$ for all $m \in \mathbb{N}$, and show that, as long u satisfies the assumptions of Theorem 6.7, these norms remain finite for $0 \le t \le T$, and all $m \in \mathbb{N}$. As $m \in \mathbb{N}$ will be arbitrary, the Sobolev embedding theorem will imply that u is infinitely differentiable in x, while the Navier-Stokes equations themselves will imply that u is infinitely differentiable in time (using the projection on the divergence free fields, the reader should convince himself that the pressure term is not a problem).

Multiplying (6.84) by $(-\Delta)^m u$ and integrating over $\mathbb{T}^3$ gives

$$(u_t, (-\Delta)^m u) - (u \cdot \nabla u, (-\Delta)^m u) = -\nu(-\Delta u, (-\Delta)^m u) + (f, (-\Delta)^m u). \tag{6.86}$$

Integrating by parts leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 - ((-\Delta)^{m/2}(u \cdot \nabla u), (-\Delta)^{m/2}u) + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 \\ &\qquad \le \|(-\Delta)^{m/2}f\|_H\|(-\Delta)^{m/2}u\|_H. \end{aligned} \tag{6.87}$$

The key inequality we will need for the nonlinear term is given by the following lemma.

Lemma 6.8 *For every $m > 3/2$ there exists a constant $C > 0$ so that for any vector-valued functions u, v such that $u_0 = v_0 = 0$, and $\nabla \cdot u = \nabla \cdot v = 0$, and $u_k = v_k = 0$ for all $k > M$, with some $M > 0$, we have*

$$\|(-\Delta)^{m/2}P(u\cdot\nabla v)\|_H \le C\|(-\Delta)^{m/2}u\|_H\|(-\Delta)^{(m+1)/2}v\|_H. \tag{6.88}$$

Here, P is the projection on divergence-free fields.

Postponing the proof of this lemma, we apply it in (6.87):

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 \le \|(-\Delta)^{m/2}f\|_H\|(-\Delta)^{m/2}u\|_H \\ &+C\|(-\Delta)^{m/2}u\|_H^2\|(-\Delta)^{(m+1)/2}u\|_H.\end{aligned} \tag{6.89}$$

Next, we use Young's inequality in the right side together with the Poincare inequality in the form

$$\|(-\Delta)^{m/2}u\|_H \le C\|(-\Delta)^{(m+1)/2}u\|_H. \tag{6.90}$$

This leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 \le \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{\nu}{4}\|(-\Delta)^{(m+1)/2}u\|_H^2 \\ &+\frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4 + \frac{\nu}{4}\|(-\Delta)^{(m+1)/2}u\|_H^2 \\ &\le \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4 + \frac{\nu}{2}\|(-\Delta)^{(m+1)/2}u\|_H^2.\end{aligned} \tag{6.91}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \frac{\nu}{2}\|(-\Delta)^{(m+1)/2}u\|_H^2 \le \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4. \tag{6.92}$$

Looking at this as the differential inequality for $y(t) = \|(-\Delta)^{m/2}u\|_H^2$, we deduce that

$$\dot{y} \le \frac{C}{\nu}\|(-\Delta)^{m/2}f\|_H^2 + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^2 y(t) \le C_f + \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^2 y(t), \tag{6.93}$$

with a finite constant C_f as $f \in C^\infty(0,T;\mathbb{T}^3)$. Grownwall's inequality implies now that $y(t)$ obeys an upper bound

$$y(t) \le y(0)\exp\Big[\frac{C}{\nu}\int_0^t \|(-\Delta)^{m/2}u(s)\|_H^2 ds\Big] + C_f\int_0^t \exp\Big[\frac{C}{\nu}\int_s^t \|(-\Delta)^{m/2}u(\tau)\|_H^2 d\tau\Big] ds. \tag{6.94}$$

In other words, if we know that

$$\int_0^T \|(-\Delta)^{m/2}u(s)\|_H^2 ds < +\infty, \tag{6.95}$$

then

$$\sup_{0\le t\le T} \|(-\Delta)^{m/2}u(s)\|_H^2 ds < +\infty. \tag{6.96}$$

This, in turn, implies that

$$\int_0^T \|(-\Delta)^{m/2} u(s)\|_H^4 ds < C, \tag{6.97}$$

which can be inserted into (6.92) to conclude that

$$\int_0^T \|(-\Delta)^{(m+1)/2} u(s)\|_H^2 ds < +\infty, \tag{6.98}$$

allowing us to build an induction argument and continue forever, meaning that

$$\sup_{0\le t\le T} \|(-\Delta)^{m/2} u(s)\|_H^2 ds < +\infty, \text{ for any } m \in \mathbb{N}. \tag{6.99}$$

This will, in turn, imply that $u \in C^\infty$ by the Sobolev embedding theorem. However, this argument uses the bound (6.88) which applies only for $m > 3/2$, and the "free" estimate for the weak solution is

$$\int_0^T \|\nabla u(s)\|_H^2 ds = \int_0^T \|(-\Delta)^{1/2} u(s)\|_H^2 ds < +\infty, \tag{6.100}$$

which corresponds to $m = 1$, and for which we may not use this argument. Hence, to start the induction we need the assumption that

$$\int_0^T \|\Delta u(s)\|_H^2 ds < +\infty, \tag{6.101}$$

which corresponds to taking $m = 2 > 3/2$, allowing us to proceed.

The proof of Lemma 6.8

Recall that

$$\|(-\Delta)^{m/2} P(u\cdot\nabla v)\|_H = \sup_{w\in H, \|w\|_H=1} ((-\Delta)^{m/2}(u\cdot\nabla v), w). \tag{6.102}$$

Let us write

$$u\cdot\nabla v(x) = \sum_{k\in\mathbb{Z}^3} (2\pi i)\Big(\sum_{j+l=k} (l\cdot u_j) v_l\Big) e^{2\pi i k\cdot x}, \tag{6.103}$$

so that

$$((-\Delta)^{m/2}(u\cdot\nabla v), w) = \sum_{k\in\mathbb{Z}^3} (2\pi i)(4\pi^2|k|^2)^{m/2}\Big(\sum_{j+l=k} (l\cdot u_j) v_l\Big)\cdot w_{-k} \tag{6.104}$$

$$= \sum_{j+l+k=0} (2\pi i)(4\pi^2|k|^2)^{m/2}(l\cdot u_j)(v_l\cdot w_k).$$

Next, we will use the inequality

$$|j+l|^m \le (|j|+|l|)^m \le C_m(|j|^m + |l|^m), \tag{6.105}$$

which implies

$$
\begin{aligned}
|((-\Delta)^{m/2}(u\cdot\nabla v),w)| &\le C\sum_{j+l+k=0}|k|^m|l||u_j||v_l||w_k| \le C\sum_{j+l+k=0}(|j|^m+|l|^m)|l||u_j||v_l||w_k| \\
&\le C\sum_{j+l+k=0}|l|^{m+1}|u_j||v_l||w_k| + C\sum_{j+l+k=0}|j|^m|l||u_j||v_l||w_k| = A+B.
\end{aligned}
\tag{6.106}
$$

For the first term, we may estimate

$$
\begin{aligned}
A &= C\sum_{j+l+k=0}|l|^{m+1}|u_j||v_l||w_k| = \sum_{j\in\mathbb{Z}^3}|u_j|\sum_{l\in\mathbb{Z}^3}|l|^{m+1}|v_l||w_{-j-l}| \\
&\le \sum_{j\in\mathbb{Z}^3}|u_j|\Big(\sum_{l\in\mathbb{Z}^3}|l|^{2m+2}|v_l|^2\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}^3}|w_l|^2\Big)^{1/2} = \|(-\Delta)^{(m+1)/2}v\|_H\|w\|_H\sum_{j\in\mathbb{Z}^3}|u_j|.
\end{aligned}
\tag{6.107}
$$

For the last sum above we may use the estimate

$$
\sum_{j\in\mathbb{Z}^3}|u_j| \le \Big(\sum_{j\in\mathbb{Z}^3}|j|^{2m}|u_j|^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}^3}\frac{1}{|j|^{2m}}\Big)^{1/2} \le C\Big(\sum_{j\in\mathbb{Z}^3}|j|^{2m}|u_j|^2\Big)^{1/2} = C\|(-\Delta)^{m/2}u\|_H.
\tag{6.108}
$$

We used in the last step the assumption that $m > 3/2$ (in a dimension n we would have needed to assume that $m > n/2$). For the second term in (6.106) we write

$$
\begin{aligned}
B &= C\sum_{j+l+k=0}|j|^m|l||u_j||v_l||w_k| = \sum_{l\in\mathbb{Z}^3}|l||v_l|\sum_{j\in\mathbb{Z}^3}|j|^m|u_j||w_{-l-k}| \\
&\le C\|(-\Delta)^{m/2}u\|_H\|w\|_H\sum_{l\in\mathbb{Z}^3}|l||v_l|,
\end{aligned}
\tag{6.109}
$$

and

$$
\sum_{l\in\mathbb{Z}^3}|l||v_l| \le \Big(\sum_{l\in\mathbb{Z}^3}|l|^{2+2m}|v_l|^2\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}^3}\frac{1}{|l|^{2m}}\Big)^{1/2} \le C\|(-\Delta)^{(m+1)/2}v\|_H,
\tag{6.110}
$$

as $m > 3/2$. This shows that for any $w \in H$ we have

$$
|((-\Delta)^{m/2}(u\cdot\nabla u),w)| \le C\|(-\Delta)^{m/2}u\|_H\|(-\Delta)^{(m+1)/2}v\|_H\|w\|_H,
\tag{6.111}
$$

and thus finishes the proof of Lemma 6.8. □

Local in time existence in higher Sobolev spaces

The arguments of the previous section imply also that the Navier-Stokes equations are locally well-posed in the higher Sobolev spaces $H^m(\mathbb{T}^3)$. We state it for simplicity for the case $f = 0$.

Theorem 6.9 *Let $u_0 \in H^m$, with $m \ge 2$, and $f = 0$. There exists a constant $C_m > 0$ which depends on ν, $m \ge 1$ and $\|u_0\|_{H^m}$, so that if*

$$
T_m \le C_m,
\tag{6.112}
$$

then the Navier-Stokes equations

$$
\begin{aligned}
&u_t + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad t > 0, \quad x \in \mathbb{T}^3, \\
&\nabla \cdot u = 0, \\
&u(0,x) = u_0(x),
\end{aligned} \tag{6.113}
$$

have a strong solution on the time interval $[0, T_m]$ *that satisfies*

$$
\|u(t)\|_{H^m}^2 \le C_0^{-1}, \tag{6.114}
$$

for all $0 \le t \le T_m$.

The proof is familiar: we start with (6.115) with $f = 0$:

$$
\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \frac{\nu}{2}\|(-\Delta)^{(m+1)/2}u\|_H^2 \le \frac{C}{\nu}\|(-\Delta)^{m/2}u\|_H^4. \tag{6.115}
$$

Looking at this as the differential inequality for $y(t) = \|(-\Delta)^{m/2}u\|_H^2$, we deduce that

$$
\dot{y} \le \frac{C}{\nu} y^2(t). \tag{6.116}
$$

As a consequence, $y(t)$ remains finite for a time that depends only on $y(0)$. □

Infinite time blow-up implies a finite time blow-up

The problem of blow-up of solutions of a nonlinear partial differential equation usually consists in two separate problems: (1) can solutions blow-up in a finite time, and (2) can they blow-up in an infinite time, in the sense that the norm of the solutions tends to infinity as $t \to +\infty$? The second notion is usually much weaker. For example, solutions of the heat equation with a linear growth term

$$
u_t = \Delta u + u, \quad t > 0, x \in \mathbb{R}^n, \tag{6.117}
$$

have the long time behavior

$$
u(t,x) \sim \frac{e^t \|u_0\|_{L^1}}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}, \tag{6.118}
$$

and thus "blow-up in an infinite time" – all its L^p-norms, $p \ge 1$ tend to infinity as $t \to +\infty$. However, one does not normally think of these solutions as really "blowing-up" – they just grow in time.

The situation is different for the Navier-Stokes equations: an infinite time blow-up implies a finite-time blow-up. More precisely, let us assume that there exists a strong solution $u(t,x)$ of the Navier-Stokes equations

$$
\begin{aligned}
&u_t + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad 0 < t \le T, \quad x \in \mathbb{T}^3, \\
&\nabla \cdot u = 0, \\
&u(0,x) = u_0(x),
\end{aligned} \tag{6.119}
$$

such that $u_0 \in H$, and

$$\lim_{t\to+\infty} \|u(t)\|_V = +\infty. \tag{6.120}$$

Assuming that such u exists, and given any $T > 0$, we will now construct an initial condition $v_0 \in V$ so that the solution of (6.119) with $v(0,x) = v_0(x)$, blows up before the time $T > 0$. That is, there will be a time $T_1 \in (0,T]$ such that

$$\lim_{t\to T} \|v(t)\|_V = +\infty. \tag{6.121}$$

The idea is to combine the blow-up assumption that there exists a sequence of times $t_j \to +\infty$ such that

$$\|u(t_j)\|_V \geq 2^j, \tag{6.122}$$

with the main result of Proposition 6.2: solutions of the Navier-Stokes are often not large. Given a sequence t_j as in (6.122), we may use the aforementioned Proposition to find a time $s_j \in [t_j - T, t_j]$ so that

$$\|u(s_j)\|_V \leq C\Big(1 + \frac{1}{T}\Big) = C'. \tag{6.123}$$

The constant C depends only on $\|u_0\|_H$, and $\nu > 0$. Thus, if we take $u(s_j)$ as the initial condition for the Navier-Stokes equations, then the corresponding solution of the Cauchy problem will have reached the V-norm that is larger than 2^j by the time T. As $\|u(s_j)\|_V$ is uniformly bounded in j, we may choose a subsequence $j_k \to +\infty$ so that $u(s_{j_k})$ converges weakly in V and strongly in H to a function $v_0 \in V$. Consider now the Cauchy problem with the initial condition v_0:

$$\begin{aligned}
&v_t + v\cdot\nabla v + \nabla p = \nu\Delta v, \quad 0 < t \leq T, \quad x \in \mathbb{T}^3, \\
&\nabla\cdot v = 0, \\
&v(0,x) = v_0(x).
\end{aligned} \tag{6.124}$$

This problem has a strong solution on some time interval $[0,T_0]$, which depends only on $\|v_0\|_V$ and ν. We will now show that (6.124) may not have a strong solution on the time interval $[0,T]$. To this end, assume that such solution exists on $[0,T]$, denote

$$r = \sup_{0\leq t\leq T} \|v(t)\|_V, \tag{6.125}$$

and consider the functions $v_k(t) = u(t + s_{j_k})$, which are solutions of

$$\begin{aligned}
&\frac{\partial v_k}{\partial t} + v_k\cdot\nabla v_k + \nabla p_k = \nu\Delta v_k, \quad 0 < t \leq T, \quad x \in \mathbb{T}^3, \\
&\nabla\cdot v_k = 0, \\
&v_k(0,x) = v_0(x).
\end{aligned} \tag{6.126}$$

Writing $w_j = v_j - v$, and expanding

$$v_j\cdot\nabla v_j - v\cdot\nabla v = (v + w_j)\cdot\nabla(v + w_j) - v\cdot\nabla v = w_j\cdot\nabla v + v\cdot\nabla w_j + w_j\cdot\nabla w_j, \tag{6.127}$$

we see that w_j satisfies (as in the proof of the uniqueness of the solutions of the Navier-Stokes equations):

$$\begin{aligned}
&\frac{\partial w_j}{\partial t} + w_j \cdot \nabla v + v \cdot \nabla w_j + w_j \cdot \nabla w_j + \nabla p' = \nu \Delta w_j, \quad 0 < t \le T, \quad x \in \mathbb{T}^3, \quad (6.128)\\
&\nabla \cdot w_j = 0,\\
&w_j(0,x) = v_j(x) - v_0(x),
\end{aligned}$$

with $p' = p_j - p$. Multiplying by w_j and integrating leads to

$$\frac{1}{2}\frac{d}{dt}\|w_j\|_H^2 + \nu\|w_j\|_V^2 = -(w_j \cdot \nabla v, w_j). \quad (6.129)$$

We estimate the right side as

$$\begin{aligned}
&|(w_j \cdot \nabla v, w_j)| \le \|w_j\|_{L^3}\|\nabla v\|_{L^2}\|w_j\|_{L^6} \le C\|w_j\|_{H^{1/2}}\|v\|_V\|w_j\|_{H^1} \quad (6.130)\\
&\le C\|w_j\|_H^{1/2}\|w_j\|_V^{1/2}\|v\|_V\|w_j\|_V = C\|v\|_V\|w_j\|_H^{1/2}\|w_j\|_V^{3/2} \le \frac{\nu}{2}\|w_j\|_V^2 + \frac{C}{\nu^3}\|v\|_V^4\|w_j\|_H^2.
\end{aligned}$$

We used Young's inequality in the last step, with $p = 4/3$, $q = 4$. Using this in (6.129) gives

$$\frac{1}{2}\frac{d}{dt}\|w_j\|_H^2 + \frac{\nu}{2}\|w_j\|_V^2 \le \frac{C}{\nu^3}\|v\|_V^4\|w_j\|_H^2. \quad (6.131)$$

As v is a strong solution, there exists $C > 0$, which depends on ν and r in (6.125), so that

$$\|w_j(t)\|_H \le \|w_j(0)\|_H e^{Ct}, \quad (6.132)$$

meaning that $w_j(t) \to 0$ strongly in H, for all $0 \le t \le T$. Furthermore, as

$$\frac{\nu}{2}\int_0^T \|w_j(t)\|_V^2 dt \le \|w_j(0)\|_H^2 + C\int_0^T \|w_j(t)\|_H^2 dt, \quad (6.133)$$

and since $\|w_j(t)\|_H \to 0$, pointwise in t, while $\|w_j(t)\|_H \le C$, we conclude that

$$\int_0^T \|w_j(t)\|_V^2 dt \to 0 \text{ as } j \to \infty. \quad (6.134)$$

In particular, possibly after extracting another subsequence, we know that $\|w_j(t)\|_V \to 0$ for a.e. $t \in [0,T]$. Take any $t \in [0,T]$ such that $\|w_j\|_V \le 1$, then

$$\|v_j(t)\|_V \le \|w_j(t)\|_V + \|v(t)\|_V \le 1 + r. \quad (6.135)$$

The local in time existence theorem implies that there exists a time T_1, which depends only on ν, so that

$$\|v_j(s)\|_V \le 10(1+r), \quad (6.136)$$

for all $s \in [t, t+T_1]$. The density of times t so that (6.135) holds, means that (6.136) holds for all $0 \le t \le T$. This, however, contradicts the assumption that

$$\|v(s_j - t_j)\|_V = \|u(s_j)\|_V \ge \frac{1}{2^j}.$$

Thus, $v(s,x)$ can not be a strong solution on the time interval $[0,T]$.

The Beale-Kato-Majda regularity criterion

We now describe a sufficient condition for the solution to remain smooth. This time, we will work in the whole space $\mathbb{R}^3$ but the existence and regularity results we have proved for the three-dimensional torus apply essentially verbatim to the whole space as well. As we have seen in Theorem 6.9, if the H^m-norms of a smooth solution $u(t,x)$ remain finite on a time interval $[0,T]$, then the solution may be extended past the time T. In other words, a time T is the maximal time of existence of a smooth solution $u(t,x)$ if and only if

$$\lim_{t\uparrow T}\|u(t)\|_{H^m}=+\infty. \tag{6.137}$$

The Beale-Kato-Majda criterion reformulates this condition in terms of the vorticity.

