Lecture notes for Math 227

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1 The Haar functions and the Brownian motion

Prologue: a probabilistic interpretation for a difference equation

A good way to understand how the probabilistic interpretation of some PDEs comes about, is to start with the discrete equations. Consider the finite difference analog of the Laplace equation:

$$u(x+1,y)+u(x-1,y)+u(x,y+1)+u(x,y-1)-4u(x,y)=0, \tag{1.1}$$

which is the discrete analog of the Laplace equation

$$-\Delta u = 0,$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

in two dimensions. In order to formulate a (discrete) boundary value problem, let U be a domain of the two-dimensional square lattice $\mathbb{Z}^2$, and let $u(x,y)$ solve the difference equation (1.1), with the boundary condition $u(x,y)=g(x,y)$ on the boundary ∂U. Here $g(x,y)$ is a prescribed non-negative function, which is positive somewhere.

We claim that the solution of this problem has the following probabilistic interpretation. Let $(X(t),Y(t))$ be the standard random walk on the lattice $\mathbb{Z}^2$ – the probability to go up down, left of right is equal to $1/4$, and let it start at the point (x,y): $X(0)=x$, $Y(0)=y$. Let $(\bar{x},\bar{y})$ be the first point where $(X(t),Y(t))$ reaches the boundary ∂U of the domain. The point $(\bar{x},\bar{y})$ is, of course, random. The beautiful observation is that the function $v(x,y)=\mathbb{E}(g(\bar{x},\bar{y})$ gives a solution of (1.1), connecting this discrete problem to the random walk. Why? First, it is immediate that if (x,y) is on the boundary then, of course, $\bar{x}=x$ and $\bar{y}=y$, so $v(x,y)=g(x,y)$ in that case. On the other hand, if (x,y) is inside U then

$$v(x,y)=\frac{1}{4}(v(x+1,y)+v(x-1,y)+v(x,y+1)+v(x,y-1))$$

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simply from the definition of the random walk, and the definition of $v(x, y)$ – we can go in any of the four possible directions with the probability equal to 1/4 and then we start afresh.

Now, if we let the mesh size be not 1 but $h > 0$ and let $h \downarrow 0$, the discrete equation (1.1) becomes the Laplace equation, while the random walk becomes the Brownian motion. More precisely, solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in U, \tag{1.2}$$

with the boundary condition $u(x, y) = g(x, y)$ for $(x, y) \in \partial U$, has the following probabilistic interpretation. Consider a Brownian motion $B(t; x, y)$ that starts at a point $(x, y) \in U$ and let $(\bar{x}, \bar{y})$ be a (random) point where $B(t; x)$ hits the boundary ∂U for the first time. Then solution of (1.2) is $u(x) = \mathbb{E}(g(\bar{x}))$.

From the heuristic point of view, we can deduce immediately some properties of the Laplace equation. For example, if $g(x, y) > 0$ in some open subset $S \subset U$, and $g(x, y) \geq 0$ for all $(x, y) \in \partial U$, then with a positive probability we have $(\bar{x}, \bar{y}) \in S$ for all $(x, y) \in U$ (this is not really obvious but is true), which means that $u(x, y) = \mathbb{E}(g(\bar{x}, \bar{y})) > 0$ for all $(x, y) \in U$.

It is also easy to deduce the maximum principle from the probabilistic interpretation: it is easy to see that $\mathbb{E}(g(\bar{x}, \bar{y})) \leq \sup_{(x,y) \in \partial U} g(x, y)$ – expected value of a function can not exceed its maximum.

In order to really use such ideas we need some probabilistic background such as what is the Brownian motion, and this is how we will start – with a construction of the Brownian motion on the real line. This will be done using the Haar functions in a very explicit way. We will rarely prove purely probabilistic results in this course but this construction is so basic and explicit that it would be criminal to omit it.

1.1 The Haar functions and their completeness

The Haar functions

The basic Haar function is

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2, \\ -1 & \text{if } 1/2 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

It has mean zero

$$\int_0^1 \psi(x) dx = 0,$$

and is normalized so that

$$\int_0^1 \psi^2(x) dx = 1.$$

The rescaled and shifted Haar functions are

$$\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

These functions form an orthonormal set in $L^2(\mathbb{R})$ because if $j = j'$ and $k \neq k'$ then

$$\int_{\mathbb{R}} \psi_{jk}(x)\psi_{jk'}(x)dx = 2^j \int_{\mathbb{R}} \psi(2^j x - k)\psi(2^j x - k')dx = 0$$

because $\psi(y-k)\psi(y-k') = 0$ for any $y \in \mathbb{R}$ and $k \neq k'$. On the other hand, if $j \neq j'$, say, $j < j'$, then

$$\begin{aligned}
&\int_{\mathbb{R}} \psi_{jk}(x)\psi_{j'k'}(x)dx = 2^{j/2+j'/2} \int_{\mathbb{R}} \psi(2^j x - k)\psi(2^{j'} x - k')dx \\
&= 2^{j'/2-j/2} \int_{\mathbb{R}} \psi(y)\psi(2^{j'-j}y + 2^{j'-j}k - k')dy \\
&= 2^{j'/2-j/2} \int_0^{1/2} \psi(2^{j'-j}y + 2^{j'-j}k - k')dy - 2^{j'/2-j/2} \int_{1/2}^1 \psi(2^{j'-j}y + 2^{j'-j}k - k')dy.
\end{aligned}$$

Both of the integrals above equal to zero. Indeed, $2^{j'-j} \geq 2$, hence, for instance,

$$\int_0^{1/2} \psi(2^{j'-j}y + 2^{j'-j}k - k')dy = 2^{j-j'} \int_0^{2^{j'-j-1}} \psi(y + 2^{j'-j}k - k')dy = 0,$$

because

$$\int_m^n \psi(y)dy = 0,$$

for all $m, n \in \mathbb{Z}$, and $j' > j$. Finally, when $j = j'$, $k = k'$ we have

$$\int_{\mathbb{R}} |\psi_{jk}(x)|^2 = 2^j \int_{\mathbb{R}} |\psi(2^j x - k)|^2 dx = \int_{\mathbb{R}} |\psi(x-k)|^2 dx = 1.$$

The Haar coefficients of a function $f \in L^2(\mathbb{R})$ are defined as the inner products

$$c_{jk} = \int f(x)\psi_{jk}(x)dx, \tag{1.4}$$

and the Haar series of f is

$$\sum_{j,k \in \mathbb{Z}} c_{jk}\psi_{jk}(x). \tag{1.5}$$

Orthonormality of the family $\{\psi_{jk}\}$ ensures that

$$\sum_{j,k} |c_{jk}|^2 \leq \|f\|_{L^2}^2 < +\infty,$$

and the series (1.5) converges in $L^2(\mathbb{R})$. In order to show that it actually converges to the function f itself we need to prove that the Haar functions form a basis for $L^2(\mathbb{R})$.

Completeness of the Haar functions

To show that Haar functions form a basis in $L^2(\mathbb{R})$ we consider the dyadic projections P_n defined as follows. Given $f \in L^2(\mathbb{R})$, and $n, k \in \mathbb{Z}$, consider the intervals

$$I_{nk} = ((k-1)/2^n, k/2^n],$$

then

$$P_n f(x) = \fint_{I_{nk}} f dx = 2^n \int_{I_{nk}} f dx, \quad \text{for } x \in I_{nk}.$$

The function $P_n f$ is constant on each of the dyadic intervals I_{nk}. In particular, each Haar function ψ_{jk} satisfies $P_n \psi_{jk}(x) = 0$ for $j \geq n$, while $P_n \psi_{jk}(x) = \psi_{jk}(x)$ for $j < n$. We claim that, actually, for any $f \in L^2(\mathbb{R})$ we have

$$P_{n+1} f - P_n f = \sum_{k \in \mathbb{Z}} c_{nk} \psi_{nk}(x), \tag{1.6}$$

with the Haar coefficients c_{nk} given by (1.4). Indeed, let $x \in I_{nk}$ and write

$$I_{nk} = \Big(\frac{(k-1)}{2^n}, \frac{k}{2^n}\Big] = \Big(\frac{2(k-1)}{2^{n+1}}, \frac{(2k-1)}{2^{n+1}}\Big] \bigcup \Big(\frac{(2k-1)}{2^{n+1}}, \frac{2k}{2^{n+1}}\Big] = I_{n+1,2k-1} \bigcup I_{n+1,2k}.$$

The function $P_n f$ is constant on the whole interval I_{nk} while $P_{n+1} f$ is constant on each of the sub-intervals $I_{n+1,2k-1}$ and $I_{n+1,2k}$. In addition,

$$\int_{I_{nk}} (P_n f) dx = \int_{I_{nk}} (P_{n+1} f) dx.$$

This means exactly that

$$P_{n+1}(x) = P_n f(x) + c_{nk} \psi_{nk}(x) \text{ for } x \in I_{nk},$$

which is (1.6).

As a consequence of (1.6) we deduce that

$$P_{n+1} f(x) - P_{-m} f(x) = \sum_{j=-m}^{n} \sum_{k \in \mathbb{Z}} c_{jk} \psi_{jk}(x), \tag{1.7}$$

for all $m, n \in \mathbb{Z}$ with $n > m$. It remains to show that for any $f \in L^2(\mathbb{R})$ we have

$$\lim_{m \to +\infty} P_{-m} f(x) = 0, \qquad \lim_{n \to +\infty} P_n f(x) = f(x), \tag{1.8}$$

both in the L^2-sense. The operators $P_n f$ are uniformly bounded because for all $n, k \in \mathbb{Z}$ we have

$$\int_{I_{nk}} |(P_n f)(x)|^2 dx = 2^{-n} 2^{2n} \left| \int_{I_{nk}} f(y) dy \right|^2 \leq \int_{I_{nk}} |f(y)|^2 dy.$$

Summing over $k \in \mathbb{Z}$ for a fixed n we get

$$\int_{\mathbb{R}} |P_n f(x)|^2 \leq \int_{\mathbb{R}} |f(x)|^2,$$

thus $\|P_n f\|_{L^2} \le \|f\|_{L^2}$. Uniform boundedness of P_n implies that it is sufficient to establish both limits in (1.8) for functions $f \in C_c(\mathbb{R})$. However, for such f we have, on one hand,

$$|P_{-m} f(x)| \le \frac{1}{2^m} \int_{\mathbb{R}} |f(x)| dx \to 0 \text{ as } m \to +\infty,$$

and, on the other, f is uniformly continuous on $\mathbb{R}$, so that $\|P_n f(x) - f(x)\|_{L^\infty} \to 0$ as $n \to +\infty$, which, as both $P_n f$ and f are compactly supported, implies the second limit in (1.8). Therefore, ψ_{jk} form an orthonormal basis in $L^2(\mathbb{R})$ and every function $f \in L^2(\mathbb{R})$ has the reperesentation

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{jk} \psi_{jk}(x), \quad c_{jk} = \int_{\mathbb{R}} f(y) \psi_{jk}(y) dy. \tag{1.9}$$

Possible lack of convergence in $L^1(\mathbb{R})$

Let us note that the Haar series need not converge in $L^1(\mathbb{R})$. In particular, it can not be integrated term-wise to conclude that

$$\int_{\mathbb{R}} f(x) dx = 0,$$

which obviously can not be true for all $f \in L^2(\mathbb{R})$. Consider, for example, the function $f(x) = \chi_{[0,1]}(x)$, that is, $f(x) = 1$ for $x \in [0,1]$ and $f(x) = 0$ otherwise. Its Haar coefficients are

$$c_{jk} = \int_0^1 \psi_{jk}(x) dx = 2^{j/2} \int_0^1 \psi(2^j x - k) dx = 2^{-j/2} \int_0^{2^j} \psi(x-k) dx = 2^{-j/2} \int_{-k}^{-k+2^j} \psi(x) dx.$$

We see immediately that $c_{jk} = 0$ for $k \ne 0$, and also for $j \ge 0$. On the other hand, for $j < 0$ and $k = 0$ we have $c_{j,0} = 2^{j/2}$. Therefore, the Haar series for the function f is

$$\sum_{j<0} 2^j \psi(2^j x) = \sum_{j>0} \frac{1}{2^j} \psi(\frac{x}{2^j}).$$

The partial sums

$$S_N(x) = \sum_{j=1}^{N} \frac{1}{2^j} \psi(\frac{x}{2^j})$$

look as follows:

$$S_N(x) = \sum_{j=1}^{N} \frac{1}{2^j} = 1 - \frac{1}{2^N}, \qquad \text{for } x \in [0,1],$$

and,

$$S_N(x) = \sum_{j=2}^{N} \frac{1}{2^j} - \frac{1}{2} = -\frac{1}{2^N} \qquad \text{for } x \in [1, 2^N],$$

while $S_N(x) = 0$ for $x > 2^N$. The function $S_N(x)$ has tails that are not uniformly integrable, and the series does not converge to $f(x) = \chi_{[0,1]}(x)$ in L^1. However, we have

$$\int_{\mathbb{R}} |S_N(x) - 1|^2 dx = \frac{1}{2^N} \to 0,$$

hence the series converges in $L^2(\mathbb{R})$, as it should. This is a general phenomenon – the Haar series converges in L^p for $1 < p < \infty$ but not in L^1 since it has fat tails that decay only as $1/x$, which is not sufficient for the L^1-convergence.