Theorem 6.10 *Let $u_0\in C_c^\infty(\mathbb{R}^3)$, so that there exists a classical solution v to the Navier-Stokes equations. If for any $T>0$ we have*

$$\int_0^T\|\omega(t)\|_{L^\infty}dt<+\infty, \tag{6.138}$$

then the smooth solution u exists globally in time. If the maximal existence time of the smooth solution is $T<+\infty$, then necessarily we have

$$\lim_{t\uparrow T}\int_0^T\|\omega(t)\|_{L^\infty}dt=+\infty. \tag{6.139}$$

The starting point in the proof is the estimate for the evolution of the H^m-norms. Let us recall the identity (6.87) with $f=0$:

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2+\nu\|(-\Delta)^{(m+1)/2}u\|_H^2=((-\Delta)^{m/2}(u\cdot\nabla u),(-\Delta)^{m/2}u). \tag{6.140}$$

Note that

$$((u\cdot\nabla(-\Delta)^{m/2}u),(-\Delta)^{m/2}u)=0,$$

hence the right side in (6.140) can be estimated by

$$C_m\|D^mu\|_2\sum_{i,j=1}^{3}\sum_{k=1}^{m}\|D^ku_j\|_{L^p}\|D^{(m+1-k)}u_i\|_{L^q}, \tag{6.141}$$

with $1/p+1/q=1/2$, and with the notation $D=(-\Delta)^{1/2}$. We recall a Gagliardo-Nirenberg inequality for $\mathbb{R}^d$:

$$\|D^jf\|_{L^p}\leq C\|D^mf\|_2^a\|f\|_{L^\infty}^{1-a}, \tag{6.142}$$

with $0\leq j<m$, and

$$\frac{1}{p}=\frac{j}{d}+a(\frac{1}{2}-\frac{m}{d}),$$

and $a=j/m$. We will use it for $f=Du$ and $1\leq k<m$:

$$\|D^{k-1}Du\|_{L^p}\leq c\|D^{m-1}Du\|_{L^2}^a\|Du\|_\infty^{1-a}, \tag{6.143}$$

that is, the terms in (6.141) with $1 \le k < m$ can be estimated as

$$\|D^k u\|_{L^p} \le c\|D^m u\|_{L^2}^a \|Du\|_\infty^{1-a}, \tag{6.144}$$

with

$$a = \frac{k-1}{m-1},$$

so that

$$\frac{1}{p} = \frac{k-1}{d} + \frac{k-1}{m-1}(\frac{1}{2} - \frac{m-1}{d}) = \frac{k-1}{2(m-1)} = \frac{a}{2}.$$

The paired terms $\|D^{m+1-k}u\|_q$ can be estimated similarly:

$$\|D^{m+1-k}u\|_{L^q} = \|D^{m-k}Du\|_{L^q} \le c\|D^m u\|_{L^2}^b \|Du\|_\infty^{1-b}, \tag{6.145}$$

with

$$b = \frac{m-k}{m-1},$$

and

$$\frac{1}{q} = \frac{m-k}{d} + \frac{m-k}{m-1}(\frac{1}{2} - \frac{m-1}{d}) = \frac{m-k}{2(m-1)} = \frac{b}{2}.$$

Luckily, we have $a + b = 1$, and

$$\frac{1}{p} + \frac{1}{q} = \frac{a+b}{2} = \frac{1}{2},$$

so that these p and q can be taken in (6.141). It follows that

$$\|D^k u_j\|_{L^p} \|D^{(m+1-k)} u_i\|_{L^q} \le C\|D^m u\|_{L^2} \|Du\|_{L^\infty}.$$

When $k = m$ or $k = 1$, we simply use $p = 1/2$ and $q = \infty$, getting the estimate

$$\|D^m u\|_{L^2} \|Du\|_{L^\infty}$$

for those terms. Altogether we conclude that

$$\frac{1}{2}\frac{d}{dt}\|D^m u\|_H^2 \le C\|D^m u\|_H^2 \|\nabla u\|_{L^\infty}. \tag{6.146}$$

Summing over m, we conclude that for any $s \in \mathbb{N}$ we have

$$\frac{d}{dt}\|u\|_{H^s} \le C_s \|\nabla u\|_{L^\infty} \|u\|_{H^s}. \tag{6.147}$$

Therefore, if $u_0 \in C_c^\infty(\mathbb{R}^3)$, then for any of the H^s-norms to become infinite by a time T it is necessary that

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt = +\infty, \tag{6.148}$$

and, in general, we have

$$\|u\|_{H^s} \le \|u_0\|_{H^s} \exp\Big\{C_s \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\Big\}. \tag{6.149}$$

In a similar vein, multiplying the vorticity equation

$$\omega_t + u \cdot \nabla\omega = \nu\Delta\omega + \omega \cdot \nabla u \tag{6.150}$$

by ω and integrating, we see that

$$\frac{d}{dt}\|\omega(t)\|_{L^2} \le \|\nabla u\|_{L^\infty}\|\omega\|_{L^2}, \tag{6.151}$$

so that

$$\|\omega(t)\|_{L^2} \le \|\omega_0\|_{L^2} \exp\Big\{\int_0^t \|\nabla u(s)\|_{L^\infty} ds\Big\}. \tag{6.152}$$

The conclusion of Theorem 6.10 would follow from (6.148) if we would know that

$$\text{“}\|\nabla u\|_{L^\infty} \le C\|\omega\|_{L^\infty}\text{”}. \tag{6.153}$$

One may expect this to be true based on its validity for L^2-norms: recall (3.26)

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx = \int_{\mathbb{R}^3} |\omega|^2 dx, \tag{6.154}$$

because

$$|\omega|^2 = \varepsilon_{ijk}\varepsilon_{imn}(\partial_j u_k)(\partial_m u_n) = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})(\partial_j u_k)(\partial_m u_n) = |\nabla u|^2 - (\partial_j u_k)(\partial_k u_j), \tag{6.155}$$

and

$$\int_{\mathbb{R}^n} (\partial_j u_k)(\partial_k u_j) dx = -\int_{\mathbb{R}^n} u_k(\partial_k\partial_j u_j) dx = 0. \tag{6.156}$$

Identity (6.153), however, is not quite true for the L^∞-norms – the relation between the gradient of the velocity and the vorticity is in terms of a singular integral operator which maps every $L^p \to L^p$ for $1 < p < +\infty$ but does not map L^∞ to L^∞. However, it is “almost true” as shown by the following lemma.

Lemma 6.11 *Let $u(x)$ be a smooth divergence free velocity field in $L^2 \cap L^\infty$, and let $\omega = \nabla \times v$. There exists a constant $C > 0$ so that*

$$\|\nabla u\|_{L^\infty} \le C(1 + \log^+ \|u\|_{H^3} + \log^+ \|\omega\|_{L^2})(1 + \|\omega\|_{L^\infty}). \tag{6.157}$$

Here, for $z > 0$, we set $\log^+ z = \log z$ if $\log z > 0$, and $\log^+ z = 0$ otherwise. The L^2-norm of $\omega(t)$ that appears in (6.157) can be estimated from (6.152) as

$$\log^+ \|\omega(t)\|_{L^2} \le \log^+ \|\omega_0\|_{L^2} + \int_0^t \|\nabla u(s)\|_{L^\infty} ds. \tag{6.158}$$

Similarly, the H^3-norm of $u(t)$ can be bounded as in (6.149):

$$\log^+ \|u(t)\|_{H^3} \le \log^+ \|\omega_0\|_{L^2} + C\int_0^t \|\nabla u(s)\|_{L^\infty} ds. \tag{6.159}$$

Assuming the result of Lemma 6.11, we deduce that $\|\nabla u\|_\infty$ satisfies the inequality

$$\|\nabla u(t)\|_{L^\infty} \leq C_0\Big(1+\int_0^t \|\nabla u(s)\|_{L^\infty} ds\Big)(1+\|\omega(t)\|_{L^\infty}, \tag{6.160}$$

with a constant C_0 that depends on the initial data u_0. Setting

$$G(t) = \int_0^t \|\nabla u(s)\|_{L^\infty} ds, \quad \beta(t) = 1+\|\omega(t)\|_{L^\infty},$$

we have from (6.160):

$$\frac{dG}{dt} \leq C_0(1+G(t))\beta(t),$$

so that

$$\frac{d}{dt}\Big(G(t)\exp\Big\{-C_0\int_0^t \beta(s)ds)\Big\}\Big) \leq C_0\beta(t)\exp\Big\{-C_0\int_0^t \beta(s)ds\Big\}.$$

Integrating in time gives

$$G(t)\exp\Big\{-C_0\int_0^t \beta(s)ds)\Big\} \leq 1-\exp\Big\{-C_0\int_0^t \beta(s)ds\Big\}, \tag{6.161}$$

so that

$$G(t) \leq \exp\Big\{C_0\int_0^t \beta(s)ds)\Big\}.$$

In other words, we have

$$\int_0^t \|\nabla u(s)\|_{L^\infty} ds \leq \exp\Big\{C_0 t + C_0\int_0^t \|\omega(s)\|_{L^\infty} ds)\Big\}. \tag{6.162}$$

As a consequence, as long as

$$\int_0^t \|\omega(s)\|_{L^\infty} ds < +\infty, \tag{6.163}$$

all H^m-norms of the velocity remain finite, hence $u(t) \in C^\infty(\mathbb{R}^3)$. Therefore, the proof of Theorem 6.10 boils down to Lemma 6.11.

The proof of the estimate on $\|\nabla u\|_{L^\infty}$

We now prove Lemma 6.11 using the ideas from the theory of singular integral operators. The velocity field is related to vorticity by the Biot-Savart law:

$$u(x) = -\int_{\mathbb{R}^3} K(x-y)\omega(y)dy = \int_{\mathbb{R}^3} K(y)\omega(x+y)dy, \tag{6.164}$$

with

$$K(x)h = \frac{1}{4\pi|x|^3} x \times h, \tag{6.165}$$

for any $h \in \mathbb{R}^3$. As the singularity in $\nabla K(x)$ is of the order $1/|x|^3$ which is not integrable in three dimensions, we have to be careful about computing the gradient of u. Let us write

$$u(x+z) - u(x) = \int_{\mathbb{R}^3} K(y)[\omega(x+z+y) - \omega(x+y)]dy. \tag{6.166}$$

As $K \in L^1_{loc}(\mathbb{R}^3)$, if, say, $\omega \in C_0^\infty(\mathbb{R}^3)$, then, passing to the limit $z \to 0$, we get

$$\frac{\partial u_k(x)}{\partial x_j} = \int_{\mathbb{R}^3} K_{km}(y)\partial_j \omega_m(x+y)dy. \tag{6.167}$$

Because of the singularity in K we can not immediately integrate by parts. Let us write this integral as

$$\begin{aligned}
&\frac{\partial u_k(x)}{\partial x_j} = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} K_{km}(y)\partial_j\omega_m(x+y)dy = \\
&= -\lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy - \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}[\partial_j K_{km}(y)]\omega_m(x+y)dy = A_{kj} + B_{kj}.
\end{aligned}$$

The first integral can be re-written as

$$\begin{aligned}
A_{kj} &= -\lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy = -\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{|y|=\varepsilon}\frac{1}{|y|^3}[y\times\omega(x+y)]_k\frac{y_j}{|y|}dy \\
&= -\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{|z|=1}\frac{1}{\varepsilon^3|z|^3}[\varepsilon z\times\omega(x+\varepsilon z)]_k\frac{z_j}{|z|}\varepsilon^2 dz = -\frac{1}{4\pi}\int_{|z|=1}[z\times\omega(x)]_k z_j dz \\
&= -\frac{1}{4\pi}\epsilon_{kmn}\int_{|z|=1} z_m\omega_n(x)z_j dz = \frac{\epsilon_{kmn}}{3}\omega_n(x)\delta_{mj} = -\frac{1}{3}\epsilon_{kjn}\omega_n(x).
\end{aligned} \tag{6.168}$$

Thus, we have

$$|A_{kj}| \leq \frac{1}{3}\|\omega\|_{L^\infty},$$

and the main focus is on the second term. We have

$$K_{km}(y) = \frac{\epsilon_{krm}}{4\pi|y|^3}y_r,$$

so that

$$\partial_j K_{km}(y) = -\frac{3\epsilon_{krm}}{4\pi|y|^5}y_j y_r + \frac{\epsilon_{kjm}}{4\pi|y|^3}.$$

We conclude that for any $h \in \mathbb{R}^3$ we have

$$\begin{aligned}
(Bh)_k &= -\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[-\frac{3\epsilon_{krm}}{4\pi|y|^5}y_j y_r + \frac{\epsilon_{kjm}}{4\pi|y|^3}\Big]\omega_m(x+y)h_j dy \\
&= \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x+y)\times h]_k\Big)dy.
\end{aligned}$$

We shall split B further as follows: take a smooth cut-off function $\rho(r)$ so that $\rho(r) = 0$ for $r > 2R$, and $\rho(r) = 1$ for $r < R$, with R to be chosen later, and write

$$(Bh)_k = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x+y)\times h]_k\Big)\rho(|y|)dy$$
$$+\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x-y)\times h]_k\Big)(1-\rho(|y|)dy = C_k + D_k.$$

The Cauchy-Schwartz inequality implies that

$$|D_k| \leq C|h|\|\omega\|_{L^2}\Big(\int_{R_0}^{\infty}\frac{1}{r^6}r^2dr\Big)^{1/2} \leq \frac{C}{R^{3/2}}\|\omega\|_{L^2}|h|. \tag{6.169}$$

The key estimate is for C_k: we will show that for any $\delta > 0$ and any Hölder regularity exponent $\gamma \in (0,1)$ we have

$$|C_k| \leq C\Big\{\delta^\gamma\|\omega\|_{C^\gamma} + \|\omega\|_{L^\infty}\max(1,\log\frac{R}{\delta}\Big)\Big\}|h|. \tag{6.170}$$

Here, $\|\omega\|_{C^\gamma}$ is the Hölder norm. The Sobolev inequality in dimension n

$$\|f\|_{C^\gamma(\mathbb{R}^n)} \leq C\|f\|_{H^{s+\gamma}(\mathbb{R}^n)}, \quad s > \frac{n}{2}$$

implies that in three dimensions we have, for all $0 < \gamma < 1/2$:

$$\|\omega\|_{C^\gamma} \leq C\|\omega\|_{H^2},$$

so that

$$|C_k| \leq C\Big\{\delta^\gamma\|\omega\|_{H^2}+\|\omega\|_{L^\infty}\max(1,\log\frac{R}{\delta}\Big)\Big\} \leq C\Big\{\delta^\gamma\|u\|_{H^3}+\|\omega\|_{L^\infty}\max(1,\log\frac{R}{\delta}\Big)\Big\}. \tag{6.171}$$

Altogether, we have

$$\|\nabla u\|_{L^\infty} \leq C\Big(\|\omega\|_{L^\infty} + \frac{C}{R^{3/2}}\|\omega\|_{L^2} + \Big\{\delta^\gamma\|u\|_{H^3} + \|\omega\|_{L^\infty}\max(1,\log\frac{R}{\delta}\Big)\Big\}\Big). \tag{6.172}$$

Thus, we set $R = \|\omega\|_{L^2}^{2/3}$. As far δ is concerned, if $\|u\|_{H^3} \leq 1$, we can take $\delta = 1$, while if $\|u\|_{H^3} \geq 1$, we can take $\delta = \|u\|_{H^3}^{-\gamma}$. In both cases, we have

$$\|\nabla v\|_{L^\infty} \leq C(1 + \log^+\|v\|_{H^3} + \log^+\|\omega\|_{L^2})(1 + \|\omega\|_{L^\infty}), \tag{6.173}$$

which is the claim of Lemma 6.11. It remains, therefore, only to prove the estimate (6.170).

A nearly $L^\infty \to L^\infty$ estimate for singular integral operators

We now prove estimate (6.170) for C_k, which we write as

$$C_k = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(\frac{3(y\cdot h)[y\times\omega(x+y)]_k}{4\pi|y|^5} + \frac{1}{4\pi|y|^3}[\omega(x+y)\times h]_k\Big)\rho(|y|)dy$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(3(\hat y\cdot h)[\hat y\times\omega(x+y)]_k+[\omega(x+y)\times h]_k\Big)\rho(|y|)\frac{dy}{|y|^3}\\
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(3\hat y_m h_m\varepsilon_{kjr}\hat y_j\omega_r(x+y)+\varepsilon_{krm}\omega_r(x+y)h_m\Big)\rho(|y|)\frac{dy}{|y|^3}\\
&= \frac{h_m}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big(3\hat y_m\varepsilon_{kjr}\hat y_j+\varepsilon_{krm}\Big)\omega_r(x+y)\rho(|y|)\frac{dy}{|y|^3}\\
&= \frac{h_m}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}P_{mkr}(y)\omega_r(x+y)\rho(|y|)dy. \qquad (6.174)
\end{aligned}
$$

We have denoted here

$$
P_{mkr}=\frac{1}{|y|^3}(3\hat y_m\varepsilon_{kjr}\hat y_j+\varepsilon_{krm}). \qquad (6.175)
$$

The kernel $Q(y)=P_{mkr}(y)$ (we fix for the moment the indices m, k and r) is homogenous of degree (-3):

$$
Q(\lambda y)=\frac{1}{\lambda^3}Q(y),\quad \text{for all } \lambda>0 \text{ and } y\in\mathbb{R}^3,\ y\neq 0. \qquad (6.176)
$$

Thus, $Q(y)$ is "barely not in L^1": if it were slightly less singular it would have been in L^1. In addition, the average of $Q(y)$ over the unit sphere (and thus over any sphere centered at $y=0$) vanishes:

$$
\int_{|y|=1}Q(y)dy=\int_{|y|=1}(3\hat y_m\varepsilon_{kjr}\hat y_j+\varepsilon_{krm})dy=4\pi[\varepsilon_{kjr}\delta_{mj}+\varepsilon_{krm}]=4\pi[\varepsilon_{kmr}=\varepsilon_{krm}]=0. \qquad (6.177)
$$

Consider now the term (again, with an index r fixed)

$$
\mathcal{Q}\omega(x)=\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}Q(y)\omega_r(x+y)\rho(|y|)dy. \qquad (6.178)
$$

We split the integration in the definition of $\mathcal{Q}\omega$ as follows:

$$
\mathcal{Q}\omega(x)=\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|y|\leq\delta}Q(y)\omega_r(x+y)\rho(|y|)dy+\int_{|y|\geq\delta}Q(y)\omega_r(x+y)\rho(|y|)dy=A+B. \qquad (6.179)
$$

The second term above is (recall that $\rho(|y|)=0$ for $|y|>2R$):

$$
B=\int_{\delta\leq|y|\leq 2R}Q(y)\omega_r(x+y)\rho(|y|)dy, \qquad (6.180)
$$

which can be estimated as

$$
|B|\leq C\|\omega\|_{L^\infty}\int_\delta^{2R}\frac{r^{n-1}}{r^n}dr\leq C\|\omega\|_{L^\infty}\log\frac{2R}{\delta}. \qquad (6.181)
$$

The first term in (6.179) is estimated using the Hölder continuity of ω: the mean-zero property (6.177) means that we can write

$$
A=\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|y|\leq\delta}Q(y)\omega_r(x+y)\rho(|y|)dy=\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|y|\leq\delta}Q(y)[\omega_r(x+y)-\omega_r(x)]\rho(|y|)dy. \qquad (6.182)
$$

The Hölder continuity of ω implies that the integrand in the last expression above has an upper bound

$$|Q(y)[\omega_r(x-y)-\omega_r(x)]\rho(|y|)| \leq \frac{C}{|y|^n}|y|^\gamma \|\omega\|_{C^\gamma} = \frac{C}{|y|^{n-\gamma}}\|\omega\|_{C^\gamma}, \tag{6.183}$$

which is integrable in y at $y=0$ for $\gamma > 0$. Therefore, we have

$$A = \int_{0\leq|y|\leq\delta} Q(y)[\omega_r(x-y)-\omega(x)]\rho(|y|)dy, \tag{6.184}$$

and

$$|A| \leq C\|\omega\|_{C^\gamma}\int_0^\delta \frac{r^{n-1}}{r^{n-\gamma}}dy \leq C\|\omega\|_{C^\gamma}\delta^\gamma. \tag{6.185}$$

Putting the bounds for A and B together gives (6.170).

7 Vortex lines and geometric conditions for blow-up

The vorticity growth equation

Here, we investigate how vorticity alignment in the regions of high vorticity can prevent blow-up in the Navier-Stokes and Euler equations. First, we obtain an equation for the magnitude of vorticity $|\omega|$ that shows that it is plausible that the vorticity alignment in the regions of high vorticity may prevent the growth of vorticity. Recall that the vorticity of the solutions of the Navier-Stokes equations satisfies the evolution equation

$$\omega_t + u\cdot\nabla\omega - \nu\Delta\omega = \omega\cdot\nabla u \tag{7.1}$$

Multiplying by 2ω, we obtain

$$\partial_t(|\omega|^2) + u\cdot\nabla(|\omega|^2) - \nu\Delta|\omega|^2 + 2\nu|\nabla\omega|^2 = 2(\omega\cdot\nabla u)\cdot\omega. \tag{7.2}$$

The right side can be written as

$$2(\omega\cdot\nabla u)\cdot\omega = 2\omega_j(\partial_j u_k)\omega_k = 2(S\omega\cdot\omega) = 2\alpha(x)|\omega|^2,$$

with

$$\alpha(x) = (S(x)\xi(x)\cdot\xi(x)),\quad \xi(x) = \frac{\omega(x)}{|\omega(x)|}, \tag{7.3}$$

and

$$S(x) = \frac{1}{2}(\nabla u + (\nabla u)^t). \tag{7.4}$$

When $\nu = 0$ we get a particularly simple form of the vortex stretching balance for the Euler equations:

$$\partial_t|\omega| + u\cdot\nabla|\omega| = \alpha(t,x)|\omega|. \tag{7.5}$$

Thus, the vorticity growth may only appear from $\alpha(x)$ large. Our next task is to express $\alpha(x)$ in terms of the vorticity alignment. We start with the Biot-Savart law

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y)dy. \tag{7.6}$$

Let us recall that

$$\frac{\partial u_k(x)}{\partial x_j} = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} K_{km}(y)\partial_j\omega_m(x+y)dy \tag{7.7}$$
$$= -\lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy - \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}[\partial_j K_{km}(y)]\omega_m(x+y)dy = A_{kj}+B_{kj}.$$

The term A_{kj} can be simplified as

$$A_{kj} = -\lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} K_{km}(y)\omega_m(x+y)\frac{y_j}{|y|}dy = -\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{|y|=\varepsilon}\frac{1}{|y|^3}[y\times\omega(x+y)]_k\frac{y_j}{|y|}dy$$
$$= -\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{|z|=1}\frac{1}{\varepsilon^3|z|^3}[\varepsilon z\times\omega(x+\varepsilon z)]_k\frac{z_j}{|z|}\varepsilon^2 dz = -\frac{1}{4\pi}\int_{|z|=1}[z\times\omega(x)]_k z_j dz$$
$$= -\frac{1}{4\pi}\epsilon_{kmn}\int_{|z|=1} z_m\omega_n(x)z_j dz = -\frac{\epsilon_{kmn}}{3}\omega_n(x)\delta_{mj} = -\frac{1}{3}\epsilon_{kjn}\omega_n(x), \tag{7.8}$$

and B can be written as

$$B_{kj} = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[\frac{3\epsilon_{krm}}{4\pi|y|^5}y_j y_r - \frac{\epsilon_{kjm}}{4\pi|y|^3}\Big]\omega_m(x+y)dy.$$

Multiplying (7.7) by ϵ_{ijk} and summing over j,k, leads now to an integral equation for the vorticity:

$$\omega_i(x) = \epsilon_{ijk}\partial_j u_k = \epsilon_{ijk}A_{kj} + \epsilon_{ijk}B_{kj} = -\frac{1}{3}\epsilon_{ijk}\epsilon_{kjn}\omega_n \tag{7.9}$$
$$+\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\epsilon_{ijk}\Big[\frac{3\epsilon_{krm}}{4\pi|y|^5}y_j y_r + \frac{\epsilon_{kmj}}{4\pi|y|^3}\Big]\omega_m(x+y)dy.$$

The first term above can be re-written as

$$-\epsilon_{ijk}\epsilon_{kjn}\omega_n = \epsilon_{ijk}\epsilon_{njk}\omega_n = 2\omega_i.$$

In the second term, we use the identities

$$\epsilon_{ijk}\epsilon_{krm}y_j y_r\omega_m = \epsilon_{kij}\epsilon_{krm}y_j y_r\omega_m = [\delta_{ir}\delta_{jm}-\delta_{im}\delta_{jr}]y_j y_r\omega_m = y_i(y\cdot\omega) - |y|^2\omega_i,$$

and

$$\epsilon_{ijk}\epsilon_{kmj}\omega_m = \epsilon_{kij}\epsilon_{kmj}\omega_m = 2\omega_i$$

Using these transformations in (7.9), gives

$$\frac{1}{3}\omega_i(x) = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[\frac{3}{4\pi|y|^5}[y_i(y\cdot\omega(x+y)) - |y|^2\omega_i(x+y)] + \frac{2\omega_i(x+y)}{4\pi|y|^3}\Big], dy.$$

so that

$$\omega(x) = \frac{3}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\sigma(\hat{y})\omega(x+y)\frac{dy}{|y|^3} \tag{7.10}$$

with the matrix $\sigma(\hat{y})$, $\hat{y} = y/|y|$, defined as

$$\sigma(\hat{y}) = 3(\hat{y}\otimes\hat{y}) - I. \tag{7.11}$$

Similarly, we may compute the symmetric part of ∇u:

$$S(x) = \frac{1}{2}(\nabla u + (\nabla u)^t).$$

We have

$$S_{kj} = \frac{1}{2}(A_{kj} + A_{jk}) + \frac{1}{2}(B_{kj} + B_{jk}).$$

It is easy to see that the matrix A_{kj} is anti-symmetric, thus

$$A_{kj} + A_{jk} = 0.$$

For the symmetric part of the matrix B we compute

$$B_{kj} + B_{jk} = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[\frac{3\epsilon_{krm}}{4\pi|y|^5}y_jy_r + \frac{3\epsilon_{jrm}}{4\pi|y|^5}y_ky_r - \frac{\epsilon_{kjm}}{4\pi|y|^3} - \frac{\epsilon_{jkm}}{4\pi|y|^3}\Big]\omega_m(x+y)dy$$
$$= \frac{3}{4\pi}\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\Big[\epsilon_{krm}\hat{y}_j\hat{y}_r + \epsilon_{jrm}\hat{y}_k\hat{y}_r\Big]\omega_m(x+y)\frac{dy}{|y|^3}.$$

We conclude that

$$S(x) = \frac{3}{4\pi}\text{P.V.}\int M(\hat{y},\omega(x+y))\frac{dy}{|y|^3}, \tag{7.12}$$

with the matrix-valued function

$$M(\hat{y},\omega) = \frac{1}{2}[(\hat{y}\times\omega)\otimes\hat{y} + \hat{y}\otimes(\hat{y}\times\omega)]. \tag{7.13}$$

Going back to (7.3), we get the following expression for the vorticity stretching coefficient $\alpha(x)$:

$$\alpha(x) = (S(x)\xi(x)\cdot\xi(x)) = \frac{3}{4\pi}\text{P.V.}\int(M(\hat{y},\omega(x+y))\xi(x)\cdot\xi(x))\frac{dy}{|y|^3}. \tag{7.14}$$

The integrand can be re-written as

$$M(\hat{y},\omega(x+y))\xi(x)\cdot\xi(x)) = \frac{1}{2}[(\hat{y}\times\omega(x+y))\otimes\hat{y} + \hat{y}\otimes(\hat{y}\times\omega(x+y))]\xi(x)\cdot\xi(x)$$
$$= (\hat{y}\times\omega(x+y)\cdot\xi(x))(\hat{y}\cdot\xi(x)) = D(\hat{y},\xi(x+y),\xi(x))|\omega(x+y)|,$$

thus

$$\alpha(x) = (S(x)\xi(x)\cdot\xi(x)) = \frac{3}{4\pi}\text{P.V.}\int D(\hat{y},\xi(x+y),\xi(x))|\omega(x+y)|\frac{dy}{|y|^3}. \tag{7.15}$$

Here, we have defined, for three unite vectors e_1, e_2 and e_3:

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3)\mathrm{Det}(e_1, e_2, e_3).$$

Geometrically, it follows that the regions where $\xi(x + y)$ is aligned with $\xi(x)$ contribute less to $\alpha(x)$. This applies also to the antiparallel vortex pairing, which is a physically observed phenomenon. That is, we expect that if the vorticity direction field is aligned or anti-aligned in the regions of high vorticity, the blow-up might be prevented by the vorticity alignment, though this requires a careful analysis which we will undertake next.

A priori bounds on the strain matrix

Let us first obtain some bounds on the strain matrix in terms of ω that we will need later. We have

$$S_{kj}(x) = \frac{3}{8\pi}\mathrm{P.V.}\int \Big[\epsilon_{krm}\hat{y}_j\hat{y}_r + \epsilon_{jrm}\hat{y}_k\hat{y}_r\Big]\omega_m(x+y)\frac{dy}{|y|^3} = \frac{3}{8\pi}\mathrm{P.V.}\int R_{kjm}(y)\omega_m(x+y)dy, \tag{7.16}$$

with the kernel

$$R_{kjm}(y) = \frac{1}{|y|^3}[\epsilon_{krm}\hat{y}_j\hat{y}_r + \epsilon_{jrm}\hat{y}_k\hat{y}_r].$$

This kernel is of the singular integral type we have seen before in the Beale-Kato-Majda criterion: it is homogeneous of degree $(-n)$ (the dimension $n = 3$), in the sense that

$$R_{kjm}(\lambda y) = \lambda^{-3}R_{kmj}(y), \tag{7.17}$$

and its integral over any sphere centered at $y = 0$ vanishes:

$$\int_{|y|=1} R_{kjm}(y)dy = \frac{1}{3}[\epsilon_{krm}\delta_{jr} + \epsilon_{jrm}\delta_{kr}] = \frac{1}{3}[\epsilon_{kjm} + \epsilon_{jkm}] = 0. \tag{7.18}$$

Let us show that (7.17) and (7.18) imply that the Fourier transform $\hat{R}_{kjm}(\xi)$ is uniformly bounded:

$$|\hat{R}_{kjm}(\xi)| \le C. \tag{7.19}$$

Indeed, let us write

$$R_{kjm}(y) = \frac{1}{|y|^3}\Phi(\hat{y}), \quad \int_{|y|=1}\Phi(y)dy = 0.$$

As $R_{kjm}(y)$ is homogeneous of degree $(-n)$ (in dimension $n = 3$), its Fourier transform is homogeneous of degree zero. Then we have:

$$\begin{aligned}\hat{R}_{kjm}(\xi) &= \lim_{\varepsilon,\delta\to 0}\int_\varepsilon^{1/\delta}\int_{\mathbb{S}^2}\frac{1}{r^3}e^{2\pi i r(\xi\cdot\hat{y})}\Phi(\hat{y})r^2drd\hat{y} = \lim_{\varepsilon\to 0}\int_\varepsilon^1\int_{\mathbb{S}^2}\Big[\cos(2\pi r(\xi\cdot\hat{y})) - 1\Big]\Phi(\hat{y})\frac{drd\hat{y}}{r}\\ &+ \lim_{\delta\to 0}\int_1^{1/\delta}\int_{\mathbb{S}^2}\cos(2\pi r(\xi\cdot\hat{y}))\Phi(\hat{y})\frac{drd\hat{y}}{r} + \lim_{\varepsilon,\delta\to 0}\int_\varepsilon^{1/\delta}\int_{\mathbb{S}^2}\sin(2\pi r(\xi\cdot\hat{y}))\Phi(\hat{y})\frac{drd\hat{y}}{r}\\ &= A_1 + A_2 + A_3.\end{aligned} \tag{7.20}$$

We used the mean-zero property of $\Phi(\hat{y})$ in the second equality above. For A_3, we may write

$$A_3(\xi) = \lim_{\varepsilon,\delta\to 0} \int_{\mathbb{S}^2} \Phi(\hat{y}) \int_{\varepsilon}^{1/\delta} \sin(2\pi r(\xi\cdot\hat{y})) \frac{dr d\hat{y}}{r} = \lim_{\varepsilon,\delta\to 0} \int_{\mathbb{S}^2} \Phi(\hat{y}) \mathrm{sgn}(\xi\cdot\hat{y}) \Big(\int_{2\pi|\xi\cdot\hat{y}|\varepsilon}^{2\pi|\xi\cdot\hat{y}|/\delta} \frac{\sin r dr}{r} \Big) d\hat{y}.$$

Recall that there exists a constant $C_0 > 0$ so that for any $a, b > 0$ we have

$$\Big| \int_a^b \frac{\sin r dr}{r} \Big| \le C_0,$$

hence $|A_3(\xi)| \le C$. For $A_1 + A_2$, we have

$$\begin{aligned}
A_1(\xi) + A_2(\xi) &= \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} \Phi(\hat{y}) \Big[\int_{\varepsilon}^{1/\varepsilon} [\cos(2\pi r(\xi\cdot\hat{y})) - 1] \frac{dr}{r} \Big] d\hat{y} \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} \Phi(\hat{y}) \Big[\int_{2\pi|\xi\cdot\hat{y}|\varepsilon}^{2\pi|\xi\cdot\hat{y}|/\varepsilon} [\cos(r) - 1] \frac{dr}{r} \Big] d\hat{y} = \int_{\mathbb{S}^2} \Phi(\hat{y}) \Big[\int_0^1 (\cos r - 1) \frac{dr}{r} + \int_1^\infty \frac{\cos r dr}{r} \Big] d\hat{y} \\
&- \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} \Phi(\hat{y}) \Big[\int_1^{2\pi|\xi\cdot\hat{y}|/\varepsilon} \frac{dr}{r} \Big] d\hat{y} = \int_{\mathbb{S}^2} \Phi(\hat{y}) \Big[\int_0^1 (\cos r - 1) \frac{dr}{r} + \int_1^\infty \frac{\cos r dr}{r} \Big] d\hat{y} \\
&- \lim_{\varepsilon\to 0} \int_{\mathbb{S}^2} \Phi(\hat{y}) \log(2\pi|\xi\cdot\hat{y}|/\varepsilon) d\hat{y} \\
&= \int_{\mathbb{S}^2} \Phi(\hat{y}) \Big[\int_0^1 (\cos r - 1) \frac{dr}{r} + \int_1^\infty \frac{\cos r dr}{r} \Big] d\hat{y} - \int_{\mathbb{S}^2} \Phi(\hat{y}) \log(|\hat{\xi}\cdot\hat{y}|) d\hat{y}. \qquad (7.21)
\end{aligned}$$

We used the mean-zero property of $\Phi(\hat{y})$ in the last step. In particular, it allowed us to replace ξ by $\hat{\xi}$ under the logarithm sign. Now, the first integral in the last line in (7.21) does not depend on ξ and is, therefore, uniformly bounded. The second is also bounded, by an application of the Cauchy-Schwartz inequality on $\mathbb{S}^2$. We conclude that the uniform bound (7.19) holds. It follows immediately that the strain matrix satisfies an L^2-bound

$$\|S\|_{L^2} \le C\|\omega\|_{L^2}, \qquad (7.22)$$

a bound we have already seen before.

The regularized system

We will follow the paper by P. Constatin and C Fefferman for the analysis of the vorticity alignment for the Navier-Stokes equations. A similar issue for the Euler equations has been studied in their joint paper with A. Majda. We will start with a regularized Navier-Stokes system, obtained by smoothing the advecting velocity:

$$\begin{aligned}
&u_t + (\phi_\delta * u)\cdot\nabla u + \nabla p = \nu\Delta u, \quad t > 0, \quad x \in \mathbb{R}^n \qquad (7.23)\\
&\nabla\cdot u = 0,\\
&u(0,x) = u_0(x).
\end{aligned}$$

The convolution is performed in space only:

$$u_\delta(t,x) = \phi_\delta * u(t,x) = \int \phi_\delta(x-y) u(t,y) dy,$$

and the kernel ϕ_δ has the form

$$\phi_\delta(x) = \frac{1}{\delta^3}\phi\Big(\frac{x}{\delta}\Big),$$

with a smooth compactly supported function $\phi(x) \geq 0$ with $\|\phi\|_{L^1} = 1$. Note that u_δ is also divergence-free: $\nabla \cdot u_\delta = 0$. Let us explain why the regularized system (7.23) has a strong solution, which is smooth if $u_0 \in C_c^\infty(\mathbb{R}^3)$. Of course, the easy bounds on $u(t,x)$ will blow-up as $\delta \to 0$. We argue as in the estimate for the evolution of the H^m-norms in the proof of the Beale-Kato-Majda criterion. First, multiplying (7.23) by u and integrating by parts we deduce that

$$\int_{\mathbb{R}^3} |u(t,x)|^2 dx + \nu \int_0^t \int |\nabla u(s,x)|^2 dx ds = \int_{\mathbb{R}^3} |u_0(x)|^2 dx, \tag{7.24}$$

hence

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}. \tag{7.25}$$

As a consequence of this estimate, we know that

$$\|u_\delta(t)\|_{C^k} \leq C_k(\delta), \tag{7.26}$$

with the constants $C_k(\delta)$ that may blow-up as $\delta \to 0$. Multiplying (7.23) by $(-\Delta)^m u$ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{m/2}u\|_H^2 + \nu\|(-\Delta)^{(m+1)/2}u\|_H^2 = ((-\Delta)^{m/2}(u_\delta \cdot \nabla u), (-\Delta)^{m/2}u). \tag{7.27}$$

As before, the leading order term in the right side vanishes:

$$((u_\delta \cdot \nabla(-\Delta)^{m/2}u), (-\Delta)^{m/2}u) = 0,$$

because $\nabla \cdot u_\delta = 0$. Hence, the right side in (7.27) can be estimated by

$$C_m\|D^m u\|_2 \sum_{i,j=1}^3 \sum_{k=1}^m \|D^k u_{\delta,j}\|_{L^\infty}\|D^{(m+1-k)}u_i\|_{L^2} \leq C(\delta)\|u\|_{H^m}^2. \tag{7.28}$$

Summing over m, we conclude that for any $s \in \mathbb{N}$ we have

$$\frac{d}{dt}\|u\|_{H^s} \leq C_s\|u\|_{H^s}. \tag{7.29}$$

Therefore, if $u_0 \in C_c^\infty(\mathbb{R}^3)$, then $u(t)$ remains in all $H^m(\mathbb{R}^3)$ for all $t > 0$. Of course, the Sobolev norms of $u(t)$ may blow-up as $\delta \to 0$.