1.2 The Brownian motion

Brownian motion is a random process $X_t(\omega)$, $t \geq 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which has the following properties:

(i) The function $X_t(\omega)$ is continuous in t for a.e. realization ω.

(ii) For all $0 \leq s < t < +\infty$ the random variable $X_t(\omega) - X_s(\omega)$ is Gaussian with mean zero and variance $t - s$:

$$\mathbb{E}(X(t) - X(s)) = 0, \quad \mathbb{E}(X(t) - X(s))^2 = t - s.$$

(iii) For any subdivision $0 = t_0 < t_1 < \ldots < t_N = t$ of the interval $[0, t]$, the random variables $X_{t_1} - X_{t_0}, \ldots, X_{t_N} - X_{t_{N-1}}$ are independent.

Construction of the Brownian motion

We will construct the Brownian motion on the interval $0 \leq t \leq 1$ – the restriction to a finite interval is a simple convenience but by no means a necessity. The Haar functions $\psi_{jk}(x)$, with $j \geq 0$, $0 \leq k \leq 2^j - 1$, form a basis for the space $L^2[0,1]$. Let us denote accordingly $\phi_n(x) = \psi_{jk}(x)$ for $n = 2^j + k$, $0 \leq k \leq 2^j - 1$, and $\phi_0(x) = 1$ so that $\{\phi_n\}$ form an orthonormal basis for $L^2[0,1]$. Let $Z_n(\omega)$, $n \geq 0$, be a collection of independent Gaussian random variables of mean zero and variance one, that is,

$$P(Z_n < y) = \int_{-\infty}^{y} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}.$$

We will show that the process

$$X_t(\omega) = \sum_{n=0}^{\infty} Z_n(\omega) \int_0^t \phi_n(s) ds \tag{1.10}$$

is a Brownian motion.

First, we need to verify that the series (1.10) converges in $L^2(\Omega)$ for a fixed $t \in [0,1]$. Note that

$$\mathbb{E}\left(\sum_{k=n}^{m} Z_k(\omega) \int_0^t \phi_k(s) ds\right)^2 = \sum_{k=n}^{m} \left(\int_0^t \phi_k(s) ds\right)^2 = \sum_{k=n}^{m} \langle \chi_{[0,t]}, \phi_k \rangle^2.$$

As ϕ_k form a basis for $L^2[0,1]$, it follows that the series (1.10) satisfies the Cauchy criterion and thus converges in $L^2(\Omega)$. Moreover, for any $0 \le s < t \le 1$ we have

$$\mathbb{E}\left(X_t - X_s\right)^2 = \mathbb{E}\left(\sum_{k=0}^{\infty} Z_k(\omega)\int_s^t \phi_k(u)du\right)^2 = \sum_{k=0}^{\infty}\left(\int_s^t \phi_k(u)du\right)^2 = \sum_{k=0}^{\infty}\langle\chi_{[s,t]},\phi_k\rangle^2$$
$$= \|\chi_{[s,t]}\|_{L^2}^2 = t - s,$$

hence the increments $X_t - X_s$ have the correct variance. Let us show that they are independent: for $0 \le t_0 < t_1 \le t_2 < t_3 \le 1$:

$$\mathbb{E}\left((X_{t_3} - X_{t_2})(X_{t_1} - X_{t_0})\right) = \mathbb{E}\left(\sum_{k=0}^{\infty}\int_{t_2}^{t_3}\phi_k(u)du\int_{t_0}^{t_1}\phi_k(u')du'\right)$$
$$= \sum_{k=0}^{\infty}\langle\chi_{[t_2t_3]},\phi_k\rangle\langle\chi_{[t_0t_1]},\phi_k\rangle = \langle\chi_{[t_2t_3]},\chi_{[t_0t_1]}\rangle = 0.$$

As the variables $X_t - X_s$ are jointly Gaussian, independence of the increments follows.

Continuity of the Brownian motion

In order to prove continuity of the process $X_t(\omega)$ defined by the series (1.10) we show that the series converges uniformly in t almost surely in ω. To this end let us show that

$$M(\omega) = \sup_n \frac{|Z_n(\omega)|}{\sqrt{\log n}} < +\infty \text{ almost surely in } \omega. \tag{1.11}$$

Note that, for each $n \ge 0$:

$$\mathbb{P}\left(|Z_n(\omega)| \ge 2\sqrt{\log n}\right) \le e^{-(2\sqrt{\log n})^2/2} = \frac{1}{n^2},$$

thus

$$\sum_{n=0}^{\infty}\mathbb{P}\left(|Z_n(\omega)| \ge 2\sqrt{\log n}\right) < +\infty.$$

The Borel-Cantelli lemma implies that almost surely the event $\left\{|Z_n(\omega)| \ge 2\sqrt{\log n}\right\}$ happens only finitely many times, so that $|Z_n(\omega)| < 2\sqrt{\log n}$ for al $n \ge n_0(\omega)$ almost surely, and (1.11) follows.

Another useful observation is that for each fixed $t \ge 0$ and $j \in \mathbb{N}$ there exists only one k so that

$$\int_0^t \phi_{2^j+k}(s)ds \ne 0,$$

and for that k we have

$$\left|\int_0^t \phi_{2^j+k}(s)ds\right| \le 2^{j/2}2^{-j} = \frac{1}{2^{j/2}}.$$

Hence, we may estimate the dyadic blocs, using (1.11):

$$\left|\sum_{k=0}^{2^j-1} Z_{2^j+k}(\omega)\int_0^t \phi_{2^j+k}(s)ds\right| \le M(\omega)\sqrt{(j+1)\log 2}\sum_{k=0}^{2^j-1}\left|\int_0^t \psi_{jk}(s)ds\right| \le \frac{\sqrt{j}M_1(\omega)}{2^{j/2}}.$$

Therefore, the dyadic blocs are bounded by a convergent series which does not depend on $t \in [0,1]$, hence the sum $X_t(\omega)$ of the series is a continuous function for a.e. ω.

Nowhere differentiability of the Brownian motion

Theorem 1.1 *The Brownian path $X_t(\omega)$ is nowhere differentiable for almost every ω.*

Proof. Let us fix $\beta > 0$. Then if $\dot{X}_s$ exists at some $s \in [0,1]$ and $|\dot{X}_s| < \beta$ then there exists n_0 so that

$$|X_t - X_s| \le 2\beta|t-s| \text{ if } |t-s| \le \frac{2}{n} \tag{1.12}$$

for all $n > n_0$. Let A_n be the set of functions $x(t) \in C[0,1]$ for which (1.12) holds for some $s \in [0,1]$. Then $A_n \subset A_{n+1}$ and the set $A = \bigcup_{n=1}^{\infty} A_n$ includes all functions $x(t) \in C[0,1]$ such that $|\dot{x}(s)| \le \beta$ at some point $s \in [0,1]$.

The next step is to replace (1.12) by a discrete set of conditions – this is a standard trick in such situations. Assume that (1.12) holds for a function $x(t) \in C[0,1]$ and let $k = \sup\{j : \ j/n \le s\}$, then

$$y_k = \max\left(\left|x\left(\frac{k+2}{n}\right) - x\left(\frac{k+1}{n}\right)\right|, \left|x\left(\frac{k+1}{n}\right) - x\left(\frac{k}{n}\right)\right|, \left|x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right|\right) \le \frac{8\beta}{n}.$$

Therefore, if we denote by B_n the set of all functions $x(t) \in C[0,1]$ for which $y_k \le 8\beta/n$ for some k, then $A_n \subseteq B_n$. Therefore, in order to show that $\mathbb{P}(A) = 0$ it suffices to check that

$$\lim_{n\to\infty} \mathbb{P}(B_n) = 0. \tag{1.13}$$

This, however, can be estimated directly, using translation invariance of the Brownian motion:

$$\begin{aligned}
\mathbb{P}(B_n) &\le \sum_{k=1}^{n-2} \mathbb{P}\left[\max\left[\left|X(\frac{k+2}{n}) - X(\frac{k+1}{n})\right|, \left|X(\frac{k+1}{n}) - X(\frac{k}{n})\right|, \left|X(\frac{k}{n}) - X(\frac{k-1}{n})\right|\right] \le \frac{8\beta}{n}\right] \\
&\le n\mathbb{P}\left[\max\left[\left|X\left(\frac{3}{n}\right) - X\left(\frac{2}{n}\right)\right|, \left|X\left(\frac{2}{n}\right) - X\left(\frac{1}{n}\right)\right|, \left|X\left(\frac{1}{n}\right)\right|\right] \le \frac{8\beta}{n}\right] \\
&= n\mathbb{P}\left[\left|X\left(\frac{1}{n}\right)\right| \le \frac{8\beta}{n}\right]^3 = n\left(\sqrt{\frac{n}{2\pi}}\int_{-8\beta/n}^{8\beta/n} e^{-nx^2/2}dx\right)^3 \le n\left(\sqrt{\frac{n}{2\pi}}\frac{16\beta}{n}\right)^3 \le \frac{C}{\sqrt{n}},
\end{aligned}$$

which implies (1.13). It follows that $\mathbb{P}(A) = 0$ as well, hence Brownian motion is nowhere differentiable with probability one. □

Corollary 1.2 *Brownian motion does not have bounded variation with probability one.*

2 Stochastic integration

We would like now to understand how to interpret and ODE of the form

$$\frac{dX}{dt} = \dot{B}(t).$$

The right side certainly does not make sense since $B(t)$ is almost surely not differentiable in time but that should not stop us from trying. More generally, we would like to look at ODE's of the form

$$\frac{dX}{dt} = b(t, X) + \sigma(t, X)\dot{B}(t).$$

A natural approach is to start with a discretized version

$$X_{k+1} - X_k = b(t_k^*, X_k)\Delta t + \sigma(t_k^*, X_k)\Delta B_k, \tag{2.1}$$

where Δ is the time step, and $\Delta B_k = B(t_{k+1}) - B(t_k)$ is the increment of the Brownian motion. The time t_k^* is taken on the interval $[t_k, t_{k+1}]$. We take for the moment $t_k^* = t_k$ – we will later discuss what happens with other choices of t_k^*. In the "integrated form" (2.1) becomes

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j. \tag{2.2}$$

The question we need to understand is whether there exists a limit as $\Delta t \to 0$ for the solutions of such discrete equations.

The Ito integral

With the above goal in mind, we we would like to define

$$\int_S^T f(t, \omega) dB_t(\omega),$$

with $0 < S < T$. As in the definition of any integral, we start with simple functions of the form

$$\phi(t, \omega) = \sum_{j \geq 0} e_j(\omega)\chi_{[j/2^n, (j+1)/2^n)}(t).$$

The only reasonable definition of the integral for simple functions is to set

$$\int_S^T \phi(t, \omega) dB_t(\omega) = \sum_{j \geq 0} e_j(\omega)[B(t_{j+1}) - B(t_j)]. \tag{2.3}$$

In order to understand what can happen with this definition, let us consider the following two examples:

$$\phi_1(t, \omega) = \sum_{j \geq 0} B(\frac{j}{2^n}, \omega)\chi_{[j/2^n, (j+1)/2^n)}(t),$$

and

$$\phi_2(t, \omega) = \sum_{j \geq 0} B(\frac{j+1}{2^n}, \omega)\chi_{[j/2^n, (j+1)/2^n)}(t).$$

Both of these functions are, supposedly, approximating $\phi(t, \omega) = B_t(\omega)$, so their integrals should have the same limit as $n \to +\infty$. Let us see (we assume for simplicity that $T = M2^{-n}$ is an integer multiple of 2^{-n}):

$$\int_0^T \phi_1(t, \omega) dB_t(\omega) = \sum_{j=0}^{M-1} B(\frac{j}{2^n})[B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})],$$

and

$$\int_0^T \phi_2(t,\omega)dB_t(\omega) = \sum_{j=0}^{M} B(\frac{j+1}{2^n})[B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})].$$

Hence, $\phi_1(t,\omega)$ corresponds to approximating $\phi(t,\omega)$ by taking $t_j^* = t_j$ while $\phi_2(t,\omega)$ corresponds to $t_j^* = t_{j+1}$. We compute:

$$\mathbb{E}\left(\int_0^T \phi_1(t,\omega)dB_t(\omega)\right) = \sum_{j=0}^{M} \mathbb{E}\left(B(\frac{j}{2^n})[B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})]\right) = 0,$$

while

$$\mathbb{E}\left(\int_0^T \phi_2(t,\omega)dB_t(\omega)\right) = \sum_{j=0}^{M} \mathbb{E}\left(B(\frac{j+1}{2^n})[B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})]\right)$$

$$= \sum_{j=0}^{M} \mathbb{E}\left((B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})[B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})]\right) = \sum_{j=0}^{M} \frac{1}{2^n} = T.$$

Therefore, the "integrals" of $\phi_1(t,\omega)$ and $\phi_2(t,\omega)$ differ in a non-trivial way that does not vanish as $n \to +\infty$ – the choice of t_j^* matters! There are two canonical choices: $t_j^* = t_j$ gives rise to the Ito integral, while $t_j^* = (t_j + t_{j+1})/2$ leads to the Stratonovich integral.