Vorticity alignment prevents blow-up

We will now show that if the direction of the vorticity of the solutions of the regularized system (7.23) is sufficiently aligned then solutions of the Navier-Stokes system itself remain regular. Let us introduce some notation: given a vector e we denote by $P_e^\perp$ the projection orthogonal to e,

$$P_e^\perp v = v - (v \cdot e)e.$$

We will denote by $u(t,x)$ the solution of the regularized system (7.23), let $\omega(t,x) = \nabla\times u(t,x)$ be its vorticity and $\xi(t,x) = \omega(t,x)/|\omega(t,x)|$, while $v(t,x)$ will be the solution of the true Navier-Stokes equations

$$\begin{aligned} & v_t + v\cdot\nabla v + \nabla p = \nu\Delta v, \quad t>0, \quad x\in\mathbb{R}^n \\ & \nabla\cdot v = 0, \\ & v(0,x) = u_0(x). \end{aligned} \tag{7.30}$$

Theorem 7.1 *Assume that there exists δ_0, $\Omega>0$ and $\rho>0$ so that for all $\delta\in(0,\delta_0)$ the solution $u(t,x)$ of the regularized system (7.23) satisfies*

$$\left|P^{\perp}_{\xi(t,x)}(\xi(t,x+y))\right| \le \frac{|y|}{\rho}, \tag{7.31}$$

for all $x,y\in\mathbb{R}^3$ and $0\le t\le T$, such that $|\omega(t,x)|>\Omega$ and $|\omega(t,x+y)>\Omega$. Then the Navier-Stokes equations (7.30) have a strong, and hence C^∞-solution on the time interval $0\le t\le T$.

The strategy will be to get a priori bounds on $u(t,x)$ that do not depend on δ and then pass to the limit $\delta\to 0$. The passage ot the limit is very similar to what we have seen before, so we focus on the a priori bounds that follow from assumption (7.30).

The a priori bounds for the regularized system

We first get a priori bounds for the regularized system that require no assumptions on the direction of the vorticity and, in particular, are independent of (7.31). Let us set $\omega_0 = \nabla\times u_0$ and

$$Q = \int_{\mathbb{R}^3} |\omega_0(x)|dx + \frac{25}{\nu}\int_{\mathbb{R}^3} |u_0(x)|^2 dx.$$

We have then the following bounds, uniform in $\delta>0$.

Lemma 7.2 *The following two bounds hold:*

$$\int_{\mathbb{R}^3} |\omega(t,x)|dx + \nu\int_0^t\int_{\{x:|\omega(s,x)|>0\}} |\omega(s,x)||\nabla\xi(s,x)|^2 dxds \le Q, \tag{7.32}$$

for all $0\le t\le T$, and for any $\Omega>0$ we have

$$\int_0^T\int_{\{x:|\omega(s,x)|>\Omega\}} |\nabla\xi(s,x)|^2 dxds \le \frac{Q}{\nu\Omega}. \tag{7.33}$$

Proof. Let us derive the equation for $\omega(t,x)$: this derivation follows that for the true Navier-Stokes equations but the vorticity equation in presence of the regularization is not identical to that of the Navier-Stokes equations. The advection term in the regularized Navier-Stokes equations can be written as

$$u_\delta\cdot\nabla u = u\cdot\nabla u + (u_\delta - u)\cdot\nabla u = u\cdot\nabla u - v_\delta\cdot\nabla u, \tag{7.34}$$

with

$$v_\delta = u - u_\delta.$$

Recall that

$$
\begin{aligned}
(\omega \times u)_i &= \varepsilon_{ijk}\omega_j u_k = \varepsilon_{ijk}\varepsilon_{jmn}(\partial_m u_n)u_k = (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn})(\partial_m u_n)u_k \\
&= (\partial_k u_i)u_k - (\partial_i u_k)u_k.
\end{aligned} \tag{7.35}
$$

We used above the identity

$$
\varepsilon_{jik}\varepsilon_{jmn} = \delta_{im}\delta_{kn} - \delta_{in}\delta_{km} \tag{7.36}
$$

and anti-symmetry of ε_{ijk}. Thus, as we have previously seen, the advection term can be written as

$$
u \cdot \nabla u = \omega \times u + \nabla\Big(\frac{|u|^2}{2}\Big). \tag{7.37}
$$

Recall also the formula

$$
\nabla \times (a \times b) = -a \cdot \nabla b + b \cdot \nabla a + a(\nabla \cdot b) - b(\nabla \cdot a), \tag{7.38}
$$

which now gives

$$
\nabla \times (u \cdot \nabla u) = \nabla \times (\omega \times u) = -\omega \cdot \nabla u + u \cdot \nabla \omega. \tag{7.39}
$$

We also had an observation that

$$
\omega \cdot \nabla u = V(t,x)\omega, \quad V_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{7.40}
$$

The matrix V can be split into its symmetric and anti-symmetric parts:

$$
V = S + P, \quad S = \frac{1}{2}(V + V^T), \quad P = \frac{1}{2}(V - V^T), \tag{7.41}
$$

The anti-symmetric part has the form

$$
\begin{aligned}
P_{ij}h_j &= \frac{1}{2}[\partial_j u_i - \partial_i u_j]h_j = \frac{1}{2}\partial_m u_k[\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}]h_j = \frac{1}{2}\varepsilon_{lij}\varepsilon_{lkm}(\partial_m u_k)h_j \\
&= -\frac{1}{2}\varepsilon_{lij}\varepsilon_{lmk}(\partial_m u_k)h_j = -\frac{1}{2}\varepsilon_{lij}\omega_l h_j = \frac{1}{2}\varepsilon_{ilj}\omega_l h_j = \frac{1}{2}[\omega \times h]_i,
\end{aligned} \tag{7.42}
$$

for any $h \in \mathbb{R}^3$. In other words, P satisfies

$$
Ph = \frac{1}{2}\omega \times h, \tag{7.43}
$$

and thus has an explicit form

$$
P = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{7.44}
$$

As a consequence, we have $P\omega = 0$, thus $V\omega = S\omega$, so that

$$
\nabla \times (u \cdot \nabla u) = u \cdot \nabla \omega - S\omega. \tag{7.45}
$$

This is, of course, identical to what we have obtained for the true Navier-Stokes equations. For the term in (7.34), which involves v_δ and comes from the regularization, we write

$$[\nabla \times (v_\delta \cdot \nabla u)]_i = \varepsilon_{ijk}\partial_j[v_{\delta,m}\partial_m u_k] = v_{\delta,m}\partial_m[\varepsilon_{ijk}\partial_j u_k] + \varepsilon_{ijk}(\partial_j v_{\delta,m})(\partial_m u_k) \quad (7.46)$$
$$= v_\delta \cdot \nabla\omega_i + \varepsilon_{ijk}(\partial_j v_{\delta,m})(\partial_m u_k)$$

Thus, we have

$$\nabla \times (u_\delta \cdot \nabla u) = u \cdot \nabla\omega - S\omega - v_\delta \cdot \nabla\omega + (\nabla u) \odot (\nabla v_\delta) = u_\delta \cdot \nabla\omega - S\omega + (\nabla u) \odot (\nabla v_\delta). \quad (7.47)$$

Here, we have introduced the following notation: given two matrices a and b, the vector $a \odot b$ has the entries

$$(a \odot b)_i = \varepsilon_{ijk} a_{km} b_{mj}. \quad (7.48)$$

Thus, the vorticity satisfies the evolution equation

$$\omega_t + u_\delta \cdot \nabla\omega - \nu\Delta\omega = S\omega - (\nabla u) \odot (\nabla v_\delta). \quad (7.49)$$

Once again, we stress that the second term in the right side comes from the regularization. Multiplying this equation by $\xi(t,x) = \omega(t,x)/|\omega(t.x)|$, and using the relation $|\xi|^2 = 1$, we get in the left side

$$|\omega|_t + u \cdot \nabla|\omega| - \Delta|\omega| - |\omega|(\xi \cdot \Delta\xi).$$

As we have

$$\xi_j \partial_k \xi_j = 0,$$

leading to

$$(\partial_k\xi_j)(\partial_k\xi_j) + \xi_j\Delta\xi_j = 0,$$

we deduce an evolution equation for $|\omega(t,x)|$ in the region where $\omega(t,x) \neq 0$:

$$\frac{\partial|\omega|}{\partial t} + u_\delta \cdot \nabla|\omega| - \nu\Delta|\omega| + \nu|\omega||\nabla\xi|^2 = \xi \cdot (S\omega - (\nabla u) \odot (\nabla v_\delta)). \quad (7.50)$$

Let now $f(z)$ be a C^2-function of a scalar variable z which vanishes in a neighborhood of $z = 0$. Multiplying (7.50) by $f'(|\omega|)$ and integrating gives

$$\frac{d}{dt}\int_{\mathbb{R}^3} f(|\omega|)dx + \nu\int_{\mathbb{R}^3} f''(|\omega|)|\nabla|\omega||^2 dx + \nu\int_{\mathbb{R}^3} |\omega| f'(|\omega|)|\nabla\xi|^2 dx \quad (7.51)$$
$$= \int_{\mathbb{R}^3} [\xi \cdot (S\omega - (\nabla u) \odot (\nabla v_\delta))] f'(|\omega|)dx.$$

Choose a function $\psi(y) \geq 0$ such that $\psi(y)$ vanishes for $|y| \leq r_0$ and $y > \Omega_0$, and such that

$$\int_0^{\Omega_0} \psi(y)dy = 1, \quad (7.52)$$

and set

$$f(z) = \int_0^z (z-y)\psi(y)dy, \quad (7.53)$$

so that

$$f'(z) = \int_0^z \psi(y)dy, \quad f''(z) = \psi(z) \geq 0. \tag{7.54}$$

In particular, we have $0 \leq f'(z) \leq 1$, $f'(z) = 0$ in a neighborhood of $z = 0$, and

$$zf'(z) = z, \text{ for } z > \Omega_0. \tag{7.55}$$

Thus, integrating (7.51) in time gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(|\omega(t,x)|)dx + \nu \int_0^t \int_{\{x:\omega(s,x)|>\Omega_0\}} |\omega(s,x)||\nabla\xi(s,x)|^2 dx \leq \int_{\mathbb{R}^3} f(|\omega_0(x)|)dx \\
&+ \int_0^t \int_{\mathbb{R}^3} [\xi \cdot (S\omega - (\nabla u) \odot (\nabla v_\delta))] f'(|\omega|) dx ds \qquad (7.56)\\
&\leq \int_{\mathbb{R}^3} |\omega_0(x)|dx + \int_0^t \int_{\mathbb{R}^3} \Big(\frac{1}{2}|S(s,x)|^2 + \frac{1}{2}|\omega(s,x)|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v_\delta|^2\Big).
\end{aligned}$$

As $\nabla \cdot u = 0$, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx = \int_{\mathbb{R}^3} |\omega|^2 dx = 2\int_{\mathbb{R}^3} \mathrm{Tr} S^2 dx.$$

The energy identity (7.24) means that

$$\int_{\mathbb{R}^3} f(|\omega(t,x)|)dx + \nu \int_0^t \int_{\{x:\omega(s,x)|>\Omega_0\}} |\omega(s,x)||\nabla\xi(s,x)|^2 dx \leq Q, \tag{7.57}$$

with

$$Q = \int_{\mathbb{R}^3} |\omega_0(x)|dx + \frac{25}{\nu} \int_{\mathbb{R}^3} |u_0(x)|^2 dx. \tag{7.58}$$

In particular, for any $\Omega > 0$ we obtain

$$\int_0^t \int_{\{x:\omega(s,x)|>\Omega\}} |\nabla\xi(s,x)|^2 dx \leq \frac{Q}{\nu\Omega}. \tag{7.59}$$

We may also let $\Omega_0 \to 0$ in (7.57), so that $f(z) \to z$, and obtain the or estimate in Lemma 7.2

$$\int_{\mathbb{R}^3} |\omega(t,x)|dx + \nu \int_0^t \int_{\{x:\omega(s,x)|>0\}} |\omega(s,x)||\nabla\xi(s,x)|^2 dx \leq Q. \tag{7.60}$$

This finishes the proof of this Lemma.

Enstrophy bounds when the vorticity direction is regular

Lemma 7.2 does not use assumption (7.31) on the vorticity direction. Now, we will use this assumption to obtain enstrophy bounds on the solution of the regularized system. We will show that the solution of the regularized system obeys the following a priori bounds. Here, we use assumption (7.31): there exists δ_0, $\Omega > 0$ and $\rho > 0$ so that for all $\delta \in (0, \delta_0)$ the solution $u(t,x)$ of the regularized system (7.23) satisfies

$$\left|P^{\perp}_{\xi(t,x)}(\xi(t,x+y))\right| \le \frac{|y|}{\rho}, \tag{7.61}$$

for all $x, y \in \mathbb{R}^3$ and $0 \le t \le T$, such that $|\omega(t,x)| > \Omega$ and $|\omega(t,x+y) > \Omega$.

Lemma 7.3 *There exists a constant C which depends on the initial data u_0, and Ω, ν, T, and the constant ρ in (7.61), so that*

$$\sup_{0\le t\le T} \int_{\mathbb{R}^3} |\omega(t,x)|^2 dx \le C, \tag{7.62}$$

and

$$\sup_{0\le t\le T} \int_{\mathbb{R}^3} |\nabla\omega(t,x)|^2 dx \le C, \tag{7.63}$$

for all $\delta \in (0, \delta_0)$.

With these a priori bounds in hand, one can find a subsequence $\delta_k \downarrow 0$, such that the solutions $u(t,x)$ of the regularized Navier-Stokes system converge to a solution $v(t,x)$ of the true Navier-Stokes equations which obeys the same bounds (7.62) and (7.63). These bounds imply that v is a strong solution and is therefore smooth if u_0 is smooth. Thus, our focus is on proving Lemma 7.3.

Multiplying the vorticity equation

$$\omega_t + u_\delta \cdot \nabla\omega - \nu\Delta\omega = S\omega - (\nabla u) \odot (\nabla v_\delta) \tag{7.64}$$

by ω and integrating gives

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2 dx + \nu \int |\nabla\omega|^2 dx = \int (S\omega\cdot\omega)dx - \int \omega\cdot((\nabla u)\odot(\nabla v_\delta))dx. \tag{7.65}$$

We will split the vorticity into the "small" and "large" components: take a cut-off function $\chi(z) \ge 0$ such that $0 \le \chi(z) \le 1$ for all $z \ge 0$, $\chi(z) = 1$ for $0 \le z \le 1$ and $\chi(z) = 0$ for $z \ge 2$. We set

$$\omega(t,x) = \omega^{(1)}(t,x) + \omega^{(2)}(t,x), \tag{7.66}$$

with

$$\omega^{(1)}(t,x) = \chi\Big(\frac{|\omega(t,x)|}{\Omega}\Big)\omega(t,x), \qquad \omega^{(2)}(t,x) = \Big(1 - \chi\Big(\frac{|\omega(t,x)|}{\Omega}\Big)\Big)\omega(t,x). \tag{7.67}$$

Recall that the strain matrix can be written in terms of vorticity as

$$S(x) = \frac{3}{4\pi}\text{P.V.}\int M(\hat{y}, \omega(x+y))\frac{dy}{|y|^3}, \tag{7.68}$$

with the matrix-valued function

$$M(\hat{y}, \omega) = \frac{1}{2}[(\hat{y}\times\omega)\otimes\hat{y} + \hat{y}\otimes(\hat{y}\times\omega)]. \tag{7.69}$$

The decomposition (7.66) induces then the corresponding decomposition

$$S(t,x) = S^{(1)}(t,x) + S^{(2)}(t,x). \tag{7.70}$$

We can then write

$$(S\omega \cdot \omega) = \sum_{i,j,k=1}^{2} (S^{(i)}\omega^{(j)} \cdot \omega^{(k)}) = X + Y + Z, \tag{7.71}$$

where X comes from the triplets where at least one of ω is "small":

$$X = \sum_{(j,k)\neq(2,2)}^{2} (S^{(i)}\omega^{(j)} \cdot \omega^{(k)}),$$

the term Y has S "small", and both ω "large":

$$Y = (S^{(1)}\omega^{(2)} \cdot \omega^{(2)}),$$

and, finally, Z has S and both ω "large":

$$Z = (S^{(2)}\omega^{(2)} \cdot \omega^{(2)}).$$

We also set

$$W = -\omega \cdot ((\nabla u) \odot (\nabla v_\delta)).$$

With this notation, (7.65) has the form

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2 dx + \nu \int |\nabla\omega|^2 dx = \int (X + Y + Z + W) dx. \tag{7.72}$$

We will estimate the size of each term in the right side of (7.72) separately.

In order to estimate X, we recall that for any incompressible flow v we have

$$\int |\nabla v|^2 dx = \int |\zeta|^2 dx, \quad \zeta = \nabla \times v.$$

As a consequence, the strain matrix

$$S_v = \frac{1}{2}(\nabla v + (\nabla v)^t)$$

satisfies

$$\|S_v\|_{L^2}^2 = \sum_{i,j=1}^{3} \int \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)^2 dx \le 4 \sum_{i,j=1}^{3} \int \Big(\frac{\partial v_i}{\partial x_j}\Big)^2 dx = 4\int |\nabla v|^2 dx = 4\int |\zeta|^2 dx. \tag{7.73}$$

Then, the term X can be estimated as follows: either $\omega^{(j)}$ or $\omega^{(k)}$ is "small" and can be bounded pointwise by Ω. This allows us to use the Cauchy-Schwartz inequality and (7.73):

$$\Big|\int X(t,x)dx\Big| \le C\Omega \|S\|_{L^2}\|\omega\|_{L^2} \le C\Omega\|\omega\|_{L^2}^2. \tag{7.74}$$

We have used the bound (7.22)

$$\|S\|_{L^2} \le C\|\omega\|_{L^2}. \tag{7.75}$$

in the second inequality above.

Next, we note that Y is bounded from above by

$$|Y(t,x)| \le |S^{(1)}(t,x)||\omega(t.x)|^2, \tag{7.76}$$

so that

$$\int |Y(t,x)|dx \le \|S^{(1)}\|_{L^2}\Big(\int |\omega(t.x)|^4 dx\Big)^{1/2}. \tag{7.77}$$

The Gagliardo-Nirenberg inequality in $\mathbb{R}^n$:

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^2}^a \|u\|_{L^2}^{1-a}, \quad \frac{1}{p} = \frac{1}{2} - \frac{a}{n},$$

implies that in $\mathbb{R}^3$ we have

$$\Big(\int |\omega(x)|^4 dx\Big)^{1/2} \le C\Big(\int |\nabla\omega(x)|^2 dx\Big)^{3/4}\Big(\int |\omega(x)|^2 dx\Big)^{1/4}. \tag{7.78}$$

Using this in (7.76) gives

$$\begin{aligned}\int |Y(t,x)|dx &\le \|S^{(1)}\|_{L^2}\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{1/2} \le C\|\omega^{(1)}\|_{L^2}\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{1/2}\\ &\le \frac{\nu}{8}\|\nabla\omega\|_{L^2}^2 + \frac{C}{\nu^3}\|\omega^{(1)}\|_{L^2}^4\|\omega\|_{L^2}^2. \end{aligned} \tag{7.79}$$

We have used Young's inequality in the last step, as well as the bound (7.75) for $\|S^{(1)}\|_{L^2}$. The second term in the right side can be bounded with the help of the estimate (7.32) in Lemma 7.2 as

$$\|\omega^{(1)}\|_{L^2}^2 \le 2\Omega \int |\omega(t,x)|dx \le 2\Omega Q. \tag{7.80}$$

Thus, the term Y can be estimated as

$$\int |Y(t,x)|dx \le \frac{\nu}{8}\|\nabla\omega\|_{L^2}^2 + \frac{C}{\nu^3}(\Omega Q)^2\|\omega\|_{L^2}^2. \tag{7.81}$$

Before looking at Z, which is the most difficult term, we bound W:

$$W = -\omega\cdot((\nabla u)\odot(\nabla v_\delta)).$$

This term is only there because of the regularization and should disappear as $\delta \to 0$. Note that

$$\begin{aligned}\|v_\delta\|_{L^2}^2 &= \|u-u_\delta\|_{L^2}^2 = \|u-\phi_\delta * u\|_{L^2}^2 = \int |1-\hat\phi_\delta(\xi)|^2|\hat u(\xi)|^2 d\xi = \int |1-\hat\phi(\delta\xi)|^2|\hat u(\xi)|^2 d\xi\\ &\le C\delta^2\int |\xi|^2|\hat u(\xi)|^2 d\xi = C\delta^2\|\nabla u\|_{L^2}^2 = C\delta^2\|\omega\|_{L^2}^2. \end{aligned} \tag{7.82}$$

The integral of W is

$$\int W(t,x)dx = -\int \omega_i \varepsilon_{ijk} (\nabla u)_{km} (\nabla v_\delta)_{mj} dx = -\int \varepsilon_{ijk} \omega_i \frac{\partial u_k}{\partial x_m} \frac{\partial v_{\delta,m}}{\partial x_j} dx \tag{7.83}$$
$$= \int \varepsilon_{ijk} v_{\delta,m} \frac{\partial \omega_i}{\partial x_j} \frac{\partial u_k}{\partial x_m} dx + \int \varepsilon_{ijk} v_{\delta,m} \omega_i \frac{\partial^2 u_k}{\partial x_j \partial x_m} dx.$$

The last integral above can be written as

$$\int \varepsilon_{ijk} v_{\delta,m} \omega_i \frac{\partial^2 u_k}{\partial x_j \partial x_m} dx = \int \omega_i v_{\delta,m} \frac{\partial}{\partial x_m} \Big(\varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \Big) dx = \int \omega_i v_{\delta,m} \frac{\partial \omega_i}{\partial x_m} dx = 0, \tag{7.84}$$

since v_δ is divergence-free. Therefore, we have a bound for W:

$$\Big| \int W(t,x)dx \Big| \le \frac{\nu}{16} \int |\nabla \omega(t,x)|^2 dx + \frac{C}{\nu} \int |v_\delta(t,x)|^2 |\nabla u(t,x)|^2 dx$$
$$\le \frac{\nu}{16} \int |\nabla \omega(t,x)|^2 dx + \frac{C}{\nu} \|v_\delta\|_{L^4}^2 \|\nabla u\|_{L^4}^2. \tag{7.85}$$

The Gagliardo-Nirenberg inequality implies that

$$\|v_\delta\|_{L^4}^2 \le C \|\nabla v_\delta\|_{L^2}^{3/2} \|v_\delta\|_{L^2}^{1/2}. \tag{7.86}$$

For the gradient term above we can simply bound

$$\|\nabla v_\delta\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 + C\|\nabla u_\delta\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 \le C\|\omega\|_{L^2}^2, \tag{7.87}$$

and we may use the estimate (7.82) for $|v_\delta\|_{L^2}$. Therefore, we have

$$\|v_\delta\|_{L^4}^2 \le C\delta^{1/2} \|\omega\|_{L^2}^2. \tag{7.88}$$

We may also use the same Gagliardo-Nirenberg inequality for $\|\nabla u\|_{L^4}$, leading to

$$\|\nabla u\|_{L^4}^2 \le C \|\nabla \omega\|_{L^2}^{3/2} \|\omega\|_{L^2}^{1/2}. \tag{7.89}$$

Altogether, this gives

$$\frac{1}{\nu} \|v_\delta\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \le \frac{C\delta^{1/2}}{\nu} \|\omega\|_{L^2}^2 \|\nabla \omega\|_{L^2}^{3/2} \|\omega\|_{L^2}^{1/2} = \frac{C\delta^{1/2}}{\nu} \|\omega\|_{L^2}^{5/2} \|\nabla \omega\|_{L^2}^{3/2}$$
$$\le \frac{\nu}{16} \|\nabla \omega\|_{L^2}^2 + \frac{C\delta^2}{\nu^7} \|\omega\|_{L^2}^{10}, \tag{7.90}$$

thus

$$\Big| \int W(t,x)dx \Big| \le \frac{\nu}{8} \int |\nabla \omega(t,x)|^2 dx + \frac{C\delta^2}{\nu^7} \|\omega\|_{L^2}^{10}. \tag{7.91}$$

Finally, we estimate the most dangerous term $Z(t,x)$,

$$Z = (S^{(2)} \omega^{(2)} \cdot \omega^{(2)}),$$

and this will be the only estimate that will involve the assumption that the direction $\xi(t,x)$ of the vorticity is Lipschitz:

$$\left|P^{\perp}_{\xi(t,x)}(\xi(t,x+y))\right| \le \frac{|y|}{\rho}, \tag{7.92}$$

We write

$$Z(t,x) = (S^{(2)}\omega^{(2)}\cdot\omega^{(2)}) = |\omega^{(2)}(t,x)|^2(S^{(2)}(t,x)\xi^{(2)}(t,x)\cdot\xi^{(2)}(t,x)) = |\omega(t,x)|^2\alpha^{(2)}(t,x), \tag{7.93}$$

with

$$\alpha^{(2)}(t,x) = \frac{3}{4\pi}\text{P.V.}\int D(\hat{y},\xi(x+y),\xi(x))|\omega^{(2)}(x+y)|\frac{dy}{|y|^3}, \tag{7.94}$$

where

$$D(e_1,e_2,e_3) = (e_1\cdot e_3)\text{Det}(e_1,e_2,e_3).$$

Assumption (7.92) means that

$$|D(\hat{y},\xi(x+y),\xi(x))| \le \frac{|y|}{\rho}, \tag{7.95}$$

so that

$$|Z(t,x)| \le \frac{3}{4\pi\rho}|\omega^{(2)}(t,x)|^2\int|\omega^{(2)}(t,x+y)|\frac{dy}{|y|^2} \le \frac{3}{4\pi\rho}|\omega(t,x)|^2\int|\omega(t,x+y)|\frac{dy}{|y|^2}. \tag{7.96}$$

Therefore, we have

$$\int|Z(t,x)|dx \le \frac{C}{\rho}\|\omega\|^2_{L^4}\left(\int|I(t,x)|^2dx\right)^{1/2}, \tag{7.97}$$

with

$$I(t,x) = \int|\omega(t,x+y)|\frac{dy}{|y|^2}.$$

In order to compute the L^2-norm of I, let us compute the Fourier transform of the function $\psi(y) = 1/|y|^2$:

$$\begin{aligned}\hat{\psi}(\xi) &= \int\frac{e^{2\pi i\xi\cdot y}dy}{|y|^2} = \int_0^\infty dr\int_{-\pi/2}^{\pi/2}d\theta\cos\theta\int_0^{2\pi}d\phi e^{2\pi i|\xi|r\sin\theta}\\ &= 2\pi\int_0^\infty dr\int_{-1}^1 due^{2\pi i|\xi|ru} = \frac{2}{|\xi|}\int_0^\infty\frac{\sin r dr}{r}.\end{aligned}$$

Hence, the L^2-norm of $I)t,x)$ can be bounded as (for any $R>0$)

$$\|I(t)\|^2_{L^2} = \int|\hat{I}(t,\xi)|^2d\xi \le C\int\frac{|\omega(\xi)|^2}{|\xi|^2}d\xi \le C\int_{|\xi|\le R}\frac{|\omega(\xi)|^2d\xi}{|\xi|^2} + C\int_{|\xi|\ge R}\frac{|\omega(\xi)|^2d\xi}{|\xi|^2} = A_R + B_R.$$

Since

$$|\hat{\omega}(\xi)| \le \|\omega\|_{L^1},$$

the first term can be bounded as,

$$|A_R| \le C\int_0^R \|\omega\|_{L^1}^2 d\xi \le CR\|\omega\|_{L^1}^2.$$

The second term can be simply bounded by

$$|B_R| \le \frac{C}{R^2}\int_{|\xi|\ge R} |\omega(\xi)|^2 d\xi = \frac{C}{R^2}\|\omega\|_{L^2}^2.$$

It follows that for any $R > 0$ we have

$$\|I\|_{L^2}^2 \le CR\|\omega\|_{L^1}^2 + \frac{C}{R^2}\|\omega\|_{L^2}^2.$$

Choosing

$$R = \Big(\frac{\|\omega\|_{L^2}^2}{\|\omega\|_{L^1}^2}\Big)^{1/3},$$

we deduce that

$$\|I\|_{L^2}^2 \le C\|\omega\|_{L^1}^{4/3}\|\omega\|_{L^2}^{2/3}. \tag{7.98}$$

Returning to (7.97), we see that

$$\int |Z(t,x)|dx \le \frac{C}{\rho}\|\omega\|_{L^4}^2\|\omega\|_{L^1}^{2/3}\|\omega\|_{L^2}^{1/3}. \tag{7.99}$$

The L^4-norm of ω is estimated using the same Gagliardo-Nirenberg inequality:

$$\|\omega\|_{L^4}^2 \le C\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{1/2}, \tag{7.100}$$

so that

$$\int |Z(t,x)|dx \le C\|\nabla\omega\|_{L^2}^{3/2}\|\omega\|_{L^2}^{5/6}\|\omega\|_{L^1}^{2/3} \le \frac{\nu}{15}\|\nabla\omega\|_{L^2}^2 + \frac{C}{\nu^3\rho^4}\|\omega\|_{L^2}^{20/6}\|\omega\|_{L^1}^{8/3}. \tag{7.101}$$

Recalling also the a priori bound (7.32) in Lemma 7.2:

$$\int_{\mathbb{R}^3} |\omega(t,x)|dx \le Q, \tag{7.102}$$

we see that Z is bounded as

$$\int |Z(t,x)|dx \le \frac{\nu}{15}\|\nabla\omega\|_{L^2}^2 + \frac{CQ^{8/3}}{\nu^3\rho^4}\|\omega\|_{L^2}^{10/3}. \tag{7.103}$$

Recollecting the starting point of our analysis (7.65)

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2 dx + \nu\int |\nabla\omega|^2 dx = \int (S\omega\cdot\omega)dx - \int \omega\cdot((\nabla u)\odot(\nabla v_\delta))dx, \tag{7.104}$$

and summarizing the bounds (7.74), (7.81), (7.91), (7.103) we have obtained for the terms X, Y. W and Z, respectively, in the right side of the above identity, we get

$$\frac{1}{2}\frac{d}{dt}\int|\omega|^2dx+\nu\int|\nabla\omega|^2dx\le C\Omega\|\omega\|_{L^2}^2+\frac{\nu}{8}\|\nabla\omega\|_{L^2}^2+\frac{C}{\nu^3}(\Omega Q)^2\|\omega\|_{L^2}^2$$
$$+\frac{\nu}{8}\|\nabla\omega\|^2+\frac{C\delta^2}{\nu^7}\|\omega\|_{L^2}^{10}+\frac{\nu}{15}\|\nabla\omega\|_{L^2}^2+\frac{CQ^{8/3}}{\nu^3\rho^4}\|\omega\|_{L^2}^{10/3}. \tag{7.105}$$

Thus, the enstrophy

$$E(t)=\int|\omega(t,x)|^2dx,$$

satisfies a differential inequality

$$\frac{dE}{dt}\le C_1(1+E^{2/3})E+C_1\delta^2E^5, \tag{7.106}$$

with a constant C_1 that depends on ν, ρ, Ω and Q. This is a nonlinear inequality and at the first glance it may seem useless as the solution of an ODE

$$\dot z=C_1(1+z^{2/3})z+C_1\delta^2z^5,\quad z(0)=z_0>0, \tag{7.107}$$

blows up in a finite time. Here, however, we re only concerned with the solution being finite until time $t=T$, and, in addition, we have an extra piece of information: the function

$$k(t)=C_1(1+E^{2/3})$$

has a bounded integral:

$$\int_0^T k(t)dt\le CT+\int_0^T\|\omega(t)\|_{L^2}^{4/3}dt\le CT+CT^{1/3}\Big(\int_0^T\|\omega(t)\|_{L^2}^2dt\Big)^{2/3}\le C(1+T)=D. \tag{7.108}$$

Crucially, the constant D does not depend on δ. Therefore, the solution of (7.107) with $\delta=0$ does remain finite until the time T, and it is reasonable to expect that so does the solution with $\delta>0$ but small. To formalize this observation, let

$$\bar E(t)=2E(0)\exp\Big\{\int_0^t k(s)ds\Big\}.$$

Then $E(0)\le\bar E(0)$, and we may define τ as the first time such that $E(\tau)=\bar E(\tau)$. Until that time, the function $E(t)$ satisfies

$$\frac{dE}{dt}\le k(t)E+C_1\delta^2\bar E^5,\quad 0\le t\le\tau. \tag{7.109}$$

Therefore, as long as $E(t)\le\bar E(t)$, we have a bound for $E(t)$:

$$E(t)\le E(0)\exp\Big\{\int_0^t k(s)ds\Big\}+C_1\delta^2\int_0^t\bar E^5(s)\exp\Big\{\int_s^t k(s')ds'\Big\}ds.$$

Thus, if δ is sufficiently small, we have $E(t) \le \bar{E}(t)$ for all $0 \le t \le T$. We conclude that there3 exists $\delta_0 > 0$ so that for all $0 < \delta < \delta_0$ the enstrophy is bounded:

$$\sup_{0 \le t \le T} \int |\omega(t,x)|^2 dx < +\infty. \tag{7.110}$$

The last step is to observe that (7.105) together with (7.110) implies that

$$\nu \int_0^T \int |\nabla \omega|^2 dx \ < +\infty. \tag{7.111}$$

This completes the proof of Lemma 7.3, and thus that of Theorem 7.1. □

8 The Caffarelli-Kohn-Nirenberg theorem

In this section, we will describe the results of Caffarelli, Kohn and Nirenberg on the Hausdorff dimension of the set where the solution of the three-dimensional Navier-Stokes equations

$$u_t + u \cdot \nabla u + \nabla p = \Delta u + f, \tag{8.1}$$

$$\nabla \cdot u = 0, \tag{8.2}$$

can possibly be singular. We consider this problem in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, with the no-slip boundary condition

$$u(t,x) = 0 \text{ on } \partial\Omega. \tag{8.3}$$

The force $f(t,x)$ is assumed to satisfy the incompressibility condition $\nabla \cdot f = 0$ – this condition is not really necessary, as otherwise we would write $f = \nabla \Phi + g$, with $\nabla \cdot g = 0$, and absorb Φ into the pressure term.

Weak solutions

Let us recall the notion of a Leray weak solution of the Navier-Stokes equations: u is a weak solution if, first, it is a solution in the sense of distributions, that is, for any smooth compactly supported vector-valued function $\psi(t,x)$ we have

$$\begin{aligned}&\int_\Omega [u(t,x)\cdot\psi(t,x) - u_0(x)\cdot\psi(0,x)]dx - \int_0^t \int_\Omega (u\cdot\psi_s)dxds - \int_0^t \int_\Omega u_k u_j \frac{\partial \psi_j}{\partial x_k} dxds \\ &- \int_0^t \int_\Omega p(\nabla\cdot\psi)dxds = \int_0^t \int_\Omega (u\cdot\Delta\psi)dxds + \int_0^t \int_\Omega (f\cdot\psi)dfxds. \end{aligned} \tag{8.4}$$

The second condition is that u satisfies the energy inequality. Note that if u is a smooth solution of the Navier-Stokes equations, then for any smooth test function ϕ we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u(t,x)|^2\phi(t,x)dx + \int_0^t \int_\Omega |\nabla u(s,x)|^2 \phi(s,x)dxds = \frac{1}{2}\int_\Omega |u_0(x)|^2 \phi(0,x)dx \quad (8.5)\\ &+\frac{1}{2}\int_0^t \int_\Omega |u(s,x)|^2(\phi_s(s,x) + \Delta\phi(s,x))dxds \\ &+ \int_0^t \int_\Omega \Big(\frac{|u(s,x)|^2}{2} + p(s,x)\Big) u\cdot\nabla\phi(s,x)dxds + \int_0^t \int_\Omega (f\cdot u)\phi(s,x)dxds.\end{aligned}$$

Taking, formally, $\phi \equiv 1$, the second condition for u to be a Leray weak solution is that it satisfies the energy inequality:

$$\frac{1}{2}\int_\Omega |u(t,x)|^2 dx + \int_0^t \int_\Omega |\nabla u(s,x)|^2 dxds \leq \frac{1}{2}\int_\Omega |u_0(x)|^2 dx + \int_0^t \int_\Omega (f \cdot u) dxds. \tag{8.6}$$

Suitable weak solutions

Caffarelli, Kohn and Nuremberg consider a slightly stronger class of solutions, which they call suitable weak solutions, defined on an open (time-space) set $D \in \mathbb{R} \times \mathbb{R}^3$. We will, obviously, require that u is a weak solution of the Navier-Stokes equations in the sense of distributions: (8.4) holds for any function ϕ supported in D. We will assume that $f \in L^q(D)$ with some $q > 5/2$ – this assumption is not very important, as the main result is interesting even for $f \in C^\infty(D)$. We will also assume that the pressure satisfies

$$p \in L^{5/4}(D), \tag{8.7}$$

and that there exist some constants E_0 and E_1 so that or any fixed time t we have

$$\int_{D_t} |u(t,x)|^2 dx \leq E_0, \tag{8.8}$$

where $D_t = D \cap (\mathbb{R}^3 \times \{t\})$, and

$$\int_D |\nabla u(s,x)|^2 dx \leq E_1. \tag{8.9}$$

In addition, we require that the generalized (or, localized) energy inequality holds: for any function $\phi \geq 0$ which is smooth and compactly supported in D, we have

$$\begin{aligned} \int_D |\nabla u(s,x)|^2 \phi(s,x) dxds &\leq \frac{1}{2}\int_D |u(s,x)|^2 (\phi_s(s,x) + \Delta\phi(s,x)) dxds \\ &+ \int_D \Big(\frac{|u(s,x)|^2}{2} + p(s,x)\Big) u \cdot \nabla\phi(s,x) dxds + \int_D (f \cdot u)\phi(s,x) dxds. \end{aligned} \tag{8.10}$$

At the moment, it is not clear that a suitable weak solution exists – we will prove it below.