Integrable functions

We begin with the following definition.

Definition 2.1 *Let $\{\mathcal{N}_t\}$ be an increasing family of σ-algebras. A process $g(t,\omega)$ is $\mathcal{N}_t$-adapted if $g(t,\omega)$ is $\mathcal{N}_t$-measurable for each $t \geq 0$.*

A typical situation when this definition is used is when $\mathcal{N}_t$ is generated by a process $X(t,\omega)$ – $\mathcal{N}_t$ is the collection of all events that depend on $X(s,\omega)$ for $0 \leq s \leq t$ but not on $X(s,\omega)$ for $s > t$. In turn, $g(t,\omega)$ is $\mathcal{N}_t$-adapted if $g(t,\omega)$ depends only on $X(s,\omega)$ for $0 \leq s \leq t$. For example,

$$g(t,\omega) = \int_0^t X(s,\omega)ds$$

is $\mathcal{N}_t$-adapted, while $g(t,\omega) = \max_{t \leq s \leq t+1} |X(s,\omega)|$ is not $\mathcal{N}_t$-adapted.

We will define the Ito integral for functions $f(t,\omega)$ (measurable in both variables) for which the following two conditions hold, on a time interval $[0,T]$:
(i) $f(t,\omega)$ is $\mathcal{F}_t$-adapted (here $\mathcal{F}_t$ is the σ-algebra of events generated by $\{B_s : \ 0 \leq s \leq t\}$, and

$$\text{(ii)} \quad \mathbb{E}\left(\int_0^T f(t,\omega)^2 dt\right) < +\infty.$$

We will denote by V the class of functions for which both conditions (i) and (ii) hold.

Ito integral for elementary functions

Definition 2.2 *A function $\phi \in V$ is elementary if it has the form*

$$\phi(t,\omega) = \sum_j e_j(\omega)\chi_{[t_j,t_{j+1})}(t).$$

For an elementary function $\phi(t,\omega)$ we set

$$\int_S^T \phi(t,\omega)dB_t(\omega) = \sum_{j\geq 0} e_j(\omega)(B(t_{j+1}) - B(t_j)).$$

If $\phi \in V$ then it is piece-wise constant in time, and $e_j(\omega)$ has to be $\mathcal{F}_{t_j}$-measurable – this follows from the fact that it is $\mathcal{F}_t$-adapted.

A very important observation is the following:

Lemma 2.3 *(Ito isometry) If $\phi(t,\omega)$ is bounded and elementary then*

$$\mathbb{E}\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2 = \mathbb{E}\left[\int_S^T \phi^2(t,\omega)dt\right]. \tag{2.4}$$

Proof. Let us take

$$\phi(t,\omega) = \sum_j e_j(\omega)\chi_{[t_j,t_{j+1})}(t),$$

and compute ($\Delta B_j = B(t_{j+1}) - B(t_j)$, and similarly for ΔB_i):

$$\mathbb{E}\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2 = \mathbb{E}\left(\sum_{i,j} e_i(\omega)e_j(\omega)\Delta B_i \Delta B_j\right). \tag{2.5}$$

Note that when $i \neq j$, say, $i > j$, then $e_i(\omega)$, $e_j(\omega)$ and ΔB_j all depend only on the Brownian motion until the time t_i, and thus these quantities are independent from the forward increment $B(t_{i+1}) - B(t_i)$, whence

$$\mathbb{E}\left(e_i(\omega)e_j(\omega)\Delta B_i \Delta B_j\right) = \mathbb{E}(e_i(\omega)e_j(\omega)\Delta B_j)\mathbb{E}(\Delta B_i) = 0, \quad \text{for } i > j.$$

Using this in (2.5) gives

$$\mathbb{E}\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2 = \mathbb{E}\left(\sum_{i=j} e_i(\omega)e_j(\omega)\Delta B_i \Delta B_j\right) = \sum_j \mathbb{E}(e_j^2(\omega)(\Delta B_j)^2). \tag{2.6}$$

Once again, $e_j(\omega)$ depends only on the Brownian motion until the time t_j and is, therefore, independent of the forward increment $\Delta B(t_j)$, giving

$$\mathbb{E}(e_j^2(\omega)(\Delta B_j)^2) = \mathbb{E}(e_j^2(\omega))\mathbb{E}((\Delta B_j)^2) = \sum_j \mathbb{E}(e_j^2(\omega))\Delta t_j.$$

We conclude that

$$\mathbb{E}\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2 = \sum_j \mathbb{E}(e_j^2(\omega))\Delta t_j, \tag{2.7}$$

which proves Lemma 2.3. □

Extension to non-elementary functions

We now extend, gingerly and slowly, the notion of the Ito integral to non-elementary functions. The idea is to show that bounded elementary functions are dense in V and then use Ito's isometry property. We will prove density of bounded elementary functions in V in three steps: (1) Show that bounded continuous functions can be approximated by bounded elementary functions, (2) Show that bounded functions in V can be approximated by bounded continuous functions, and (3) Show that bounded functions are dense in V.

Step 1. Let $g \in V$ be bounded, and also continuous in t for each ω. We claim that there exists a sequence of elementary functions ϕ_n so that

$$\mathbb{E}\left(\int_S^T (g-\phi_n)^2 dt\right) \to 0 \text{ as } n \to +\infty. \tag{2.8}$$

This is true for the natural piece-wise constant approximation

$$\phi_n(t,\omega) = \sum_j g(t_j,\omega)\chi_{[t_j,t_{j+1})}(t).$$

To see that, note first that

$$\int_S^T (g-\phi_n)^2 ds \to 0 \text{ as } n \to +\infty, \tag{2.9}$$

for each realization ω fixed, because $g(t,\omega)$ is continuous for each ω fixed. As the function $g(t,\omega)$ is bounded, (2.9) and the Lebesgue dominated convergence theorem imply that

$$\mathbb{E}\int_S^T (g-\phi_n)^2 ds \to 0 \text{ as } n \to +\infty,$$

which is (2.8).

Step 2. Let $h \in V$ be bounded. We claim that there exists a sequence of bounded continuous functions ψ_n so that

$$\mathbb{E}\left(\int_S^T (h-\psi_n)^2 dt\right) \to 0 \text{ as } n \to +\infty. \tag{2.10}$$

Indeed, suppose that $|h(t,\omega)| \le M$ and take a smooth function $g_n(t) \ge 0$ such that $g_n(t) = 0$ for t outside $[-1/n, 0]$ and

$$\int_{\mathbb{R}} g_n(t)dt = 1.$$

Now, set

$$\psi_n(t) = \int_0^t g_n(s-t)h(s,\omega)ds,$$

then $\psi_n(t,\omega)$ is continuous in t, and $|\psi_n(t)| \le M$. Moreover, as $g_n(t) = 0$ for $t \ge 0$, and $h(t,\omega)$ is $\mathcal{F}_t$-adapted, the functions $\psi_n(t,\omega)$ are also $\mathcal{F}_t$-adapted. It is also easy to check that

$$\int_S^T (h(t,\omega) - \psi_n(t,\omega))^2 dt \to 0 \text{ as } n \to +\infty,$$

for each realization ω fixed. Hence, by bounded convergence theorem we also have

$$\mathbb{E}\left(\int_S^T (h-\psi_n)^2 ds\right) \to 0 \text{ as } n \to +\infty,$$

Step 3. Finally, given any $f \in V$ we find a sequence $h_n \in V$ such that each $h_n(t,\omega)$ is bounded, and

$$\mathbb{E}\left(\int_S^T (f-h_n)^2 ds\right) \to 0 \text{ as } n \to +\infty.$$

This is done as follows: set

$$h_n = \begin{cases} -n \text{ if } f(t,\omega) \le -n, \\ f(t,\omega) \text{ if } -n \le f(t,\omega) \le n, \\ n \text{ if } f(t,\omega) \ge n. \end{cases}$$

Note that on the set $\{f \ge n\}$ we have

$$(f-h_n)^2 = (f-n)^2 \le f^2,$$

and similarly for the set $\{f \le -n\}$. It follows that

$$\mathbb{E}\left(\int_S^T (f-h_n)^2 ds\right) \le \mathbb{E}\left(\int_S^T f^2(t,\omega)\chi_{|f(t,\omega)| \ge n} dt\right) \to 0 \text{ as } n \to +\infty,$$

by the Lebesgue dominated convergence theorem, since

$$\mathbb{E}\left(\int_S^T f^2(t,\omega) dt\right) < +\infty.$$

Together, steps (1)-(3) show that for any function $f \in V$ we can find a sequence of bounded elementary functions ϕ_n such that

$$\mathbb{E}\left(\int_S^T (f-\phi_n)^2 ds\right) \to 0 \text{ as } n \to +\infty, \tag{2.11}$$

that is, ϕ_n converges to f in the space $L^2([S,T] \times \Omega)$. Then, given $f \in V$ we choose such sequence ϕ_n and define

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{n \to +\infty} \int_S^T \phi_n(t,\omega) dB_t. \tag{2.12}$$

We now need to check two things: (i) the limit in the right side exists for any sequence ϕ_n for which (2.11) holds, and (ii) it does not depend on the particular choice of the sequence ϕ_n. Actually, (ii) follows from (i) (this is a simple exercise). The reason the limit exists is that the sequence

$$\alpha_n(\omega) = \int_S^T \phi_n(t,\omega) dB_t$$

is a Cauchy sequence in $L^2(\Omega)$. Indeed, we have

$$\mathbb{E}(\alpha_n - \alpha_m)^2 = \mathbb{E}\left(\int_S^T (\alpha_n(t,\omega) - \alpha_m(t,\omega))^2 dt\right) \to 0 \text{ as } n,m \to +\infty, \tag{2.13}$$

because the sequence $\phi_n(t)$ is convergent (hence, Cauchy) in $L^2([S,T] \times \Omega)$. The first equality in (2.13) follows from Ito's isometry.