The parabolic Hausdorff measure

In order to formulate the main results, we need to define an analog of the Hausdorff measure $\mathcal{H}^1$ but suitable for the parabolic problems. For any set $X \subset \mathbb{R} \times \mathbb{R}^3$, $\delta > 0$ and $k \geq 0$ we define

$$\mathcal{P}_\delta^k(X) = \inf \Big\{ \sum_{i=1}^\infty r_i^k : \ X \subset \bigcup_i Q_{r_i}, \ r_i < \delta \Big\}. \tag{8.11}$$

Here, Q_r is a parabolic cylinder: it has the form

$$Q_r = [t - r^2, t] \times B_r(x),$$

where $B_r(x)$ is a ball of radius r centered at the point x. Then we set

$$\mathcal{P}^k(X) = \lim_{\delta \downarrow 0} \mathcal{P}^k_\delta(X). \tag{8.12}$$

The standard Hausdorff measure is defined in the same way but with Q_r replaced by an arbitrary closed subset of $\mathbb{R} \times \mathbb{R}^3$ of diameter at most r_i, thus we have

$$\mathcal{H}^1 \leq C_k \mathcal{P}^k.$$

The main results

We may now describe the main results of the Caffarelli-Kohn-Nirenberg paper. We say that a point (t, x) is singular if u is not in L^∞_{loc} in any neighborhood of (t, x). Otherwise, we say that (t, x) is a singular point. We will denote by S the set of all singular points of $u(t, x)$. Their first result shows that the singularity set has zero Hausdorff measure $\mathcal{H}^1$.

Theorem 8.1 *Assume that either $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, and let $D = (0, T) \times \Omega$. Suppose that for some $q > 5/2$ we have*

$$f \in L^2(D) \cap L^q_{loc}(D) \quad \nabla \cdot f = 0$$

and

$$u_0 \in L^2(\Omega), \quad \nabla \cdot u_0 = 0, \quad u_0 \cdot \nu|_{\partial\Omega} = 0.$$

If Ω is bounded, we require, in addition, that $u_0 \in W^{-2/5}_{5/4}(\Omega)$. Then the initial boundary value problem has a suitable weak solution in D whose singular set S satisfies $\mathcal{P}^1(S) = 0$.

Their second result concerns absence of singularities outside of a ball of radius $1/\sqrt{t}$.

Theorem 8.2 *Consider the Navier-Stokes equations in $\mathbb{R}^3$ with $f = 0$ and assume that the initial data satisfies $\nabla \cdot u_0 = 0$, and*

$$G = \frac{1}{2} \int_{\mathbb{R}^3} |u_0(x)|^2 |x| dx < +\infty. \tag{8.13}$$

Then there exists a weak solution of the initial value problem which is regular in the region $\{|x| \geq K_1/\sqrt{t}\}$, with the constant K_1 which depends only on G and E, where

$$E = \int_{\mathbb{R}^3} |u_0(x)|^2 |x| dx < +\infty.$$

Assumption (8.13) means that u is small at infinity, and this smallness, so to speak, invades the whole space as t grows. If we assume that u is "small near the origin", in the sense, that

$$L = \int_{\mathbb{R}^3} \frac{|u_0|^2}{|x|} dx = L < +\infty, \tag{8.14}$$

then we have the following result.

Theorem 8.3 *Consider the Navier-Stokes equations in $\mathbb{R}^3$ with $f = 0$ and assume that the initial data satisfies $\nabla \cdot u_0 = 0$, and (8.14) holds. There exists a universal constant L_0 so that if $L < L_0$, then u is regular in the region $\{|x| \leq \sqrt{(L_0 - L)t}\}$.*

The first key estimate: localizing "small data regularity"

We will denote the cylinders labeled by the top as

$$Q_r(t,x) = \{(s,y):\ |y-x|<r,\ t-r^2<s<t\},$$

and those labeled by a point slightly below the top as

$$Q_r^*(t,x) = \{(s,y):\ |y-x|<r,\ t-\frac{7}{8}r^2<s<t+\frac{1}{8}r^2\}.$$

It is well known that if the initial condition u_0 and the force f are small in an appropriate norm, then the solution of the Navier-Stokes equations remains regular for a short time. The main issue in proving the partial regularity theorems is to localize this result. The first step in this direction is an estimate showing that if u, p and f are sufficiently small on the unit cylinder $Q_1 = Q_1(0,0)$, then u is regular in the smaller cylinder $Q_{1/2} = Q_{1/2}(0,0)$ – this is a very common theme in the parabolic regularity theory.

Proposition 8.4 *There exist absolute constants $C_1 > 0$ and $\varepsilon_1 > 0$ and a constant $\varepsilon_2(q) > 0$, which depends only on q with the following property. Suppose that (u,p) is a suitable weak solution of the Navier-Stokes system on Q_1 with $f \in L^q$, with $q > 5/2$. Assume also that*

$$\int_{Q_1} (|u|^3 + |u||p|)dxdt + \int_{-1}^{0} \Big(\int_{|x|<1} |p|dx \Big)^{5/4} dt \le \varepsilon_1, \tag{8.15}$$

and

$$\int_{Q_1} |f|^q dxdt \le \varepsilon_2. \tag{8.16}$$

Then we have $|u(t,x)| \le C_1$ for Lebesgue-almost every $(t,x) \in Q_{1/2}$. In particular, u is regular in $Q_{1/2}$.

In order to see how we may scale this result to a parabolic cylinder of length r, let us investigate the dimension of various terms in the Navier-Stokes equations

$$u_t + u\cdot\nabla u + \nabla p = \Delta u + f. \tag{8.17}$$

Let us assign dimension L to the spatial variable x. As all individual terms in (8.17) should have the same dimension, looking at the terms u_t and Δu we conclude that time should have dimension L^2. Comparing the terms u_t and $u\cdot\nabla u$ we see that u should have the dimension L^{-1}. Then, f should have the same dimension as u_t, which is L^{-3}. Finally, the dimension of the pressure term should be L^{-2}. Summarizing, we have

$$[x] = L,\ [t] = L^2,\ [u] = L^{-1},\ [f] = L^{-3},\ [p] = L^{-2}. \tag{8.18}$$

Let us look at the dimension of each term in the estimate (8.15): the term involving $|u|^3$ has the dimension

$$[x]^3[t][u]^3 = L^2,$$

the term involving $|u||p|$ has the same dimension:

$$[x]^3[t][u][p] = L^2,$$

while the last term in the left side has the dimension

$$[t][x]^{15/4}[p]^{5/4} = L^{23/4}L^{-10/4} = L^{13/4}.$$

We also should note that the dimension of the L^q-norm of f (to the power q) is

$$[x]^3[t][f]^q = L^{5-3q}.$$

Accordingly, for a parabolic cylinder $Q_r(t,x)$ we set

$$M(r) = \frac{1}{r^2}\int_{Q_r}(|u|^3 + |u||p|)dxdt + \frac{1}{r^{13/4}}\int_{t-r^2}^{t}\Big(\int_{|y-x|<r}|p|dx\Big)^{5/4}dt, \tag{8.19}$$

and

$$F_q(r) = r^{3q-5}\int_{Q_r}|f|^q dyds. \tag{8.20}$$

Therefore, Proposition 8.4 has the following corollary.

Corollary 8.5 *Suppose hat (u,p) is a suitable weak solution of the Navier-Stokes system on a cylinder Q_r with $f \in L^q$, with $q > 5/2$. Assume also that*

$$M(r) \le \varepsilon_1, \tag{8.21}$$

and

$$F_q(r) \le \varepsilon_2. \tag{8.22}$$

Then we have $|u(t,x)| \le C_1/r$ for Lebesgue-almost every $(t,x) \in Q_{r/2}$. In particular, u is regular in $Q_{r/2}$.

The second key estimate: the blow-up rate

One can deduce from Corollary 8.5 a heuristic estimate on the possible blow-up rate of the solution. Assume that (t_0,x_0) is a singular point. Then, (8.21) has to fail for all $Q_r(t,x)$ such that $(t_0,x_0) \in Q_{r/2}(t,x)$. Therefore, we must have

$$M(r) = M(r;t,x) > \varepsilon_1$$

for a family of parabolic cylinders shrinking to the point (t_0,x_0). Let us assume that

$$u(t,x) \sim r^{-m},$$

near x_0, with

$$r = (|x-x_0|^2 + |t-t_0|)^{1/2}.$$

Then we have

$$M(r) \sim \frac{1}{r^2}\frac{1}{r^{3m}}r^2r^3 = r^{3-3m}.$$

hence, a natural guess is $m = 1$, which translates into

$$|\nabla u| \ge \frac{C}{r^2}, \quad \text{as } (t,x) \to (t_0,x_0). \tag{8.23}$$

The next key estimate verifies that this is qualitatively correct.

Proposition 8.6 *There is an absolute constant $\varepsilon_3 > 0$ with the following property. If u is a suitable weak solution of the Navier-Stokes equations near (t, x), and if*

$$\limsup_{r \downarrow 0} \frac{1}{r} \int_{Q_r^*(t,x)} |\nabla u|^2 dyds \leq \varepsilon_3, \tag{8.24}$$

then (t, x) is a regular point.

Let us explain how Theorem 8.1 would follow. Take any (t, x) in the singular set, then, by Proposition 8.6 we have

$$\limsup_{r \downarrow 0} \frac{1}{r} \int_{Q_r^*(t,x)} |\nabla u|^2 dyds > \varepsilon_3. \tag{8.25}$$

Take a neighborhood V of the singular set S and $\delta > 0$. For each $(t, x) \in S$ we may choose a parabolic cylinder $Q_r^*(t, x)$ with $r < \delta$ and such that

$$\frac{1}{r} \int_{Q_r^*(t,x)} |\nabla u|^2 dyds > \varepsilon_3, \tag{8.26}$$

and $Q_r^*(t, x) \subset V$. We will make use of the following covering lemma.

Lemma 8.7 *Let $\mathcal{J}$ be a collection of parabolic cylinders $Q_r^*(t, x)$ contained in a bounded set V. Then there exists an at most countable sub-collection $\mathcal{J}' = \{Q_i^* = Q_{r_1}^*(t_i, x_i)\}$ of non-overlapping cylinders such that for any $Q^* \in \mathcal{J}$ there exists Q_i^* so that*

$$Q^* \subset Q_{5r_i}^*(t_i, x_i).$$

The proof is very similar to that of the classic Vitali lemma and we leave it to the reader as an exercise. Using this lemma, we obtain a disjoint collection of cylinders $Q_{r_i}^*(t_i, x_i)$ such that

$$S \subset \bigcup_i Q_{5r_i}^*(t_i, x_i),$$

and

$$\sum_i r_i \leq \frac{1}{\varepsilon_3} \int_{Q_{r_i}^*} |\nabla u|^2 dxdt \leq \frac{1}{\varepsilon_3} \int_V |\nabla u|^2 dxdt.$$

We deduce that

$$\mathcal{P}^1(S) \leq \frac{1}{\varepsilon_3} \int_V |\nabla u|^2 dxdt. \tag{8.27}$$

In particular, we deduce that the (three-dimensional) Lebesgue measure of S is zero. Then, as V is an arbitrary neighborhood of S, and the function $|\nabla u|^2$ is integrable, we can make the right side of (8.27) arbitrarily small. It follows that $\mathcal{P}^1(S) = 0$, proving Theorem 8.1. Thus, the crux of the matter is the proof of Propositions 8.4 and 8.6.

Serrin's interior regularity result

Before we proceed with the further discussion of the proofs of the theorems of Caffarelli, Kohn and Nirenberg, let us explain why we say a solution is regular if it is just bounded, and do not require further differentiability. The reason is a result of Serrin on the interior regularity of the weak solutions of the Navier-Stokes equations

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \Delta u + f, \\ &\nabla \cdot u = 0. \end{aligned} \tag{8.28}$$

We will assume for simplicity that $f = 0$ – the reader should consider the generalization to the case $f \neq 0$ as an exercise, or consult Serrin's original paper. Let us borrow the following very simple observation from Serrin's paper: if $\psi(x)$ is a harmonic function, then any function of the form

$$u(t, x) = a(t)\nabla\psi(x)$$

is a weak solution of the Navier-Stokes equations, as long as the function $a(t)$ is integrable. Therefore, boundedness of $u(t, x)$ can not, in general, imply any information on the time derivatives of u. On the other hand, this example does not rule out the hope that relatively weak assumptions on u would guarantee its spatial regularity.

Here is one version of Serrin's result, which says that bounded solutions of the force-less Navier-Stokes equations are essentially as good as the solutions of the heat equation.

Theorem 8.8 *Let u be a Leray weak solution of the Navier-Stokes equations in an open region $R = (t_1, t_2) \times \Omega$ of space-time, with $f = 0$, and such that*

$$\int_{t_1}^{t_2} \int_\Omega |\omega(t, x)|^2 dxdt < +\infty, \quad \sup_{t \in [t_1, t_2]} \int_\Omega |u(t, x)|^2 dx < +\infty, \tag{8.29}$$

where $\omega = \nabla \times u$ is the vorticity. Assume, in addition, that $u \in L^\infty(R)$. Then, u is of the C^∞ class in the space variables on every compact subset of R.

The full statement of the Serrin theorem says that if $u \in L^{s,s'}(R)$, with

$$\|u\|_{L^{s,s'}} = \Big(\int_{t_1}^{t_2} \|u\|_{L^s(\Omega)}^{s'} dt\Big)^{1/s'},$$

with (in three dimensions)

$$\frac{3}{s} + \frac{2}{s'} < 1, \tag{8.30}$$

then u is C^∞ in the spatial variables. If, in addition, we know that $u_t \in L^{2,p}$ with $p \geq 1$, then the spatial derivatives of u are absolutely continuous in time. We will not need these results for our purposes, so we will leave them out for now. Let us make one comment, however: if we take $s' = \infty$, then condition (8.30) is satisfied, as long as $s > 3$. That is, if we would have known a priori that

$$\int_{\mathbb{R}^3} |u(t, x)|^3 dx \leq \text{const},$$

then we could conclude that u is a smooth solution. Of course, we have this information only for the L^2-norm of the Leray weak solutions, and not for the L^3-norm.

For the proof of Theorem 8.8, let us recall the vorticity equation in three dimensions:

$$\omega_t + u \cdot \nabla\omega - \Delta\omega = \omega \cdot \nabla u. \tag{8.31}$$

Written in the components, this equation is

$$\frac{\partial \omega_k}{\partial t} - \Delta\omega_k = \omega_j \frac{\partial u_k}{\partial x_j} - u_j \frac{\partial \omega_k}{\partial x_j}, \tag{8.32}$$

or

$$\frac{\partial \omega_k}{\partial t} - \Delta\omega_k = \frac{\partial}{\partial x_j}(\omega_j u_k - u_j\ \omega_k). \tag{8.33}$$

Let $\bar{\Omega}_1$ be a compact subset of Ω, and $t_1 < s_1 < s_2 < t_2$, so that $S = (s_1, s_2) \times \Omega_1$ is a proper subset of R, and define, for $s_1 \le t \le s_2$:

$$\begin{aligned}\tilde{\omega}_k(t,x) &= \frac{\partial}{\partial x_j}\int_{s_1}^t \int_{\Omega_1} G(t-s, x-y)[\omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)]dyds \\ &= \int_{s_1}^t \int_{\Omega_1} \frac{\partial G(t-s,x-y)}{\partial x_j}[\omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)]dyds.\end{aligned}$$

Here, $G(t,x)$ is the standard heat kernel. The functions

$$m_{kj}(t,x) = \int_{s_1}^t \int_{\Omega_1} G(t-s, x-y)[\omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)]dyds$$

satisfy

$$\frac{\partial m_{kj}}{\partial t} - \Delta m_{kj} = (\omega_j u_k - u_j\ \omega_k)\chi_{[s_1,s_2]}(t)\chi_{\bar{\Omega}_1}(x). \tag{8.34}$$

Thus, for $(t,x) \in S$, the function $\tilde{\omega}$ is the solution of

$$\frac{\partial \tilde{\omega}_k}{\partial t} - \Delta\tilde{\omega}_k = \frac{\partial}{\partial x_j}(\omega_j u_k - u_j\ \omega_k). \tag{8.35}$$

It follows that the difference

$$B(t,x) = \omega(t,x) - \tilde{\omega}(t,x)$$

satisfies the standard heat equation

$$B_t - \Delta B = 0,$$

on the set S.

We will now show that $\omega \in L^\infty(S)$, that is, if u is uniformly bounded on R, then the vorticity is uniformly bounded on any compact subset of R.

Exercise 8.9 *Use the convolution with the heat kernel to show that if $\phi(t,x)$ satisfies*

$$\phi_t - \Delta\phi = \frac{\partial g}{\partial x_j},$$

in the whole space $\mathbb{R}^n$, then

$$\|\phi\|_{L^r} \le C\|g\|_{L^q},$$

as long as

$$(n+2)\Big(\frac{1}{q} - \frac{1}{r}\Big) < 1.$$

The norms are take in space-time.

As u is a Leray weak solution, we know that $\omega \in L^2(R)$. As $u \in L^\infty(R)$, it follows that the functions

$$g_{jk}(s,y) = \omega_j(s,y)u_k(s,y) - u_j(s,y)\omega_k(s,y)$$

are also in $L^2(R)$. The result of the above exercise says that then $\tilde{\omega} \in L^r$ with

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

But then $g \in L^6$, as well, and, as $1/6 < 1/3$, it follows that $\tilde{\omega} \in L^\infty(R)$. We also know that $B \in L^\infty(S)$ by the regularity estimates for the heat equation, as $B \in L^2(R)$ – it is the difference of two functions in $L^2(R)$. Moreover, we know that B is Hölder continuous.

Now that we know that $\omega \in L^\infty(R)$, we recall that the velocity and the vorticity are related by the stream vector ψ, defined as the solution of

$$-\Delta\psi = \omega, \quad \nabla \cdot \psi = 0,$$

and

$$u = -\nabla \times \psi.$$

Therefore, if $\omega \in L^\infty(R)$, then ψ is $C^{1,\alpha}$ in the spatial variable, hence u is Hölder in x, and, in particular, in L^∞. Then the functions m_{kj} are $C^{1,\alpha}$ in x, thus ω is Hölder in x. Then, the functions g_{kj} are Hölder in x, so ω_x is Hölder in x, continuing this argument we deduce that both ω and u are C^∞.

Existence of suitable weak solutions

We now prove the existence of suitable weak solutions, in the sense of Caffarelli, Kohn and Nirenberg. We will restrict ourselves to the whole space: $\Omega = \mathbb{R}^3$. Let us first define the appropriate function spaces. As usual, we will denote by $\mathcal{V}$ the space of smooth divergence-free vector fields u, by H the closure of $\mathcal{V}$ in $L^2(\mathbb{R}^3)$, by V the closure of $\mathcal{V}$ in $H^1(\mathbb{R}^3)$, and by V' the dual space of V. The Sobolev spaces $W_q^l(\mathbb{R}^3)$ with $q \ge 1$ and $0 < l < 1$ consists of functions with l derivatives in L^q, and with the norm

$$\|u\|_{W_q^l} = \|u\|_{L^q} + \|(-\Delta)^{l/2}u\|_{L^q}.$$

We will make the standard assumptions:

$$\Omega = \mathbb{R}^3,\ u_0 \in H,\ f \in L^2(0,T;H^{-1}(\mathbb{R}^3)). \tag{8.36}$$

Theorem 8.10 *Assume that $\Omega = \mathbb{R}^3$, u_0 and f satisfy (8.36). Then there exists a suitable weak solution*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H),$$

of the Navier-Stokes equations with the force f and the initial condition u_0, in the sense that $u(t) \to u_0$ weakly in H as $t \to 0$. The pressure satisfies $p \in L^{5/3}((0,T) \times \mathbb{R}^3)$. In addition, if $\phi \in C^\infty([0,T] \times \mathbb{R}^3)$, $\phi \geq 0$ and is compactly supported, then

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} |u(t,x)|^2 \phi(t,x)dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(s,x)|^2 \phi(s,x) dxds \leq \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 \phi(0,x)dx \\
&+\frac{1}{2}\int_0^t \int_{\mathbb{R}^3} |u(s,x)|^2 (\phi_s(s,x) + \Delta\phi(s,x))dxds \qquad (8.37)\\
&+\int_0^t \int_{\mathbb{R}^3} \Big(\frac{|u(s,x)|^2}{2} + p(s,x)\Big) u \cdot \nabla\phi(s,x)dxds + \int_0^t \int_{\mathbb{R}^3} (f \cdot u)\phi(s,x)dxds.
\end{aligned}$$

The proof is done via a "retarded mollification". The (standard) idea is to take $\Psi_\delta(u)$ to be a mollifier of u such that $\Psi_\delta(u)$ is divergence-free and depends only on the values of $u(s,x)$ with $s \leq t - \delta$. The mollified system

$$u_t + \Psi_\delta(u) \cdot \nabla u + \nabla p = \Delta u + f \qquad (8.38)$$

is then linear on each time interval of the firm $(m\delta, (m+1)\delta)$. We will get uniform in δ a priori bounds on u, and then pass to the limit $\delta \to 0$.