Corollary 2.4 *If $f(t,\omega) \in V$ then*

$$\mathbb{E}\left(\int_S^T f(t,\omega)dB_t\right)^2 = \mathbb{E}\left(\int_S^T f^2(t,\omega)dt\right).$$

Corollary 2.5 *If $f(t,\omega) \in V$ and $f_n(t,\omega) \in V$ are such that*

$$\mathbb{E}\left(\int_S^T (f_n(t,\omega) - f(t,\omega))^2 dt\right) \to 0,$$

then

$$\int_S^T f_n(t,\omega)dB_t \to \int_S^T f(t,\omega)dB_t \text{ in } L^2(\Omega).$$

An explicit example

Let us show that

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t. \tag{2.14}$$

Consider the elementary functions

$$\phi_n(s,\omega) = \sum_j B(t_j,\omega)\chi_{[t_j,t_{j+1})}(s),$$

where $t_j = jt/n$, $j = 0,\dots,n-1$. Then, we have

$$\begin{aligned}\mathbb{E}\left(\int_0^t (\phi_n(s) - B_s)^2 ds\right) &= \mathbb{E}\sum_j \int_{t_j}^{t_{j+1}} (B(t_j) - B(s))^2 ds = \sum_j \int_{t_j}^{t_{j+1}} (t_j - s)ds \\ &= \sum_j \frac{(t_{j+1} - t_j)^2}{2} \to 0,\end{aligned}$$

as $\Delta t_j \to 0$. Hence,

$$\int_0^t B_s dB_s = \lim_{n\to+\infty} \int_0^t \phi_n(s,\omega)dB_s = \lim_{n\to+\infty} \sum_{j=0}^{n-1} B(t_j)(B(t_{j+1}) - B(t_j)),$$

with the limit understood in $L^2(\Omega)$-sense. Note that

$$B^2(t_{j+1}) - B^2(t_j) = (B(t_{j+1}) - B(t_j))^2 + 2B(t_j)(B(t_{j+1}) - B(t_j)),$$

and thus

$$\sum_{j=0}^{n-1} B(t_j)(B(t_{j+1}) - B(t_j)) = \frac{1}{2}\sum_{j=0}^{n-1}(B^2(t_{j+1}) - B^2(t_j)) - \frac{1}{2}\sum_{j=0}^{n-1}(B(t_{j+1}) - B(t_j))^2. \tag{2.15}$$

The first sum above is telescoping:

$$\sum_{j=0}^{n-1}(B^2(t_{j+1}) - B^2(t_j)) = B^2(t).$$

For the second sum in the right side of (2.15) we claim that

$$\sum_{j=0}^{n-1}(B(t_{j+1})-B(t_j))^2 \to t \text{ as } j\to+\infty, \text{ in } L^2(\Omega). \tag{2.16}$$

This is verified by a direct computation:

$$\mathbb{E}\left(\sum_j (B(t_{j+1})-B(t_j))^2 - t\right)^2 = \mathbb{E}\sum_{j,k}((B(t_{j+1})-B(t_j))^2-\Delta t_j)((B(t_{k+1})-B(t_k))^2-\Delta t_k). \tag{2.17}$$

Independence of the increments of the Brownian motion implies that the terms with $j\neq k$ in (2.17) vanish since

$$\mathbb{E}((B(t_{k+1})-B(t_k))^2-\Delta t_k)=0,$$

and the same for k replaced by j. It follows that

$$\mathbb{E}\left(\sum_j (B(t_{j+1})-B(t_j))^2 - t\right)^2 = \mathbb{E}\left(\sum_j ((B(t_{j+1})-B(t_j))^2-\Delta t_j)^2\right) \tag{2.18}$$
$$= \sum_j \mathbb{E}\left((B(t_{j+1})-B(t_j))^4 + (t_{j+1}-t_j)^2 - 2(B(t_{j+1}-B(t_j)^2(t_{j+1}-t_j)))\right)$$
$$= \sum_j (3(t_{j+1}-t_j)^2 + (t_{j+1}-t_j)^2 - 2(t_{j+1}-t_j)^2) = 2\sum_j (t_{j+1}-t_j)^2 \to 0 \text{ as } \Delta t_j\to 0.$$

Therefore, (2.16) holds, and we have proved (2.14).

Martingales

Definition 2.6 *Let $\mathcal{F}_t$ be a family of σ-algebras such that $\mathcal{F}_s\subseteq\mathcal{F}_t$ for all $0\le s\le t$. A random process M_t is an $\mathcal{F}_t$-martingale if*
(i) The process M_t is $\mathcal{F}_t$-measurable for all $t\ge 0$.
(ii) The expectation $\mathbb{E}(|M_t|)<+\infty$ for all $t\ge 0$, and
(iii) The conditional expectation $\mathbb{E}(M_s|\mathcal{F}_t)=M_t$ a.s. for all $s\ge t$.

The main non-technical assumption here is the last one: the conditional expectation of a martingale is its present value. One may think of a martingale as a "fair game." If M_t represents a gambler's account balance at time t, then the condition $E[M_t|\mathcal{F}_s]=X_s$ says that the expected future balance, given the current balance, is unchanged. So the game favors neither the gambler nor the house. Of course, the change M_t-M_s may be positive or negative, but its expected value conditioned on M_s is zero. On the other hand, if M_t is a sub-martingale, which means that (iii) above is replaced by

$$\mathbb{E}(M_s|\mathcal{F}_t)\ge M_t \text{ for all } s\ge t, \tag{2.19}$$

then given the gambler's current account balance, he may expect his earnings to increase. If X_t is a super-martingale, that is,

$$\mathbb{E}(M_s|\mathcal{F}_t)\le M_t \text{ for all } s\ge t, \tag{2.20}$$

then given the gambler's current account balance, he may expect his earnings to decrease (this seems to be the most realistic model, given the success of many casinos).

A useful observation is that if X_s is an $\mathcal{F}_s$-martingale and a function $\phi(x)$ is convex, then $\phi(X_s)$ is a sub-martingale. This follows from Jensen's inequality. For instance, if X_s is a martingale, $|X_s|$, X_s^2, and all X_s^{2m} with $m \in \mathbb{N}$, are sub-martingales. Continuous martingales satisfy a remarkable property that estimates the maximum of a process by the terminal time statistics.

Theorem 2.7 *(Continuous Doob inequality) If M_t is a continuous in time martingale such that $\mathbb{E}(|M_T|^p) < +\infty$, then for all $p \geq 1$, $T \geq 0$ and $\lambda > 0$ we have*

$$P\left[\sup_{0\leq t\leq T} |M_t| \geq \lambda\right] \leq \frac{1}{\lambda^p}\mathbb{E}(|M_T|^p).$$

We will not prove this result here but rather prove it only for discrete martingales. A sequence X_j is a martingale with respect to a sequence of σ-algebras $\mathcal{F}_j$ if (i) $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, (ii) X_n is $\mathcal{F}_n$-measurable, (iii) $E[|X_n|] < +\infty$, and (iv) $E(X_{n+1}|\mathcal{F}_n) = X_n$ almost surely. It follows that $E(X_m|\mathcal{F}_n) = X_n$ almost surely for all $m \geq n$. The discrete Doob's inequality is the following estimate that bounds the supremum of X_j in terms of the expectation of the last element:

Theorem 2.8 *(Discrete Doob's inequality) Suppose $(X_j, \mathcal{F}_j)$, $1 \leq j \leq n$, is a martingale sequence such that $\mathbb{E}(|X_n|^p) < +\infty$, then for any $l > 0$ and any $p \geq 1$we have*

$$P\Big\{\omega : \sup_{1\leq j\leq n} |X_j| \geq l\Big\} \leq \frac{1}{l^p}E(|X_n|^p).$$

Proof. Let us define $S(\omega) = \sup_{1\leq j\leq n} |X_j(\omega)$. Then the event $E = \{\omega : \ S(\omega) \geq l\}$ can be decomposed as a disjoint union of the sets

$$E_j = \{\omega : \ |X_1(\omega)| < l, \ldots, |X_{j-1}(\omega)| < l, |X_j(\omega) \geq l\},$$

that is, $E = \bigcup_{j=1}^n E_j$ and $E_j \cap E_m = \emptyset$ for $j \neq m$. Note that, as $|X_j| \geq l$ on the set E_j we have an inequality

$$P(E_j) \leq \frac{1}{l^p}\int_{E_j} |X_j|^p dP.$$

The function $\phi(x) = |x|^p$ is convex for $p \geq 1$, hence, as we mentioned above, the sequence $|X_j|^p$ is a sub-martingale, thus $|X_j|^p \leq \mathbb{E}(|X_n|^p|\mathcal{F}_j)$, and

$$P(E_j) \leq \frac{1}{l^p}\int_{E_j} |X_j|^p dP \leq \frac{1}{l^p}\int_{E_j} \mathbb{E}(|X_n|^p|\mathcal{F}_j)dP$$

Moreover, the set E_j is $\mathcal{F}_j$-measurable as follows immediately from the way E_j is defined, hence

$$P(E_j) \leq \frac{1}{l^p}\int_{E_j} \mathbb{E}(|X_n|^p|\mathcal{F}_j)dP = \frac{1}{l^p}\int_{E_j} |X_n|^p dP,$$

simply from the definition of the conditional expectation $\mathbb{E}(|X_n|^p|\mathcal{F}_j)$. Now, summing over all j and using the fact that E_j are disjoint we obtain

$$P(E) = \sum_{j=1}^n P(E_j) \leq \frac{1}{l^p}\sum_{j=1}^n \int_{E_j} |X_n|^p dP \leq \frac{1}{l^p}\int_E |X_n|^p dP \leq \frac{1}{l^p}\int_\Omega |X_n|^p dP = \frac{1}{l^p}\mathbb{E}(|X_n|^p),$$

and we are done. □

Ito integral as a martingale

It turns out that Ito integral is always a martingale (this is a great advantage of the Ito integral compared to Stratonovich and other definitions of the stochastic integral).

Theorem 2.9 *Let $f \in V(0,T)$ for all $T > 0$ then the process $M_t(\omega) = \int_0^t f(s,\omega)dB_s(\omega)$ is an $\mathcal{F}_t$-martingale, and*

$$P\left(\sup_{0\le t\le t} |M_t| \ge \lambda\right) \le \frac{1}{\lambda^2}\mathbb{E}\left(\int_0^T f^2(s,\omega)ds\right),$$

for all $\lambda > 0$.

Another important property of the Ito integral is that it has a continuous in time version.

Theorem 2.10 *Let $f \in V(0,T)$ then there exists a t-continuous version of*

$$M_t(\omega) = \int_0^t f(s,\omega)dB_s(\omega).$$

We will not prove these results here (the proofs are not long or difficult – see Oksendal's book) – both are consequences of the continuous Doob inequality.

In order to understand why Theorems 2.9 and 2.10 hold let is consider an elementary function

$$f(t) = \sum_j e_j(\omega)\chi_{[t_j,t_{j+1})}(t).$$

Then, for each t we find N such that $t \in [t_N, t_{N+1}]$ and write

$$M_t = \int_0^t f(s,\omega)dB_s(\omega) = \sum_{j=1}^{N-1} e_j(\omega)(B_{t_{j+1}} - B_{t_j}) + e_N(\omega)(B_t - B_{t_N}). \tag{2.21}$$

Since B_t is almost surely continuous, it follows immediately that $M_t(\omega)$ is almost surely continuous on each interval (t_j, t_{j+1}). It is also clear that there is no jump at the times t_j, hence $M_t(\omega)$ is a.s. continuous for all $t \ge 0$. In order to verify that

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s, \tag{2.22}$$

let us assume that $s = 0$, then (2.22) is simply

$$\mathbb{E}(M_t) = M_0 = 0. \tag{2.23}$$

Using (2.24) we write

$$\begin{aligned}\mathbb{E}(M_t) = \mathbb{E}\left(\int_0^t f(s,\omega)dB_s(\omega)\right) &= \sum_{j=1}^{N-1} \mathbb{E}(e_j(\omega)(B_{t_{j+1}} - B_{t_j})) + \mathbb{E}(e_N(\omega)(B_t - B_{t_N})) \\ &= \sum_{j=1}^{N-1} \mathbb{E}(e_j(\omega))\mathbb{E}(B_{t_{j+1}} - B_{t_j}) + \mathbb{E}(e_N(\omega))\mathbb{E}(B_t - B_{t_N})) = 0,\end{aligned} \tag{2.24}$$

hence (2.23), indeed, holds. Once again, we used above the independence of $e_j(\omega)$ and the forward increment $B_{t_{j+1}} - B_{t_j}$ (recall that this is true because $f(t,\omega)$ is $\mathcal{F}_t$-adapted). If $s > 0$ we use essentially the same argument starting from

$$M_t - M_s = \int_s^t f(\tau,\omega)dB_\tau. \tag{2.25}$$

The Ito formula

We begin with the definition of the Ito processes.

Definition 2.11 *A one-dimensional Ito process is a stochastic process of the form*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s,$$

with $u(s,\omega)$ and $v(s,\omega)$ such that

$$P\left(\int_0^t v^2(s,\omega)ds < +\infty \ \ \text{for all } t \geq 0.\right) = 1,$$

and

$$P\left(\int_0^t |u(s,\omega)|ds < +\infty \ \ \text{for all } t \geq 0.\right) = 1.$$

A shorter notation is

$$dX_t = udt + vdB_t.$$

Example. Our computation of $\int_0^t B_s dB_s$ shows that the process $X_t = B_t^2$ can be written as

$$B_t^2 = t + 2\int_0^t B_s dB_s,$$

or

$$dX_t = dt + 2B_t dB_t.$$

The Ito formula gives a recipe on how to express a function of an Ito process as another Ito process.

Theorem 2.12 *(The Ito formula) Let X_t be an Ito process given by*

$$dX_t = udt + vdB_t,$$

and let $g(t,x) \in C^2([0,+\infty) \times \mathbb{R})$. Then $Y_t = g(t,X_t)$ is also an Ito process, and

$$dY_t = \frac{\partial g(t,X_t)}{\partial t}dt + \frac{\partial g(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 g(t,X_t)}{\partial x^2}(dX_t)^2.$$

Here $(dX_t)^2 = dX_t \cdot dX_t$ with the convention $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$, and $dB_t \cdot dB_t = dt$.

An interlude on the random walk

The Ito formula has an extra term involving $\partial^2 g(t,X_t)/\partial x^2$ in the right side that is absent in the deterministic case $v = 0$. As this term is what really matters in the connection to elliptic and parabolic partial differential equations, let us explain where it comes from. We need to look no further than the random walk that approximates the Brownian motion. Recall that the correct scaling is to take the time step $\Delta t = h^2$ and the spatial step $\Delta x = h$, with h small. That is, the random walk satisfies

$$P(X_{t_{n+1}} = X_{t_n} \pm h | X_{t_n}) = 1/2,$$

where $t = nh^2$ and $x = mh$ with some integers n and m. Then, given a twice continuously differentiable function $f(t, x)$, and since $t_{n+1} - t_n = h^2$, we have

$$
\begin{aligned}
f(t_{n+1}, X_{t_{n+1}}) &= f(t_n, X_{t_n}) + \frac{\partial f(t_n, X_{t_n})}{\partial x}(X_{t_{n+1}} - X_{t_n}) \\
&+ \frac{\partial f(t_n, X_{t_n})}{\partial t} h^2 + \frac{1}{2}\frac{\partial^2 f(t_n, X_{t_n})}{\partial x^2}(X_{t_{n+1}} - X_{t_n})^2 \\
&+ \frac{\partial^2 f(t_n, X_{t_n})}{\partial t^2} h^4 + \frac{\partial^2 f(t_n X_{t_n})}{\partial x \partial t} h^2 (X_{t_{n+1}} - X_{t_n}) + \dots
\end{aligned} \tag{2.26}
$$

Note that $(X_{t_{n+1}} - X_{t_n})^2 = h^2$ with probability one – this is where the convention $dB_t \cdot dB_t = dt$ comes from! Hence the two terms in the second line above are of the same order, and we can actually rewrite it as

$$
\begin{aligned}
f(t_{n+1}, X_{t_{n+1}}) &= f(t_n, X_{t_n}) + \frac{\partial f(t_n, X_{t_n})}{\partial x}(X_{t_{n+1}} - X_{t_n}) \\
&+ \left[\frac{\partial f(t_n, X_{t_n})}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t_n, X_{t_n})}{\partial x^2}\right](t_{n+1} - t_n) \\
&+ \frac{\partial^2 f(t_n X_{t_n})}{\partial t^2}(t_{n+1} - t_n)^2 + \frac{\partial^2 f(t_n, X_{t_n})}{\partial x \partial t}(t_{n+1} - t_n)(X_{t_{n+1}} - X_{t_n}) + \dots
\end{aligned} \tag{2.27}
$$

The terms in the last line in (2.27) vanish as $h \to 0$. Summing over n and formally passing to the limit $h \to 0$ in (2.27) we get

$$
f(B_t) = f(B_0) + \int_0^t \frac{\partial f(s, B_s)}{\partial x} dB_s + \int_0^t \left[\frac{\partial f(t, B_t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t, B_t)}{\partial x^2}\right] ds.
$$

This is, of course, nothing but the Ito formula.

Sketch of proof of Ito's formula

Let us assume that

$$
dX_t = udt + vdB_t.
$$

Using Taylor's formula we get, for any partition $0 = t_0 < t_1 < \ldots < t_N = t$, and $X_j = X_{t_j}$

$$
\begin{aligned}
g(t, X_t) &= g(0, X_0) + \sum_j (g(t_{j+1}, X_{j+1}) - g(t_j, X_j)) \\
&= g(0, X_0) + \sum_j \frac{\partial g(t_j, X_j)}{\partial t}\Delta t_j + \sum_j \frac{\partial g(t_j, X_j)}{\partial x}\Delta X_j \\
&+ \frac{1}{2}\sum_j \frac{\partial^2 g(t_j, X_j)}{\partial t^2}(\Delta t_j)^2 + \sum_j \frac{\partial^2 g(t_j, X_j)}{\partial t \partial x}\Delta t_j \Delta X_j + \frac{1}{2}\sum_j \frac{\partial^2 g(t_j, X_j)}{\partial x^2}(\Delta X_j)^2 + \sum_j R_j,
\end{aligned}
$$

where $R_j = o(|\Delta t_j|^2 + |\Delta X_j|^2)$. If we let $\Delta t_j \to 0$, we get

$$
\sum_j \frac{\partial g(t_j, X_j)}{\partial t}\Delta t_j \to \int_0^t \frac{\partial g(s, X_s)}{\partial s} ds,
$$

and

$$\sum_j \frac{\partial g(t_j, X_j)}{\partial x} \Delta X_j \to \int_0^t \frac{\partial g(s, X_s)}{\partial x} dX_s.$$

In order to understand what happens to the term

$$\frac{1}{2} \sum_j \frac{\partial^2 g(t_j, X_j)}{\partial x^2} (\Delta X_j)^2,$$

we assume that u and v are elementary. In the general case we may approximate u and v by elementary functions and use a density argument. If u and v are elementary, then, after possibly refining the partition $\{t_j\}$to make sure that u_j and v_j are constant on the intervals (t_j, t_{j+1}), we have

$$\Delta X_j = u_j \Delta t_j + v_j \Delta B_j,$$

hence

$$\sum_j \frac{\partial^2 g(t_j, X_j)}{\partial x^2} (\Delta X_j)^2 = \sum_j \frac{\partial^2 g(t_j, X_j)}{\partial x^2} u_j^2 (\Delta t_j)^2 + 2 \sum_j \frac{\partial^2 g(t_j, X_j)}{\partial x^2} u_j v_j (\Delta t_j) \Delta B_j$$
$$+ \sum_j \frac{\partial^2 g(t_j, X_j)}{\partial x^2} v_j^2 (\Delta B_j)^2 = I + II + III.$$

It is easy to check that

$$\mathbb{E}(I^2) \to 0, \quad \mathbb{E}(II^2) \to 0 \text{ as } \Delta t_j \to 0.$$

Indeed, we have:

$$\mathbb{E}(II^2) = 4 \sum_{i,j} \mathbb{E}\left(\frac{\partial^2 g(t_j, X_j)}{\partial x^2} \frac{\partial^2 g(t_i, X_i)}{\partial x^2} u_j v_j u_i v_i (\Delta t_j) \Delta B_j (\Delta t_i) \Delta B_i \right) = 8 \sum_{i>j} + 4 \sum_{i=j}.$$

But using our beloved independence of the increments from the past and the fact that u_j and v_j are $\mathcal{F}_t$-adapted, we get that

$$\sum_{i>j} \mathbb{E}\left(\frac{\partial^2 g(t_j, X_j)}{\partial x^2} \frac{\partial^2 g(t_i, X_i)}{\partial x^2} u_j v_j u_i v_i (\Delta t_j) \Delta B_j (\Delta t_i) \Delta B_i \right)$$
$$= \sum_{i>j} \mathbb{E}\left(\frac{\partial^2 g(t_j, X_j)}{\partial x^2} \frac{\partial^2 g(t_i, X_i)}{\partial x^2} u_j v_j u_i v_i (\Delta t_j) \Delta B_j (\Delta t_i) \right) \mathbb{E}(\Delta B_i) = 0.$$

Therefore, we have, again using independence of the increments from the past and the fact that u_j and v_j are $\mathcal{F}_t$-adapted:

$$\mathbb{E}(II^2) = 4 \sum_j \mathbb{E}\left(\left(\frac{\partial^2 g(t_j, X_j)}{\partial x^2} \right)^2 u_j^2 v_j^2 (\Delta t_j)^2 (\Delta B_j)^2 \right)$$
$$= 4 \sum_j \mathbb{E}\left(\left(\frac{\partial^2 g(t_j, X_j)}{\partial x^2} \right)^2 u_j^2 v_j^2 (\Delta t_j)^2 \right) \mathbb{E}((\Delta B_j)^2)$$
$$= 4 \sum_j \mathbb{E}\left(\left(\frac{\partial^2 g(t_j, X_j)}{\partial x^2} \right)^2 u_j^2 v_j^2 \right) (\Delta t_j)^3 \to 0 \text{ as } \Delta t_j \to 0.$$

A very similar computation shows that $\mathbb{E}(I^2) \to 0$ as $\Delta t_j \to 0$. The last step is to show that

$$III \to \int_0^t \frac{\partial^2 g(s, X_s)}{\partial x^2} v^2(s,\omega)ds \text{ in } L^2(\Omega) \text{ as } \Delta t_j \to 0. \tag{2.28}$$

In order to check this, we set

$$a(t) = \frac{\partial^2 g(t,x)}{\partial x^2} v^2(t,\omega),$$

and $a_j = a(t_j)$. Consider then

$$\mathbb{E}\Big(\sum_j a_j(\Delta B_j)^2 - \sum_j a_j \Delta t_j\Big)^2 = \sum_{ij} \mathbb{E}(a_i a_j((\Delta B_i)^2 - \Delta t_i)((\Delta B_j)^2 - \Delta t_j)). \tag{2.29}$$

As before, if $i > j$ then the forward increment ΔB_i is independent of the other terms in (2.29). Since $\mathbb{E}(\Delta B_i^2) = \Delta t_i$, the terms with $i \neq j$ in (2.29) vanish, and we get

$$\mathbb{E}\Big(\sum_j a_j(\Delta B_j)^2 - \sum_j a_j \Delta t_j\Big)^2 = \sum_j \mathbb{E}\Big(a_j^2((\Delta B_j)^2 - \Delta t_j)^2\Big). \tag{2.30}$$

As a_j is $\mathcal{F}_{t_j}$ measurable, we deduce that ΔB_j is independent from a_j, hence

$$\mathbb{E}\Big(\sum_j a_j(\Delta B_j)^2 - \sum_j a_j \Delta t_j\Big)^2 = \sum_j \mathbb{E}(a_j^2)\mathbb{E}((\Delta B_j)^2 - \Delta t_j)^2) \tag{2.31}$$

$$= \sum_j \mathbb{E}(a_j^2)\mathbb{E}((\Delta B_j)^4 - 2(\Delta B_j)^2 \Delta t_j + (\Delta t_j)^2) = \sum_j \mathbb{E}(a_j^2)(3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2)$$

$$= 2\sum_j \mathbb{E}(a_j^2)(\Delta t_j)^2 \to 0 \text{ as } \Delta t_j \to 0. \tag{2.32}$$

Example 1: Let $X_t = B_t$ and $\phi(x) = x^2$. Then by applying Itô's formula to the process $Y_t = (B_t)^2$ we find that

$$(B_t)^2 = \int_0^t 2B_s\, dB_s + t,$$

as we have seen already.

Example 2: Let $X_t = B_t$ and $\phi(x) = e^{\alpha x}$. Itô's formula applied to the process $Y_t = e^{\alpha B_t}$ gives

$$Y_t = 1 + \int_0^t \alpha Y_s\, dB_s + \frac{\alpha^2}{2}\int_0^t Y_s\, ds$$

which may be expressed as

$$dY_t = \alpha Y_t\, dB_t + \frac{\alpha^2}{2} Y_t dt$$

Similarly, the function $Z_t = e^{\alpha B_t - \alpha^2 t/2}$ satisfies

$$dZ_t = \alpha Z_t\, dB_t$$

This shows that Z_t is a martingale since

$$Z_t = 1 + \int_0^t \alpha Z_s \, dB_s$$

and the Itô integral is a martingale. (Actually one can easily compute directly that Z is a martingale without the help of stochastic calculus.) It follows that

$$\mathbb{E}(Z_t) = Z_0 = 1,$$

whence

$$\mathbb{E}(e^{\alpha B_t}) = e^{\alpha^2 t/2}. \tag{2.33}$$

This are many other ways to compute the expectation in (2.33) but this probably is the simplest.