Let us recall some basic facts about the linear Stokes equation, whose proof is very similar to what we have done on the torus previously.

$$u_t + \nabla p = \Delta u + f, \quad \nabla \cdot u = 0. \qquad (8.39)$$

Lemma 8.11 *Suppose that $f \in L^2(0,T;V')$, $u \in L^2(0,T;V)$, p is a distribution and (8.39) holds. Then $u_t \in L^2(0,T;V')$,*

$$\frac{d}{dt}\int_\Omega |u|^2 dx = 2\int_\Omega (u_t \cdot u)dx,$$

in the sense of distributions on $(0,T)$, and $u \in C([0,T],H)$, possibly after a modification on a set of measure zero.

Lemma 8.12 *Suppose that $f \in L^2(0,T;V')$, $u_0 \in H$, and $w \in C^\infty([0,T];\Omega)$ are prescribed, and $\nabla \cdot w = 0$. Then there exists a unique function $u \in L^2(0,T;V) \cap C([0,T];H)$, and a distribution p so that*

$$u_t + w \cdot \nabla u + \nabla p = \Delta u + f, \quad \nabla \cdot u = 0, \qquad (8.40)$$

in the sense of distributions, and $u(0) = u_0$.

Some pressure bounds and interpolation on the velocity

Note that if u solves (8.40) in the whole space, then the pressure satisfies the Poisson equation

$$\Delta p = -\sum_{i,j=1}^{3} \partial_{ij}^2 (w_i u_j). \tag{8.41}$$

The singular integral operator corresponding to the Fourier multiplier

$$\frac{\xi_i \xi_j}{|\xi|^2}$$

is bounded $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, thus, in particular, we have the bound

$$\int_0^T \int_{\mathbb{R}^3} |p|^{5/3} dxds \leq C \int_0^T \int_{\mathbb{R}^3} |w|^{5/3} |u|^{5/3} dxds \tag{8.42}$$

$$\leq C\Big(\int_0^T \int_{\mathbb{R}^3} |w|^{10/3} dxds\Big)^{1/2} \Big(\int_0^T \int_{\mathbb{R}^3} |u|^{10/3} dxds\Big)^{1/2}. \tag{8.43}$$

We will now use a Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^3} |u|^q dx \leq C\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^a \Big(\int_{\mathbb{R}^3} |u|^2 dx\Big)^{q/2-a}, \tag{8.44}$$

with $2 \leq q \leq 6$ and $a = 3(q-2)/4$. Note that when $q = 2$, $a = 0$, this is a tautology, and when $q = 6$, $a = 3$, this is the familiar Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^3} |u|^6 dx \leq C\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^3. \tag{8.45}$$

Taking $q = 10/3$, and $a = 1$ gives

$$\int_{\mathbb{R}^3} |u|^{10/3} dx \leq C\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Big(\int_{\mathbb{R}^3} |u|^2 dx\Big)^{2/3} \tag{8.46}$$

Integrating in time and using the a priori assumptions (8.8) and (8.9) leads to

$$\int_0^T \int_{\mathbb{R}^3} |u|^{10/3} dxdt \leq CE_1(u) E_0^{2/3}(u). \tag{8.47}$$

Another useful estimate, obtained, once again, by taking $q = 10/3$ and $a = 1$, is

$$\int_0^T \int_{\mathbb{R}^3} |w \cdot \nabla u|^{5/4} dxdt \leq \Big(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 dxdt\Big)^{5/8} \Big(\int_0^T \int_{\mathbb{R}^3} |w|^{10/3} dxdt\Big)^{3/8} \tag{8.48}$$

$$\leq CE_1(u)^{5/8} E_1(w)^{3/8} E_0(w)^{1/4}, \tag{8.49}$$

which can be restated as

$$\|w \cdot \nabla u\|_{L^{5/4}} \leq CE_1(u)^{1/2} E_1(w)^{3/10} E_0(w)^{1/5}. \tag{8.50}$$

We will also use the following bound, which follows from (8.45) with $q = 5/2$ and $a = 3/8$:

$$\int_{\mathbb{R}^3} |u|^{5/2} dx \le C E_0^{7/8} \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{3/8}. \tag{8.51}$$

As a consequence, we have

$$\begin{aligned} \int_0^T \Big(\int_{\mathbb{R}^3} |u|^{5/2} dx \Big)^2 dt &\le C E_0(u)^{7/4} \int_0^T \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{3/4} dt \\ &\le C E_0(u)^{7/4} T^{1/4} \Big(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 dx dt \Big)^{3/4} \le C T^{1/4} E_0^{7/4} E_1(u)^{3/4}. \end{aligned} \tag{8.52}$$

This can be restated as

$$\|u\|_{L^5(0,T;L^{5/2})} \le C T^{1/20} E_0^{7/20} E_1(u)^{3/20}. \tag{8.53}$$

These bounds allow us to take a solution (in the sense of distributions) $u \in C([0,T];H) \cap L^2(0,T;V)$ of the Stokes advection equation

$$u_t + w \cdot u - \Delta u + \nabla p = f, \tag{8.54}$$

with $w \in C^\infty$, multiply by a test function ϕ and obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^2(T,x)\phi(T,x) dx + 2 \int_0^T \int_{\mathbb{R}^3} |\nabla u(t,x)|^2 \phi(t,x) dx dt = \int_{\mathbb{R}^3} |u_0(x)|^2 \phi(0,x) dx \\ + \int_0^T \int_{\mathbb{R}^3} |u|^2 (\phi_t + \Delta\phi) dx dt + \int_0^T \int_{\mathbb{R}^3} (|u|^2 w + 2pu) \cdot \nabla\phi dx dt + 2 \int_0^T \int_{\mathbb{R}^3} (u \cdot f) dx dt. \end{aligned} \tag{8.55}$$

Exercise 8.13 Justify the integration by parts above by mollifying (in time and space) each term in the Stokes equation, multiplying by ϕ, integrating by parts and then removing the mollification using the a priori bounds obtained above.

The retarded mollifier

We take a C^∞ function $\psi(t,x) \ge 0$ such that

$$\int \psi(t,x) dx dt = 1,$$

and

$$\mathrm{supp}\psi \subset \{(t,x) : \ |x|^2 < t, \ 1 < t < 2\}.$$

We also extend $u(t,x)$ by zero to $t < 0$, and set

$$\Psi_\delta(u)(t,x) = \frac{1}{\delta^4} \int_{\mathbb{R}^4} \psi(\frac{s}{\delta}, \frac{y}{\delta}) \tilde{u}(x-y, t-s) dy ds. \tag{8.56}$$

The mollified u is divergence-free:

$$\nabla \cdot \Psi_\delta(u) = 0,$$

and it inherits the a priori bounds on u:

$$\sup_{0\le t\le t}\int_{\mathbb{R}^3}|\Psi_\delta(u)|^2(t,x)dx \le CE_0(u), \tag{8.57}$$

and

$$\int_0^T\int_{\mathbb{R}^3}|\Psi_\delta(u)|^2(t,x)dxdt \le CE_1(u). \tag{8.58}$$

The approximants

We will use the approximants

$$\begin{aligned}&\frac{\partial u_N}{\partial t}+\Psi_\delta(u_N)\cdot\nabla u_N+\nabla p_N=\Delta u_N+f,\\ &\nabla\cdot u_N=0,\\ &u_N(0,x)=u_0(x),\end{aligned} \tag{8.59}$$

with $\delta = T/N$. We may apply inductively the existence result for the Stokes equation with a prescribed advection, on the time intervals of the form $(m\delta,(m+1)\delta)$, $0\le m\le N-1$. Then we have

$$\int_{\mathbb{R}^3}|u_N(t,x)|^2dx+2\int_0^t\int_{\mathbb{R}^3}|\nabla u_N(s,x)|^2dxds=\int_{\mathbb{R}^3}|u_0(x)|^2dx+2\int_0^t\int_{\mathbb{R}^3}(f\cdot u_N)dxds. \tag{8.60}$$

In particular, we have

$$\int_{\mathbb{R}^3}|u_N(t,x)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla u_N(s,x)|^2dxds\le\int_{\mathbb{R}^3}|u_0(x)|^2dx+\int_0^t\|f\|_{V'}^2ds. \tag{8.61}$$

We conclude that u_N is uniformly bounded in $L^\infty(0,T;V)\cap L^\infty(0,T;H)$, the usual Leray bound. In addition, we know that p_N is bounded in $L^{5/3}([0,T]\times\mathbb{R}^3)$. It follows that, after an extraction of a sub-sequence, we have that $p_N\to p_*$ weakly in $L^{5/3}([0,T]\times\mathbb{R}^3)$, and $u_N\to u_*$, weak-star in $L^\infty(0,T;H)$, and weakly in $L^2(0,T;V)$.

Exercise 8.14 Show that if u_N is bounded in $L^\infty(0,T;V)\cap L^\infty(0,T;H)$, and $\frac{\partial u_N}{\partial t}$ is bounded in $L^2(0,T;H^{-2})$, then u_N has a convergent subsequence in $L^2([0,T]\times\mathbb{R}^3)$.

Exercise 8.15 *Show that if $u_N\to u_*$ strongly in L^q and u_N is bounded in L^r, $1\le q<r$, then $u_N\to u_*$ strongly in L^s for all $q,s<r$.*

We may use this with $q=2$ and $r=10/3$ to conclude that $u_N\to u_*$ strongly in $L^s([0,T]\times\mathbb{R}^3)$ for all $2\le s<10/3$. Then one may easily check that (u_*,p_*) is the sought suitable weak solution of the Navier-Stokes equations.

The proof of Proposition 8.4

We now turn to the proof of the two main auxiliary results, and begin with Proposition 8.4. We recall its statement:

Proposition 8.16 *There exist two absolute constants $C_1 > 0$ and $\varepsilon_1 > 0$ and another constant $\varepsilon_2(q) > 0$, which depends only on q with the following property. Suppose that (u, p) is a suitable weak solution of the Navier-Stokes system on $Q_1(0, 0)$ with $f \in L^q$, with $q > 5/2$. Assume also that*

$$\int_{Q_1} (|u|^3 + |u||p|)dxdt + \int_{-1}^{0} \Big(\int_{|x|<1} |p|dx \Big)^{5/4} dt \leq \varepsilon_1, \tag{8.62}$$

and

$$\int_{Q_1} |f|^q dxdt \leq \varepsilon_2. \tag{8.63}$$

Then we have $|u(t, x)| \leq C_1$ for Lebesgue-almost every $(t, x) \in Q_{1/2}(0, 0)$. In particular, u is regular in $Q_{1/2}$.

Outline of the proof

Let us take an arbitrary point $(s, x_0) \in Q_{1/2}(0, 0)$, where we want to show that $|u(s, x_0)| \leq C_1$. As $Q_{1/2}(s, x_0) \subset Q_1(0, 0)$, we have an integral estimate

$$\int_{Q_{1/2}(s,x_0)} (|u|^3 + |u||p|)dxdt + \int_{s-1/4}^{s} \Big(\int_{|x-x_0|<1/2} |p|dx \Big)^{5/4} dt \leq \varepsilon_1. \tag{8.64}$$

We will consider a sequence of shrinking parabolic cylinders $Q_k = Q_{r_k}(s, x_0)$, "centered" at the point (s, x_0) with $r_k = 2^{-k}$. Our goal will be to show that for all $k \geq 2$ we have

$$\fint_{|x-x_0|<r_k} |u(s, x)|^2 dx \leq C_0 \varepsilon_1^{2/3}, \tag{8.65}$$

where $\fint_S f$ denotes the average of a function f over the set S. Then, if (s, x_0) is a Lebesgue point for u, it follows that

$$|u(s, x_0)|^2 \leq C_0 \varepsilon_1^{2/3}, \tag{8.66}$$

hence (8.66) holds for Lebesgue almost every point in $Q_{1/2}(0, 0)$, which is exactly the claim of Proposition 8.16.

In order to prove (8.65) we will show that for all $k \geq 2$ we have a more general estimate

$$\sup_{s-r_k^2<t\leq s} \fint_{|x-x_0|\leq r_k} |u(t, x)|^2 dx + \frac{1}{r_k^3} \int_{Q_k} |\nabla u(t, x)|^2 dxdt \leq C_0 \varepsilon_1^{2/3}, \tag{8.67}$$

where

$$\bar{p}_k(t) = \fint_{|x-x_0|<r_k} p(t, x)dx. \tag{8.68}$$

Note that (8.65) follows immediately from (8.67). Thus, the conclusion of Proposition 8.4 follows from (8.67).

The induction base. We will prove (8.67) by induction, starting with $k = 2$. For $k = 2$, we may use the localized energy inequality: for every smooth test function $\phi(t, x) \ge 0$, that vanishes near $|x| = 1$ and $t = -1$, we have, for $-1 < s < 0$, with $B_1 = B_1(0, 0)$:

$$\int_{B_1} |u(s,x)|^2\phi(s,x)dx + 2\int_{-1}^{s}\int_{B_1} |\nabla u(t,x)|^2\phi(t,x)dxdt \le \int_{-1}^{s}\int_{B_1} |u(t,x)|^2(\phi_t + \Delta\phi)dxdt$$
$$+ \int_{-1}^{s}\int_{B_1} (|u|^2 + 2p)u\cdot\nabla\phi(t,x)dtdx + 2\int_{-1}^{s}\int_{B_1} (f\cdot u)\phi(t,x)dxdt. \quad (8.69)$$

Taking ϕ such that $0 \le \phi \le 1$, $\phi \equiv 1$ on $Q_{1/2}(0, 0)$ and ϕ is supported in $Q_1(0, 0)$, we deduce that

$$\int_{|x-x_0|\le 1/4} |u(s,x)|^2dx + \int_{Q_2} |\nabla u(t,x)|^2dxdt \le C\int_{Q_1(0,0)} (|u|^2 + |u|^3 + |u||p| + |u||f|)dxdt. \quad (8.70)$$

Now, we may use Young's inequality on the term $|u||f|$, together with the L^q-bound on f, with $q > 5/2$, the Hölder inequality, as well as our assumption (8.64), to conclude that the left side of (8.70) is smaller than $C\varepsilon_1^{2/3}$, provided that ε_1 and ε_2 are both sufficiently small. Thus, (8.67) holds for $k = 2$.

The induction step. The induction step in the proof of (8.67) will be split into two sub-steps. First, we will show that if (8.67) holds for all $2 \le k \le n - 1$, and $n \ge 3$, then we have

$$\frac{1}{|Q_n|}\int_{Q_n} |u|^3dxdt + \frac{r_n^{3/5}}{|Q_n|}\int_{Q_n} |u||p - \bar{p}_n|dxdt \le \varepsilon_1^{2/3}. \quad (8.71)$$

Next, we will show that if (8.71) holds for all $3 \le k \le n$, then (8.67) holds for $k = n$. That is, we have the following two lemmas.

Lemma 8.17 *Assume that ε_1 and ε_2 are sufficiently small, and $n \ge 3$, and (8.67) holds for all $2 \le k \le n - 1$, then (8.71) holds.*

Lemma 8.18 *Assume that (8.71) holds for all $3 \le k \le n$, and ε_1 and ε_2 are sufficiently small, then (8.67) holds for $k = n$.*

The proof of these lemmas is the heart of the argument.

The proof of Lemma 8.17

We set

$$A(r) = \sup_{s-r^2<t<s} \frac{1}{r}\int_{B_r(x_0)} |u(t,x)|^2dx, \qquad G(r) = \frac{1}{r^2}\int_{Q_r} |u|^3dxdt,$$

and

$$\delta(r) = \frac{1}{r}\int_{Q_r} |\nabla u(t,x)|^2dxdt, \quad K(r) = \frac{1}{r^{13/4}}\int_{s-r^2}^{s}\Big(\int_{B_r(x_0)} |p|dx\Big)^{5/4}dt.$$

Recalling that the dimension of u is $1/L$, and the dimension of t is L^2, while the dimension of p is $1/L^2$, we see that, $A(r)$, $G(r)$, $K(r)$ and $\delta(r)$ are all dimensionless. The induction hypothesis is

$$A(r_k) + \delta(r_k) \le C\varepsilon_1^{2/3}r_k^2, \quad 2 \le k \le n - 1. \quad (8.72)$$

In addition, we know that

$$G(r_1) + K(r_1) \leq C\varepsilon_1, \tag{8.73}$$

which is part of (8.64).