Theorem 2.13 (Itô Product Rule) *Suppose that $X_t(\omega)$ and $Y_t(\omega)$ two stochastic processes satisfying*

$$\begin{aligned} dX_t &= F(X_t, t)dt + G(X_t, t)dB_t \\ dY_t &= H(Y_t, t)dt + K(Y_t, t)dB_t. \end{aligned}$$

Then the process $Z_t(\omega) = X_t(\omega)Y_t(\omega)$ satisfies

$$\begin{aligned} dZ_t &= \left(F(X_t, t)Y_t + H(Y_t, t)X_t + G(X_t, t)K(Y_t, t)\right) dt + (G(X_t, t)Y_t + K(Y_t, t)X_t)dB_t \\ &= Y_t dX_t + X_t dY_t + G(X_t, t)K(Y_t, t)dt. \end{aligned}$$

Itô's formula in multiple dimensions

We can also define vector-valued stochastic integrals using a m-dimensional Brownian motion. Suppose that $G(s, \omega)$ is a matrix valued process such that

$$G^{ij}(s, \omega) \in \mathcal{L}^2([0, T]), \quad i = 1, \ldots d, \quad j = 1, \ldots, m.$$

If B_t is a m-dimensional Brownian motion, then

$$X_t = \int_0^t G(s, \omega) \, dB_t$$

defines a d-dimensional stochastic process whose components are

$$X_t^{(i)} = \sum_{j=1}^m \int_0^t G^{ij}(s, \omega) \, dB_t^{(j)}, \quad i = 1, \ldots, d$$

Itô's formula extends to multiple dimensions in the following way.

Theorem 2.14 *Suppose that B_t is a m-dimensional Brownian motion and that $X_t(\omega) = (X_t^{(i)}(\omega))_i$ is a d-dimensional stochastic process satisfying*

$$X_t^{(i)}\omega) = X_0^{(i)}(\omega) + \int_0^t F^{(i)}(s,\omega)\,ds + \sum_{j=1}^m \int_0^t G^{ij}(s,\omega)\,dB_t^{(j)},$$

If $\phi(x_1,\ldots,x_d,t)$ is twice-differentiable in the spatial variables, differentiable in t, then the one-dimensional process $Y_t := \phi(X_t(\omega),t)$ satisfies

$$dY_t = [F(t,\omega)\cdot\nabla\phi(X_t(\omega),t) + \phi_t(X_t(\omega),t)]\,dt + \sum_{j=1}^m\sum_{i=1}^d \frac{\partial\phi}{\partial x_i}(X_t(\omega),t)\;G^{ij}(t,\omega)dB_t^{(j)}$$
$$+\frac{1}{2}\left(\sum_{k=1}^m \sum_{i,j=1}^d \phi_{x_ix_j}(X_t(\omega),t)G^{(ik)}(t,\omega)G^{(jk)}(t,\omega)\right)dt.$$

You can remember the last term by Taylor's formula and the heuristic formula

$$dB_t^{(i)}dB_t^{(j)} \sim dt, \quad \text{if } i=j, \qquad (=0,\text{otherwise})$$

so that

$$(G^{(kh)}\,dB_t^{(h)})(G^{(qp)}\,dB_t^{(p)}) \sim \delta_{hp}G^{(kh)}G^{(qp)}dt$$

Thus, off-diagonal terms ($p \neq h$) vanish in the formula.

Stochastic differential equations

Example. Let us consider a stochastic differential equation

$$dN_t = rN_tdt + \alpha N_tdB_t. \tag{2.34}$$

Ito's formula says that

$$d(\ln N_t) = \frac{dN_t}{N_t} - \frac{1}{2N_t^2}(dN_t)^2 = \frac{dN_t}{N_t} - \frac{\alpha^2}{2N_t^2}N_t^2dt = \frac{dN_t}{N_t} - \frac{\alpha^2}{2}dt.$$

It follows that

$$\int_0^t \frac{dN_s}{N_s} = \ln N_t - \ln N_0 + \alpha^2 t.$$

On the other hand, (2.34) implies that

$$\int_0^t \frac{dN_s}{N_s} = rt + \alpha B_t.$$

Therefore, we have an explicit solution

$$N_t = N_0\exp((r-\frac{\alpha^2}{2})t + \alpha B_t).$$

As a consequence, we also have

$$\mathbb{E}(N_t) = N_0e^{rt},$$

as can also be seen immediately from (2.34). An interesting property of N_t is that if $r < \alpha^2/2$ then $N_t \to 0$ as $t \to \infty$, almost surely, even though $\mathbb{E}(N_t) \to +\infty$.

In the general case we have the following result.

Theorem 2.15 *Let $T > 0$, and $b(t,x)$ and $\sigma(t,x)$ satisfy*

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|),$$

and

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|,$$

for all $x, y \in \mathbb{R}^n$. Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad X_0 = x_0,$$

has a unique continuous in t solution $X_t(\omega)$ that is $\mathcal{F}_t$-adapted, and

$$\mathbb{E}\left(\int_0^T |X_t|^2 dt\right) < +\infty.$$

The Lipschitz continuity and at most linear growth of b and σ are needed even for existence and uniqueness theorems for ordinary differential equations. For instance, solutions of the ODE $\dot{X} = X^2$ blow up in a finite time if $X(0) > 0$ while the ODE $\dot{X} = 2\sqrt{X}$ with the initial data $X(0) = 0$ has two solutions: $X_1(t) \equiv 0$, and $X_2(t) = t^2$.

3 Representations of solutions of PDEs

We now develop representation formulas for solutions of various PDEs – we will for now take existence and regularity of solutions for granted but will later address them separately.

Poisson's equation in the whole space

Consider an elliptic operator

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i}. \tag{3.1}$$

We assume that $a_{ij}(x)$ are sufficiently smooth and bounded, and the matrix $a_{ij}(x)$ is symmetric: $a_{ij} = a_{ji}$. A more important assumption is that the operator $\mathcal{L}$ is uniformly elliptic. This means that there exists a constant $c > 0$ so that for all $\xi \in \mathbb{R}^n$ we have

$$\sum_{i,j=1}^n a_{ij}(x)\xi\xi_j \geq c|\xi|^2. \tag{3.2}$$

Let σ_{ij} be a matrix such that $a = \sigma\sigma^T$, and each component of σ is bounded and in $C^1(\mathbb{R}^n)$. Consider X_t, the solution to the SDE

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s.$$

The Ito formula for a function $f(X_t)$ is

$$f(X_t) - f(X_0) = \int_0^t \mathcal{L}f(s,X_s)ds + \sum_{j,k=1}^n \int_0^t \sigma_{jk}(s,X_s)\frac{\partial f(s,X_s)}{\partial x_j}dB_k(s). \tag{3.3}$$

Theorem 3.1 *Let $\lambda > 0$ and $f(x)$ be a C^1 function of compact support. Suppose $u(x)$ is a $C_b^2(\mathbb{R}^n)$ solution of the Possion equation*

$$-\mathcal{L}u + \lambda u(x) = f(x), \quad x \in \mathbb{R}^n. \tag{3.4}$$

Then

$$u(x) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(X_t) dt. \tag{3.5}$$

Proof. Let $u(x)$ be the solution to (3.4), then, using the Ito formula (3.3) we obtain

$$u(X_t) - u(X_0) = \int_0^t \mathcal{L}u(X_s)ds + \sum_{j,k=1}^n \int_0^t \sigma_{jk}(s, X_s) \frac{\partial u(s, X_s)}{\partial x_j} dB_k(s).$$

Moreover, if we set $v(t, x) = e^{-\lambda t} u(x)$, then

$$\begin{aligned} v(t, X_t) - v(0, X_0) = &\int_0^t e^{-\lambda s} \mathcal{L}u(X_s)ds - \lambda \int_0^t e^{-\lambda s} u(X_s)ds \\ &+ \sum_{j,k=1}^n \int_0^t e^{-\lambda s} \sigma_{jk}(s, X_s) \frac{\partial u(s, X_s)}{\partial x_j} dB_k(s). \end{aligned}$$

Taking the expectation gives

$$e^{-\lambda t} \mathbb{E}(u(X_t)) - u(x) = -\int_0^t e^{-\lambda s} \mathbb{E}(f(X_s))ds.$$

Letting $t \to +\infty$ leads to (3.5) since the functions $u(x)$ and $f(x)$ bounded. □

In order to see why we need to take $\lambda > 0$ in (3.4), consider the one-dimensional case:

$$-u'' + k^2 u = f(x), \tag{3.6}$$

so that $\mathcal{L} = d^2/dx^2$, and X_t is simply the one-dimensional Brownian motion B_t. If $k = 0$ then (3.6) need not have a bounded solution (though it might for some $f(x)$) – this is easily seen if $f(x) \geq 0$ since then $u(x)$ is a concave function and that contradicts the fact that $u(x)$ is bounded unless $u(x) \equiv$ const. This can also be seen from the probabilistic formula (3.5): if $\lambda = 0$ and $f(x) \geq 0$ is a compactly supported function then the integral in (3.5) diverges since B_t is recurrent in one dimension. On the other hand, when $k > 0$, a bounded solution of (3.6) is given by an explicit formula

$$u(x) = \frac{1}{2k} \int_{-\infty}^\infty e^{-k|x-y|} f(y) dy. \tag{3.7}$$

An interesting exercise is to compare (3.5) and (3.7) (with $\lambda = k^2$) and deduce the law of the Brownian motion in one dimension.

The Feynman-Kac Formula

Let us now look at time dependent problems. In what follows, we will work with solutions to initial value problems and with solutions to terminal value problems. One can switch between these two perspectives through a simple change of variables: $t \to T - t$. Suppose that $w(t,x) \in C^{2,1}([0,\infty) \times \mathbb{R})$ solves the initial value problem

$$w_t = \frac{\sigma^2(x)}{2} w_{xx} + b(x) w_x \quad x \in \mathbb{R}, \ \ t > 0 \tag{3.8}$$

with initial data $w(x,0) = f(x)$, which is smooth and compactly supported. We also assume that $b(x)$ and $\sigma(x)$ are Lipschitz continuous and bounded, and that w is bounded. Then, for $t > 0$ fixed, the function $u(s,x) = w(t-s,x)$ satisfies the terminal value problem

$$u_s + \frac{\sigma^2(x)}{2} u_{xx} + b(x) u_x = 0, \quad x \in \mathbb{R}, \ \ s < t \tag{3.9}$$

with terminal condition $u(t,x) = f(x)$. Moreover, $u \in C^{2,1}((-\infty,t] \times \mathbb{R})$. Now, let $B_s(\omega)$ be a standard Brownian motion with filtration $(\mathcal{F}_s)_{s \geq 0}$. Suppose that, $X_s(\omega)$ is an $\mathcal{F}_s$-adapted solution to the stochastic ODE

$$dX_s = b(X_s)ds + \sigma(X_s)dB_s \tag{3.10}$$

with the initial condition $X_0 = x$. The existence and uniqueness of such a solution is guaranteed by our assumptions about b and σ.

Now, a direct application of Ito's formula shows us that,

$$\begin{aligned} u(t,X_t) - u(0,X_0) &= \int_0^t \left(u_s(s,X_s) + b(X_s)u_x(s,X_s) + \frac{\sigma^2(X_s)}{2} u_{xx}(s,X_s) \right) ds \\ &+ \int_0^t \sigma(X_s) u_x(s,X_s) dB_s = \int_0^t \sigma(X_s) u_x(s,X_s) dB_s. \end{aligned} \tag{3.11}$$

We used the PDE (3.9) in the last step. Therefore, taking the expectation, we find that

$$\mathbb{E}\left[u(t,X_t)\right] = \mathbb{E}\left[u(0,X_0)\right] = u(0,x), \tag{3.12}$$

since the Itô integral has zero mean. In terms of w, this shows that

$$w(t,x) = u(0,x) = \mathbb{E}\left[u(t,X_t)\right] = \mathbb{E}\left[f(X_t)\right] \tag{3.13}$$

In summary, these arguments demonstrate the following:

Theorem 3.2 *(i)* **Initial value problem:** *Suppose that $w(t,x) \in C^{2,1}([0,\infty) \times \mathbb{R})$ is bounded and satisfies*

$$w_t = \frac{\sigma^2(x)}{2} w_{xx} + b(x) w_x \quad x \in \mathbb{R}, \ \ t > 0 \tag{3.14}$$

with initial condition $w(0,x) = f(x) \in C_0^2(\mathbb{R})$. Then $w(t,x)$ is represented by

$$w(t,x) = \mathbb{E}_x\left[f(X_t)\right] \tag{3.15}$$

where

$$dX_s = b(X_s)ds + \sigma(X_s)dB_s \text{ for } s \geq 0 \text{ and } X_0(\omega) = x.$$

(ii) **Terminal value problem:** *Suppose that $u(t,x) \in C^{2,1}((-\infty, T] \times \mathbb{R})$ is bounded and satisfies*

$$u_t + \frac{\sigma^2(x)}{2} u_{xx} + b(x)u_x = 0 \quad x \in \mathbb{R}, \ \ t < T \tag{3.16}$$

with terminal condition $u(T,x) = f(x) \in C_0^2(\mathbb{R})$. Then $u(0,x)$ is represented by

$$u(0,x) = E\left[f(X_T)\right]. \tag{3.17}$$

If we need to find $u(t,x)$ for $t \in (0,T)$ we simply consider the process

$$dX_s = b(X_s)ds + \sigma(X_s)dB_s,$$

that starts at time $s = t$ at the point x: $X_t = x$. Then we have

$$u(t,x) = \mathbb{E}(f(X_T)).$$

Generalizations

To avoid technical difficulties, we have been rather conservative in our assumptions about the initial conditions and the coefficients. In fact, these representations hold under milder conditions on the initial data and the coefficients. Now let us suppose that $u(t,x)$ satisfies the second-order linear PDE

$$u_t + \sum_{i,j=1}^{d} \frac{1}{2} a_{ij}(t,x) u_{x_i x_j} + \sum_{j=1}^{d} b_j(t,x) u_{x_j} + c(t,x)u = 0, \quad x \in \mathbb{R}^d, \ \ t < T \tag{3.18}$$

with terminal condition $u(x,T) = f(x)$ which is continuous (but not necessarily differentiable or bounded). We also assume

- The matrix a_{ij} is given by $a_{ij} = \sum_k \sigma_{ik}\sigma_{kj} = \sigma\sigma^T$ for some matrix $\sigma_{jk}(t,x)$.
- The matrix $a_{ij} = a_{ij}(t,x)$ is uniformly positive definite: $\sum_{ij} a_{ij}\xi_j\xi_i \geq \mu|\xi|^2$ for some constant $\mu > 0$, independent of (t,x).
- Both $\sigma_{ij}(t,x)$ and $b(t,x) = (b_j(t,x))$ are Lipschitz continuous in x, continuous in t, and grow at most linearly in x.
- The function $c(t,x)$ is continuous in (t,x) and bounded in x.
- The terminal condition $f(x)$ satisfies the growth condition $|f(x)| \leq Ce^{p|x|^2}$ for some constant $p > 0$ sufficiently small.
- $u(t,x)$ satisfies the growth condition $|u(t,x)| \leq Ce^{p|x|^2}$ for $x \in \mathbb{R}$, $t \in [t_0, T]$ and some constant $p > 0$ sufficiently small.

Suppose that for a given (x,t), the process $X_s^{x,t}(\omega) : [t,T] \times \Omega \to \mathbb{R}^d$ satisfies

$$dX_s^{x,t} = b(s, X_s^{x,t})\, ds + \sum_j \sigma_{ij}(s, X_s^{x,t})\, dB_s^{(j)}, \quad s \in [t,T], \tag{3.19}$$

with $X_t^{x,t}(\omega) = x$. The superscripts indicate that the process $X_s^{x,t}$ starts at the point x at time t. Notice that $X_s^{x,t}$ is a vector, and $b(s,x)$ is also a vector. Then one can prove:

Theorem 3.3 *Under the assumptions given above, $u(x,t)$ satisfies*

$$u(t,x) = \mathbb{E}\left[f(X_T^{x,t}) e^{\int_t^T c(X_s^{x,t},s)\, ds} \right]. \tag{3.20}$$

Sketch of proof: To prove this statement, one may apply Itô's formula and the product rule to the process defined by

$$H_r(\omega) = u(X_r^{x,t}, r) e^{\int_t^r c(X_s^{x,t},s)\, ds}, \quad r \in [t,T]. \tag{3.21}$$

The fact that the terminal data $f(x)$ may not be smooth or bounded causes some difficulty that may be overcome by using Itô's formula with stopping times. For $n > 0$, let $S_n(\omega)$ be the stopping time $S_n = \inf\{s \geq t |\, |X_s^{x,t}| \geq n\}$. Then we conclude that for $r \in (t,T)$,

$$\begin{aligned} H_{r\wedge S_n} - H_t &= u(X_{r\wedge S_n}^{x,t}, r \wedge S_n) e^{\int_t^{r\wedge S_n} c(X_s^{x,t},s)\, ds} - u(X_t^{x,t}, t) \\ &= \int_t^{r\wedge S_n} e^{\int_t^s c(X_\tau^{x,t},\tau)\, d\tau} \left(u_s + \sum_j b_j u_{x_j} + \frac{1}{2} a_{ij} u_{x_i x_j} + c(X_s^{x,t}, s) u \right) ds \\ &\quad + \int_t^{r\wedge S_n} e^{\int_t^s c(X_\tau^{x,t},\tau)\, d\tau} \sum_{i,j} u_{x_i} \sigma_{ij} dB_s^{(j)} \\ &= \int_t^{r\wedge S_n} e^{\int_t^s c(X_\tau^{x,t},\tau)\, d\tau} \sum_{i,j} u_{x_i} \sigma_{ij} dB_s^{(j)} \quad \text{(using (3.18))} \end{aligned}$$

Notice that arguments inside the integrals are evaluated at $(X_s^{x,t}, s)$. Taking the expectation as before, we conclude that

$$u(x,t) = \mathbb{E}\left[u(X_t^{x,t}, t) \right] = \mathbb{E}\left[u(X_{r\wedge S_n}^{x,t}, r) e^{\int_t^{r\wedge S_n} c(X_s^{x,t},s)\, ds} \right]. \tag{3.22}$$

Notice that if u itself is not bounded, then the expectation on the right is not obviously finite. This explains our use of the stopping time – the stopping time restricts $X_{r\wedge S_n}^{x,t}$ to a bounded region, over which u must be bounded since u is continuous. The next step is to take $n \to \infty$. Using the growth assumptions on u and the coefficients one can show that as $n \to \infty$, the above expression remains finite since $P(S_n < r) = O(e^{-\alpha n^2})$ as $n \to \infty$. This shows that

$$u(x,t) = \mathbb{E}\left[u(X_r^{x,t}, r) e^{\int_t^r c(X_s^{x,t},s)\, ds} \right] \tag{3.23}$$

Then we let $r \to T$. If we knew that u were sufficiently smooth and bounded at $r = T$, then we could apply Itô's formula with $r = T$ in the above formula. This was our approach in the first section, since we assumed the initial (or terminal) data was C^2. In general, however, this is not the case. Nevertheless, one may use the dominated convergence theorem to show that as $r \to T$,

$$\lim_{r \to T} \mathbb{E}\left[u(X_r^x, r) e^{\int_t^r c(X_s^x, s)\, ds}\right] = \mathbb{E}\left[f(X_T^x) e^{\int_t^T c(X_s^x, s)\, ds}\right] \tag{3.24}$$

even when f is merely continuous and satisfies a growth condition (see Karatzas and Shreve for more details). $\square$

Next we formulate a similar result for the initial value problem. Suppose that $w(x,t)$ satisfies

$$w_t = \sum_{i,j=1}^{d} \frac{1}{2} a_{ij}(x,t) u_{x_i x_j} + \sum_{j=1}^{d} b_j(x,t) u_{x_j} + c(x,t) u, \quad x \in \mathbb{R}^d, \ \ t > 0 \tag{3.25}$$

with initial condition $w(x,0) = f(x)$. Then the function $\tilde{u}(x,-t) := w(x,t)$ satisfies (3.19) with $T = 0$, and coefficients given by $\tilde{a}_{ij}(x,t) = a_{ij}(x,-t)$, $\tilde{b}(x,t) = b(x,-t)$, $\tilde{c}(x,t) = c(x,-t)$. For given (x,t) let $X_s^{x,t}(\omega)$ satisfy

$$dX_s^{x,t} = b(X_s^{x,t}, t-s)\, ds + \sum_j \sigma_{ij}(X_s^{x,t}, t-s)\, dB_s^{(j)}, \quad s \in [0,t] \tag{3.26}$$

Then the analysis above shows that

$$w(x,t) = \mathbb{E}\left[f(X_t^{x,t}) e^{\int_0^t c(X_s^{x,t}, t-s)\, ds}\right]. \tag{3.27}$$

In particular, if $c \equiv 0$, then

$$w(x,t) = E\left[f(X_t^{x,t})\right]. \tag{3.28}$$

These are very elegant formulas which have a natural physical interpretation. Here is how one can think about it. The equation (3.25) models the diffusion, transport, and reaction of a scalar quantity $w(x,t)$. The vector field b is the "drift" or wind. The matrix a_{ij} determines the rates of diffusion in a given direction. The process $X_t^{x,t}$ may be thought of as the paths of particles diffusing in this velocity field. The function $c(x,t)$ represents a reaction rate. So, imagine hot, reactive particles being carried in the wind. Now, consider the formula (3.28) for the case $c \equiv 0$ (no reaction). What determines the temperature at a point (x,t)? The temperature at this point is determined by which particles arrive at point x at time t and how hot those particles were initially. The quantity $f(X_t^{x,t})$ represents the initial "temperature" evaluated at the "end" of the path $f(X_t^{x,t})$. Notice that $X_s^{x,t}$ actually runs backwards in the time-frame associated with the PDE. Roughly speaking, $f(X_t^{x,t})$ tells us what information propagates to the point x at time t. The paths are random; formula (3.28) says that the solution is determined by the expectation over all such particles. In the case that $c \neq 0$, formula (3.27) tells us that the reaction heats up each particle along its trajectory, increasing (or decreasing) its temperature by a factor of $e^{\int_0^t c(X_s^{x,t}, t-s)\, ds}$. Notice that when a_{ij}, b, and c are independent of t, we can replace $t-s$ in the above expressions with s.

Poisson's equation

Here we use Itô's formula to derive a representation for solutions to Poisson's equation with a variable zero-order term. Suppose that $w(x)$ is C^2 and bounded, and satisfies

$$\sum_{i,j} \frac{1}{2} a_{ij}(x) w_{x_i x_j} + \sum_j b_j(x) w_{x_j} - c(x) w = f(x), \quad x \in \mathbb{R}^d \tag{3.29}$$

with $c(x) \geq c_0 > 0$ for some constant $c_0 > 0$. As before, we assume $a_{ij} = \sigma\sigma^T$ is uniformly positive definite, and that a, b, and c satsify the continuity criteria given earlier.

Theorem 3.4 *Suppose that $X_t(\omega)$ solves the stochastic differential equation*

$$dX_t = b(X_t)\, dt + \sum_j \sigma_{ij}(X_t)\, dB_t^{(j)}, \quad t \geq 0 \tag{3.30}$$

with $X_0(\omega) = x \in \mathbb{R}^d$, almost surely. The solution $w(x)$ is represented by

$$w(x) = \mathbb{E}\left[\int_0^\infty e^{-\int_0^s c(X_\tau)\, d\tau} f(X_s)\, ds\right]. \tag{3.31}$$

Proof: Now apply Itô's formula and the product rule to the process

$$H_t(\omega) = e^{-\int_0^t c(X_s) ds} w(X_t^x). \tag{3.32}$$

We compute:

$$\begin{aligned} H_t - H_0 &= w(X_t) e^{-\int_0^t c(X_s)\, ds} - w(X_0) \\ &= \int_0^t e^{-\int_0^s c(X_\tau)\, d\tau} \left(\sum_j b_j w_{x_j} + \frac{1}{2} \sum_{i,j} a_{ij} w_{x_i x_j} - c(X_s) w \right) ds \\ &+ \int_0^t e^{-\int_0^s c(X_\tau)\, d\tau} \sum_{i,j} w_{x_i} \sigma_{ij} dB_s^{(j)} \\ &= \int_0^t e^{-\int_0^s c(X_\tau)\, d\tau} f(X_s)\, ds + \int_0^t e^{-\int_0^s c(X_\tau)\, d\tau} \sum_{i,j} w_{x_i} \sigma_{ij} dB_s^{(j)}. \end{aligned} \tag{3.33}$$

Now we take the expectation of both sides and let $t \to \infty$. Due to the lower bound on $c(x)$,

$$\lim_{t \to \infty} \left| \mathbb{E}\left[w(X_t) e^{-\int_0^t c(X_s)\, ds} \right] \right| \leq \lim_{t \to \infty} e^{-c_0 t} \|w\|_\infty = 0. \tag{3.34}$$

Therefore,

$$w(x) = -\mathbb{E} \int_0^\infty e^{-\int_0^s c(X_\tau)\, d\tau} f(X_s^x)\, ds, \tag{3.35}$$

and we are done. □

Problems in bounded domains

So far we have considered solutions to partial differential equations posed in the whole space $x \in \mathbb{R}^d$. Itô's formula also leads to representation formulas for solutions to PDE's posed in a bounded domain with appropriate boundary conditions. We consider two types of problems: boundary value problems for elliptic equations and initial value/terminal value problems for parabolic equations.