Bound on the first term in (8.71). The two terms in the left side of (8.71) will be estimated separately. We will extensively use the Gagliardo-Nirenberg inequality in a ball

$$\int_{B_r} |u|^q dx \leq C\Big(\int_{B_r} |\nabla u|^2 dx\Big)^a \Big(\int_{B_r} |u|^2\Big)^{q/2-a} + \frac{C}{r^{2a}}\Big(\int_{B_r} |u|^2 dx\Big)^{q/2}, \tag{8.74}$$

with $2 \leq q \leq 6$, and $a = 3(q-2)/4$ – this is the only choice of a which makes (8.74) dimensionally correct. Taking $q = 3$ and $a = 3/4$ gives a bound on the L^3-norm that appears in the left side of (8.71):

$$\int_{B_r} |u|^3 dx \leq C\Big(\int_{B_r} |\nabla u|^2 dx\Big)^{3/4} \Big(\int_{B_r} |u|^2\Big)^{3/4} + \frac{C}{r^{3/2}}\Big(\int_{B_r} |u|^2 dx\Big)^{3/2}. \tag{8.75}$$

Integrating in time and using Hölder's inequality leads to

$$\begin{aligned} \int_{Q_r} |u|^3 dxdt &\leq C\int_{s-r^2}^{s} \Big(\int_{B_r} |\nabla u|^2 dx\Big)^{3/4} \Big(\int_{B_r} |u|^2 dx\Big)^{3/4} dt + \frac{C}{r^{3/2}}\int_{s-r^2}^{s} \Big(\int_{B_r} |u|^2 dx\Big)^{3/2} dt \\ &\leq C\Big(\int_{Q_r} |\nabla u|^2 dxdt\Big)^{3/4} \Big(\int_{s-r^2}^{s} \Big(\int_{B_r} |u|^2 dx\Big)^{3} dt\Big)^{1/4} + \frac{C}{r^{3/2}}\int_{s-r^2}^{s} \Big(\int_{B_r} |u|^2 dx\Big)^{3/2} dt \\ &\leq C\Big(r\delta(r)\Big)^{3/4} r^{1/2}[rA(r)]^{3/4} + Cr^{1/2}[rA(r)]^{3/2} = Cr^2 A(r)^{3/4}[\delta(r)^{3/4} + A(r)^{3/4}]. \end{aligned} \tag{8.76}$$

Dividing by $|Q_r|$ gives

$$\begin{aligned} \frac{1}{|Q_{r_{n-1}}|}\int_{Q_{r_{n-1}}} |u|^3 dxdt &\leq \frac{C}{r_{n-1}^5}\int_{Q_{r_{n-1}}} |u|^3 dxdt \leq \frac{C}{r_{n-1}^3} A(r_{n-1})^{3/4}[\delta(r_{n-1})^{3/4} + A(r_{n-1})^{3/4}] \\ &\leq \frac{C}{r_{n-1}^3}(A(r_{n-1}) + \delta(r_{n-1}))^{3/2} \leq C\varepsilon_1, \end{aligned} \tag{8.77}$$

which, in turn, means that

$$\frac{1}{|Q_{r_n}|}\int_{Q_{r_n}} |u|^3 dxdt \leq \frac{C'}{|Q_{r_{n-1}}|}\int_{Q_{r_{n-1}}} |u|^3 dxdt \leq C''\varepsilon_1. \tag{8.78}$$

Hence, if ε_1 is so small that

$$C''\varepsilon^{1/3} \leq \frac{1}{2},$$

then

$$\frac{1}{|Q_{r_n}|}\int_{Q_{r_n}} |u|^3 dxdt \leq \frac{1}{2}\varepsilon_1^{2/3}. \tag{8.79}$$

This is the estimate we need on the first term in the left side of (8.71). Note that (8.78) can be also restated as

$$G(r_n) \leq C\varepsilon_1 r_n^3. \tag{8.80}$$

Bound on the second term in (8.71). In order to get a bound on the second term in the left side of (8.71), we need to show that, under the assumption

$$A(r_k) + \delta(r_k) \le C\varepsilon_1^{2/3} r_k^2, \quad 2 \le k \le n-1, \tag{8.81}$$

we have

$$\frac{r_n^{3/5}}{|Q_n|} \int_{Q_n} |u||p - \bar{p}_n| dx dt \le \frac{\varepsilon_1^{2/3}}{2}, \tag{8.82}$$

provided that ε_1 is sufficiently small. The main issue is bounding the pressure. Recall that p satisfies the Poisson equation

$$-\Delta p = \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j). \tag{8.83}$$

For any cut-off function ϕ we can write

$$\phi(x)p(t,x) = -\frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\Delta_y(\phi p) dy = -\frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}(p\Delta\phi + 2\nabla\phi\cdot\nabla p + \phi\Delta p) dy.$$

Using (8.83) and integrating by parts, we may write the above as

$$\phi p = p_1 + p_2 + p_3,$$

where

$$\begin{aligned}
p_1 &= \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{\partial^2}{\partial y_i \partial y_j}\Big[\frac{1}{|x-y|}\Big]\phi u_i u_j dy,\\
p_2 &= \frac{3}{2\pi}\int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3}\frac{\partial \phi}{\partial y_j} u_i u_j dy + \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\frac{\partial^2 \phi}{\partial y_i \partial y_j} u_i u_j dy,\\
p_3 &= \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} p\Delta\phi dy + \frac{3}{2\pi}\int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3} p \frac{\partial \phi}{\partial y_j} dy.
\end{aligned}$$

We will take a function ϕ so that $\phi(y) \equiv 1$ for $|y - x_0| \le 3/16$ and $\phi(y) = 0$ if $|y - x_0| \ge 1/4$. Let us split p_1 as

$$p_1 = p_{11} + p_{12},$$

with

$$\begin{aligned}
p_{11} &= \frac{3}{4\pi}\int_{|y-x_0|<2r_n} \frac{\partial^2}{\partial y_i \partial y_j}\Big[\frac{1}{|x-y|}\Big]\phi u_i u_j dy,\\
p_{12} &= \frac{3}{4\pi}\int_{|y-x_0|>2r_n} \frac{\partial^2}{\partial y_i \partial y_j}\Big[\frac{1}{|x-y|}\Big]\phi u_i u_j dy.
\end{aligned}$$

We can write (dropping the subscript n for the moment)

$$|p - \bar{p}| \le |p_{11} - \bar{p}_{11}| + |p_{12} - \bar{p}_{12}| + |p_3 - \bar{p}_3| + |p_4 - \bar{p}_4|.$$

To estimate p_{11}, recall that the operators

$$T_{ij}(\psi) = \Big(\nabla^2_{ik}\frac{1}{|x|}\Big) \star \psi$$

are Calderon-Zygmund operators, hence they are uniformly bounded in L^q, $1 < q < \infty$. It follows that (we denote $r = r_n$ and $B_r = B_{r_n}(x_0)$)

$$\|p_{11}\|_{L^{3/2}(B_r)} \le C\Big(\int_{B_{2r}} |u|^3 dx\Big)^{2/3},$$

and

$$\bar{p}_{11} \le \frac{1}{|B_r|}\int_{B_r} |p| dx \le \frac{1}{|B_r|^{2/3}}\Big(\int_{B_r} |p|^{3/2} dx\Big)^{2/3},$$

hence

$$\int_{B_r} |\bar{p}_{11}|^{3/2} dx \le \int_{B_r} |p|^{3/2} dx.$$

We conclude that

$$\int_{B_r} |u||p_{11} - \bar{p}_{11}| dx \le C\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\Big(\int_{B_{2r}} |u|^3 dx\Big)^{2/3}. \tag{8.84}$$

The terms $|p_i - \bar{p}_i|$ for p_{12}, p_2 and p_3 are estimated using the following bounds on the gradients ∇p_i for $|x - x_0| < r$ (recall that $\phi \equiv 1$ in the ball $B_{3/16}(x_0)$ so that $\nabla\phi = 0$ in that ball):

$$\begin{aligned}
&|\nabla p_{12}(x)| \le C\int_{2r<|y-x_0|<1/4} \frac{|u|^2}{|y-x|^3} dy \le C\int_{2r<|y-x_0|<1/4} \frac{|u|^2}{|y-x_0|^3} dy,\\
&|\nabla p_2(x)| \le C\int_{B_{1/4}(x_0)} |u|^2 dy,\\
&|\nabla p_3(x)| \le C\int_{B_{1/4}(x_0)} |p| dy.
\end{aligned}$$

This leads to

$$\begin{aligned}
\int_{B_r} |u||p_{12} - \bar{p}_{12}| &\le Cr[\sup_{x\in B_r} |\nabla p_{12}(x)|](r^3)^{2/3}\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\\
&\le Cr^3\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\int_{2r<|y-x_0|<1/4} \frac{|u|^2}{|y-x_0|^3} dy,
\end{aligned} \tag{8.85}$$

and

$$\int_{B_r} |u||p_2 - \bar{p}_2| \le Cr[\sup_{x\in B_r} |\nabla p_2(x)|](r^3)^{2/3}\Big(\int_{B_r} |u|^3 dx\Big)^{1/3} \tag{8.86}$$

$$\le Cr^3\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\int_{B_{1/4}(x_0)} |u|^2 dy \le Cr^3\Big(\int_{B_r} |u|^3 dx\Big)^{1/3}\Big(\int_{B_{1/4}(x_0)} |u|^3 dy\Big)^{2/3}.$$

For p_3, we write

$$\begin{aligned}
\int_{B_r} |u||p_3 - \bar{p}_3| &\le Cr\Big(\int_{B_r} |u| dy\Big)\Big(\int_{B_{1/4}(x_0)} |p|\Big) \qquad (8.87)\\
&\le Cr(r^3)^{3/5}\Big(\int_{B_r} |u|^2 dy\Big)^{1/5}\Big(\int_{B_r} |u|^3 dy\Big)^{1/5}\Big(\int_{B_{1/4}(x_0)} |p|\Big)\\
&\le Cr^3 A(r)^{1/5}\Big(\int_{B_r} |u|^3 dy\Big)^{1/5}\Big(\int_{B_{1/4}(x_0)} |p|\Big).
\end{aligned}$$

Integrating the above estimates over the time interval $s - r^2 \le t \le s$, and collecting all the terms we get

$$\int_{Q_r} |u||p - \bar{p}_r| dx dt \le W_1 + W_2 + W_3 + W_4. \tag{8.88}$$

The term

$$W_1 = C\Big(\int_{Q_r} |u|^3 dx dt\Big)^{1/3}\Big(\int_{Q_{2r}} |u|^3 dx dt\Big)^{2/3} = Cr^2 G(r)^{1/3} G(2r)^{2/3} \tag{8.89}$$

comes from (8.84) and using Hölder's inequality. Using (8.80), W_1 can be bounded as

$$W_1 \le C\varepsilon_1 r_n^2 r_n^3 = C\varepsilon_1 r_n^5. \tag{8.90}$$

The second term arises from (8.85) and also using Hölder's inequality (note that $13/3 = 3 + 2(2/3)$),

$$W_2 = Cr^{13/3}\Big(\int_{Q_r} |u|^3 dx dt\Big)^{1/3} \sup_{s-r^2<t<s} \int_{2r<|y-x_0|<1/4} \frac{|u(t,y)|^2}{|y - x_0|^3} dy. \tag{8.91}$$

Note that for $r = r_n = 2^{-n}$, the last factor in (8.91) can be estimated with the help of the induction hypothesis (8.81) as

$$\begin{aligned}
&\int_{2r_n<|y-x_0|<1/4} \frac{|u(t,y)|^2}{|y - x_0|^3} dy \le \sum_{k=3}^{n-1} \int_{2^{-k}<|y-x_0|<2^{-(k-1)}} \frac{|u(t,y)|^2}{|y - x_0|^3} dy \\
&\le \sum_{k=3}^{n-1} 2^{3k} \int_{2^{-k}<|y-x_0|<2^{-(k-1)}} |u(t,y)|^2 dy \le \sum_{k=3}^{n-1} r_k^{-3} A(r_{k-1}) \le C\varepsilon_1^{2/3} \sum_{k=3}^{n-1} r_k^{-1} \le \frac{C\varepsilon_1^{2/3}}{r_n}.
\end{aligned}$$

Using this inequality, together with (8.80) in (8.91) gives

$$W_2 \le Cr_n^{13/3}(r_n^2 G(r_n))^{1/3} \frac{\varepsilon_1^{2/3}}{r_n} \le Cr_n^4 G(r_n)^{1/3} \varepsilon_1^{2/3} \le Cr_n^5 \varepsilon_1. \tag{8.92}$$

The third term

$$W_3 = Cr^3\Big(\int_{Q_r} |u|^3 dx dt\Big)^{1/3}\Big(\int_{Q_{1/4}} |u|^3 dx dt\Big)^{2/3} \tag{8.93}$$

comes from (8.86) and, of course, using Hölder's inequality once again, and can be bounded with the help of (8.80) as

$$W_3 \le Cr_n^3 (r_n^2 G(r_n)))^{1/3} G(1/4)^{2/3} \le Cr_n^{14/3} \varepsilon_1. \tag{8.94}$$

Finally, the last term in (8.88) comes from (8.87):

$$W_4 = Cr^3 A(r)^{1/5}\Big(\int_{Q_r} |u|^3 dx dt\Big)^{1/5}\Big(\int_{-1/16}^{0}\Big(\int_{B_{1/4}} |p| dx\Big)^{5/4} dt\Big)^{4/5}. \tag{8.95}$$

It can be bounded as (assuming that $\varepsilon_1 \le 1$):

$$W_4 \le Cr_n^3 A(r_n)^{1/5}(r_n^2 G(r_n))^{1/5}\varepsilon_1^{4/5} \le Cr_n^3(r_n^2\varepsilon_1^{2/3})^{1/5}(r_n^5\varepsilon_1)^{1/5}\varepsilon_1^{4/5} \le Cr_n^{22/5}\varepsilon_1. \tag{8.96}$$

Altogether, we conclude that

$$\int_{Q_n} |u||p-\bar{p}_{r_n}|dxdt \le Cr_n^{22/5}\varepsilon_1. \tag{8.97}$$

We conclude that

$$\frac{r_n^{3/5}}{|Q_n|}\int_{Q_n} |u||p-\bar{p}_{r_n}|dxdt \le C\varepsilon_1 \le \frac{\varepsilon_1^{2/3}}{2}, \tag{8.98}$$

provided that ε_1 is small enough. This bounds the second term in (8.71) and finishes the proof of Lemma 8.17.

Proof of Lemma 8.18

We now assume that

$$\frac{1}{|Q_k|}\int_{Q_k} |u|^3dxdt + \frac{r_k^{3/5}}{|Q_k|}\int_{Q_k} |u||p-\bar{p}_n|dxdt \le \varepsilon_1^{2/3}, \tag{8.99}$$

for all $3 \le k \le n$, and show that then

$$\sup_{s-r_n^2<t\le s} \fint_{|x-x_0|\le r_n} |u(t,x)|^2dx + \frac{1}{r_n^3}\int_{Q_n} |\nabla u(t,x)|^2dxdt \le C_0\varepsilon_1^{2/3}. \tag{8.100}$$

We will shift the origin so that $(s,x_0)=(0,0)$, to simplify the notation. The idea is to use the generalized energy inequality

$$\begin{aligned}\int_{B_1} |u(s,x)|^2\phi(s,x)dx + 2\int_{-1}^s\int_{B_1} |\nabla u(t,x)|^2\phi(t,x)dxdt \le \int_{-1}^s\int_{B_1} |u(t,x)|^2(\phi_t+\Delta\phi)dxdt\\ +\int_{-1}^s\int_{B_1}(|u|^2+2p)u\cdot\nabla\phi(t,x)dtdx + 2\int_{-1}^s\int_{B_1}(f\cdot u)\phi(t,x)dxdt,\end{aligned} \tag{8.101}$$

with a suitable test function ϕ_n. We will set

$$\phi_n(t,x) = \chi(x)\psi_n(t,x),$$

with the backward heat kernel

$$\psi_n(t,x) = \frac{1}{(r_n^2-t)^{3/2}}\exp\Big\{-\frac{|x|^2}{4(r_n^2-t)}\Big\},$$

and a smooth function $\chi(x)\ge 0$ so that $\chi(x)\equiv 1$ on $Q_2 = Q_{1/4}(0,0)$ and $\chi = 0$ outside of $Q_{1/3}(0,0)$. Then we have

$$\frac{\partial\phi_n}{\partial t} + \Delta\phi_n = 0, \quad \text{on } Q_2,$$

and

$$\Big|\frac{\partial\phi_n}{\partial t}+\Delta\phi_n\Big|\le C,\quad \text{everywhere},$$

and the following bounds hold:

$$\frac{1}{Cr_n^3}\le\phi_n\le\frac{C}{r_n^3},\quad |\nabla\phi_n|\le\frac{C}{r_n^4},\quad \text{on } Q_n,\ n\ge 2 \tag{8.102}$$

and

$$\frac{1}{Cr_k^3}\le\phi_n\le\frac{C}{r_k^3},\quad |\nabla\phi_n|\le\frac{C}{r_k^4},\quad \text{on } Q_{k-1}\setminus Q_k,\ n\ge 2. \tag{8.103}$$

We may now insert this ϕ_n into (8.101), and use the lower bound for ϕ_n on Q_n to get

$$\begin{aligned}&\sup_{-r_n^2\le t\le 0}\frac{1}{r_n^3}\int_{|x|<r_n}|u(t,x)|^2dx+\frac{1}{r_n^3}\int_{Q_n}|\nabla u|^2dxdt\le C\int_{Q_1}|u|^2|\frac{\partial\phi_n}{\partial t}+\Delta\phi_n|dxdt\\ &+C\int_{Q_1}|u|^3|\nabla\phi_n|dtdx+C\Big|\int_{Q_1}p(u\cdot\nabla\phi_n)dtdx\Big|+C\int_{Q_1}^s|f||u||\phi|dxdt\\ &=C(I_1+I_2+I_3+I_4).\end{aligned} \tag{8.104}$$

To estimate I_1 we simply use Hölder's inequality:

$$|I_1|\le C\int_{Q_1}|u|^2dxdt\le C\Big(\int_{Q_1}|u|^3dxdt\Big)^{2/3}\le C\varepsilon_1^{2/3}. \tag{8.105}$$

The second term is estimated as

$$|I_2|\le C\sum_{k=1}^n\frac{1}{r_k^4}\int_{Q_k}|u|^3dxdt\le C\sum_{k=1}^n\frac{1}{r_k^4}\varepsilon_1^{2/3}r_k^5\le C\varepsilon_1^{2/3}. \tag{8.106}$$

The last term in (8.104) is also easy:

$$\begin{aligned}|I_4|&\le C\sum_{k=1}^n\frac{1}{r_k^3}\int_{Q_k}|u||f|dxdt\le C\sum_{k=1}^n\frac{1}{r_k^3}\Big(\int_{Q_k}|u|^3\Big)^{1/3}\Big(\int_{Q_k}|f|^{3/2}\Big)^{2/3}\\ &\le C\sum_{k=1}^n\frac{1}{r_k^3}(\varepsilon_1^{2/3}r_k^5)^{1/3}\|f\|_{L^q(Q_1)}r_k^{10/3-5/q}\le C\varepsilon_2^{1/q}\varepsilon_1^{2/9}\sum_{k=1}^n r_k^{2-5/q}\le C\varepsilon_2^{1/q}\varepsilon_1^{2/9},\end{aligned} \tag{8.107}$$

as $q>5/2$. Therefore, if ε_2 is sufficiently small, we have

$$|I_4|\le C\varepsilon_1^{2/3}. \tag{8.108}$$

Finally, we deal with I_3. Here, we will use the condition that u is a divergence-free flow. Let us take smooth functions $0\le\chi_k\le 1$ such that $\chi_k\equiv 1$ on $Q_{7r_k/8}$, and $\chi_k\equiv 0$ outside of Q_{r_k}, and

$$|\nabla\chi_k|\le\frac{C}{r_k}.$$

Then, as $\chi_1\phi_n = \phi_n$, we can write I_3 as a telescoping sum:

$$I_3 = \int_{Q_1} p(u \cdot \nabla\phi_n) dtdx = \sum_{k=1}^{n-1} \int_{Q_1} pu \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) + \int_{Q_1} pu \cdot (\chi_n\phi_n). \qquad (8.109)$$

Since u is divergence-free, and $\chi_k - \chi_{+1}$ vanishes outside of Q_k, we can write for $k \geq 3$:

$$\int_{Q_1} pu \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) = \int_{Q_k} pu \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) = \int_{Q_k} (p - \bar{p}_k)u \cdot \nabla((\chi_k - \chi_{k+1})\phi_n).$$

For $k = 1, 2$ we simply have

$$\Big| \int_{Q_1} pu \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) \Big| \leq c \int_{Q_1} |p||u| \leq C\varepsilon_1^{2/3},$$

while for the last term in (8.109) we have

$$\int_{Q_1} pu \cdot (\chi_n\phi_n) = \int_{Q_n} (p - \bar{p}_n)u \cdot \nabla(\chi_n\phi_n).$$

Putting these together, we have

$$I_3 \leq C\varepsilon_1^{2/3} + C\sum_{k=3}^{n} \frac{1}{r_k^4} \int_{Q_k} |p - \bar{p}_k||u| \leq C\varepsilon_1^{2/3} + C\sum_{k=3}^{n} \frac{1}{r_k^4} \varepsilon_1^{2/3} r_k^{5-3/5} \leq C\varepsilon_1^{2/3}. \qquad (8.110)$$

This finishes the proof of Lemma 8.18, and thus that of Proposition 8.4.