Boundary value problems for elliptic operators

Suppose that $D \subset \mathbb{R}^d$ is a smooth, bounded domain. Let $w(x) \in C^2(\bar{D})$ satisfy

$$\sum_{i,j} \frac{1}{2} a_{ij}(x) w_{x_i x_j} + \sum_j b_j(x) w_{x_j} - c(x) w = f(x), \quad x \in D, \tag{3.36}$$

with boundary condition $w(x) = g(x)$ for $x \in \partial D$. The function $g(x)$ is prescribed. As usual, we assume that the operator in the left side is elliptic and coefficients are sufficiently regular and bounded.

In addition, we need to assume that the function $c(x) \geq 0$. The need for this extra assumption can be seen on the very simple one-dimensional example: consider the problem

$$u'' - cu = 0, \quad u(0) = 0, \quad u(\pi) = 0.$$

If $c = -1$ this problem has two linearly independent solutions: $u_1(x) \equiv 0$, and $u_2(x) = \sin x$. On the other hand, if $c(x) \geq 0$ this can not happen: solution of the boundary value problem is always unique. Indeed, let w_1 and w_2 be two solutions of (3.36) with some prescribed functions $f(x)$ and $g(x)$. The difference $w(x) = w_1(x) - w_2(x)$ satisfies

$$\sum_{i,j} \frac{1}{2} a_{ij}(x) w_{x_i x_j} + \sum_j b_j(x) w_{x_j} - c(x) w = 0, \quad x \in D, \tag{3.37}$$

with the boundary condition $w(x) = 0$ for $x \in \partial D$. Since $c(x) \geq 0$ the maximum principle applies and shows that $w(x) = 0$ in D meaning that solution is unique. The reason why the maximum principle principle applies if $c(x) \geq 0$ is easily seen if we impose a slightly stronger condition $c(x) > 0$. Then, if $w(x)$ solves (3.37) and attains its maximum at some interior point $x_0 \in D$, the Hessian matrix $H(x)$ with the entries

$$H_{ij}(x_0) = \frac{\partial^2 w(x_0)}{\partial x_i \partial x_j}$$

is non-positive definite. Therefore, since $a_{ij}(x_0)$ is a positive-definite matrix (that follows from ellipticity), we have

$$\sum_{ij} a_{ij}(x_0) \frac{\partial^2 w(x_0)}{\partial x_i \partial x_j} \leq 0.$$

In addition, at x_0 we have

$$\frac{\partial u(x_0)}{\partial x_j} = 0$$

for all $j = 1, \ldots, n$. Using this in (3.37) we obtain

$$-c(x)w(x_0) \geq 0,$$

whence $w(x_0) \leq 0$ if $c(x) > 0$ in D. It follows that $w(x) \leq 0$ for all $x \in D$. A similar analysis at the point x_1 where $w(x)$ attains it minimum shows that

$$-c(x)w(x_1) \leq 0,$$

meaning that $w(x_1) \geq 0$ Therefore, $w(x) \geq 0$ in D, and we conclude that $w(x) = 0$. The weaker assumption $c(x) \geq 0$ requires a slightly more subtle analysis but the basic idea is the same.

How can we represent the solution of the boundary value problem? If $X_t(\omega)$ solves the stochastic differential equation

$$dX_t = b(X_t)\,dt + \sum_j \sigma_{ij}(X_t)\,dB_t^{(j)}, \quad t \geq 0 \tag{3.38}$$

with $X_0(\omega) = x \in D$, then the trajectories will travel outside of the set D, where the function w is not defined. To overcome this difficulty, we define the stopping time

$$\gamma_D(\omega) = \inf\{t \mid X_t \in \mathbb{R}^d \setminus D\}.$$

This is the first hitting time to the boundary ∂D. A basic result in the SDE theory (see Richard Bass book "Diffusions and Elliptic Operators", Proposition I.8.2) says that $\gamma_D(\omega) < +\infty$ a.s. if the domain D is bounded. Then, we can define the process

$$H_t(\omega) = e^{-\int_0^{t\wedge\gamma_D} c(X_s)ds} w(X_{t\wedge\gamma_D}^x). \tag{3.39}$$

Here, we denote $t \wedge \gamma_D := \min(t, \gamma_D)$. Itô's formula and the product rule then imply that

$$\begin{aligned}
H_t - H_0 &= w(X_{t\wedge\gamma_D})e^{-\int_0^{t\wedge\gamma_D} c(X_s)\,ds} - w(X_0) \\
&= \int_0^{t\wedge\gamma_D} e^{-\int_0^s c(X_\tau)\,d\tau} \left(\sum_j b_j w_{x_j} + \frac{1}{2}\sum_{i,j} a_{ij} w_{x_i x_j} - c(X_s) w \right) ds \\
&\quad + \int_0^{t\wedge\gamma_D} e^{-\int_0^s c(X_\tau)\,d\tau} \sum_{i,j} w_{x_i}\sigma_{ij} dB_s^{(j)} \\
&= \int_0^{t\wedge\gamma_D} e^{-\int_0^s c(X_\tau)\,d\tau} f(X_s)\,ds + \int_0^{t\wedge\gamma_D} e^{-\int_0^s c(X_\tau)\,d\tau} \sum_{i,j} w_{x_i}\sigma_{ij} dB_s^{(j)}
\end{aligned} \tag{3.40}$$

As before, we now take the expectation of both sides and let $t \to \infty$. As $\gamma_D(t)$ is finite a.s., we have $\lim_{t\to\infty} \gamma_D(\omega) \wedge t = \gamma_D(\omega)$, also almost surely. Consequently, the fact that w is bounded and that $c \geq 0$, allows us to use the dominated convergence theorem to show that

$$\lim_{t\to\infty} \mathbb{E}\left[w(X_{t\wedge\gamma_D})e^{-\int_0^{t\wedge\gamma_D} c(X_s)\,ds}\right] = \mathbb{E}\left[w(X_{\gamma_D})e^{-\int_0^{\gamma_D} c(X_s)\,ds}\right] = \mathbb{E}\left[g(X_{\gamma_D})e^{-\int_0^{\gamma_D} c(X_s)\,ds}\right]. \tag{3.41}$$

Similarly, using the fact that f is bounded and $E[\gamma_D] < \infty$, we may use the dominated convergence theorem to show that

$$\lim_{t\to\infty} \mathbb{E}\left[\int_0^{t\wedge\gamma_D} e^{-\int_0^s c(X_\tau)\,d\tau} f(X_s)\,ds\right] = \mathbb{E}\left[\int_0^{\gamma_D} e^{-\int_0^s c(X_\tau)\,d\tau} f(X_s)\,ds\right] \tag{3.42}$$

Therefore, taking $t \to \infty$, we obtain a representation for $w(x)$:

$$w(x) \;=\; E\left[g(X_{\gamma_D})e^{-\int_0^{\gamma_D} c(X_s)\,ds}\right] - E\left[\int_0^{\gamma_D} e^{-\int_0^s c(X_\tau)\,d\tau} f(X_s)\,ds\right]. \tag{3.43}$$

Notice that with the stronger assumption $c(x) \geq c_0 > 0$, we could lift the condition that $E[\gamma_D] < \infty$, which was used in the application of the dominated convergence theorem to obtain (3.42). We could also lift the restriction that $w \in C^2(\bar{D})$, and require only that $w \in C^2(D) \cap C(\bar{D})$ (thus, the second derivatives might blow up at that boundary). To handle this case, stop the process when it is distance ϵ from the boundary. Then let $\epsilon \to 0$.

Example 1: In particular, this representation shows that if $w(x)$ solves $\Delta w = 0$ in D with $w(x) = g(x)$ for $x \in \partial D$, then

$$w(x) = E\left[g(x + \sqrt{2}B_{\gamma_D})\right] \tag{3.44}$$

The quantity $g(x+\sqrt{2}B_{\gamma_D})$ is the boundary function evaluated at the point where the process first hits the boundary. The solution to the PDE is the expectation of these values.

Initial boundary value problems

Suppose that $D \subset \mathbb{R}^d$ is a smooth bounded domain. Let $D_T = D \times (0,T]$ denote the parabolic cylinder. Suppose that $w(t,x) \in C^{2,1}(D_T) \cap C(\bar{D}_T)$ satisfies the initial value problem

$$\begin{aligned}
&w_t = \sum_{i,j} \frac{1}{2} a_{ij}(t,x) w_{x_i x_j} + \sum_j b_j(t,x) w_{x_j} + c(t,x) w, \quad x \in D, \;\; t > 0 \\
&w(0,x) = f(x) \quad x \in D \\
&w(t,x) = g(t,x) \quad x \in \partial D, t \geq 0.
\end{aligned}$$

Here we assume that $c(x,t)$ is bounded and continuous. For given $(t,x) \in D_T$, let $X_s^{x,t}(\omega)$ satisfy

$$dX_s^{x,t} = b(t-s, X_s^{x,t})\,ds + \sum_j \sigma_{ij}(t-s, X_s^{x,t})\,dB_s^{(j)}, \quad s \in [0,t]. \tag{3.45}$$

Define the stopping time $\gamma_D^{x,t} = \inf\{s \geq 0 \,|\, X_s^{x,t} \in \mathbb{R} \setminus D\}$. This is the first time the process hits the boundary of the set D. Then define $\gamma^{x,t} = \gamma_D^{x,t} \wedge t$ This is also a stopping time, and it represents the time at which the process $(X_s^{x,t}, t-s)$ hits the parabolic boundary $(D \times \{0\}) \cup (\partial D \times [0,T])$, which is the boundary of the set D_T. For convient notation, let us define the function

$$k(t,x) = \begin{cases} f(x), & \text{if } t = 0,\; x \in \bar{D} \\ g(t,x), & \text{if } t > 0,\; x \in \partial D \end{cases} \tag{3.46}$$

This function is equal to $f(x)$ at the base of the parabolic boundary, and it is equal to $g(t,x)$ on the sides of the parabolic boundary.

Theorem 3.5 *Under the above assumptions, $w(x,t)$ satisfies*

$$w(x,t) = E\left[k(X^{x,t}_{\gamma^{x,t}}, \gamma^{x,t})e^{\int_0^{\gamma^{x,t}} c(X^{x,t}_s, t-s)\,ds}\right] \tag{3.47}$$

Proof: I leave this as an exercise. It may be proved as in the other cases. □

Transition Densities

Consider the vector-valued stochastic process defined by

$$dX_t = b(X_t)\,dt + \sigma^{ij}(X_t)\,dW^j_t \quad \text{for } \; t > 0, \qquad X_0(\omega) = x. \tag{3.48}$$

Suppose that $a_{ij} = \sigma\sigma^T$ is uniformly positive. Suppose also that a and b satisfy the continuity conditions described previously. Because of the Markov property of Brownian motion, one can show that X_t is a Markov process satisfying

$$P(X_t \in A \,|\, \mathcal{F}_s) = P(X_t \in A \,|\, X_s), \quad \forall\, s \in [0,t). \tag{3.49}$$

Suppose that X_t has a smooth transition density $p(x,s;y,t)$. This means that

$$P(X_t \in A \,|\, X_s = x) = \int_A p(x,s;y,t)\,dy \tag{3.50}$$

and

$$E\left[f(X_t) \,|\; X_s = x\right] = \int_{\mathbb{R}^d} f(y)p(x,s;y,t)\,dy \tag{3.51}$$

for suitable functions f. What equation does $p(x,s;y,t)$ satisfy?

Here is a formal computation that can be made rigorous under suitable smoothness and growth assumptions on the coefficients b and σ^{ij}. If $f(x)$ is smooth and compactly supported, then Itô's formula tells us that

$$f(X_t) - f(X_s) = \int_s^t \mathcal{A}f(X_r)\,dr + \int_s^t \sum_{ij} \frac{\partial f}{\partial x_i}(X_r)\sigma^{ij}(X_r)dW^j_r \tag{3.52}$$

where $\mathcal{A}$ denotes the differential operator

$$\mathcal{A}f(y) := \frac{1}{2}\sum_{ij} a_{ij}(y) f_{y_i y_j} + b(y)\cdot\nabla_y f \tag{3.53}$$

Conditioning on the event $X_s = x$ and taking the expectation, we obtain

$$E[f(X_t) \,|\, X_s = x] - E[f(X_s) \,|\, X_s = x] = \int_s^t E[\mathcal{A}f(X_r) \,|\, X_s = x]\,dr. \tag{3.54}$$

Now using the definition of the transition density, we may write this expression as

$$\int_{\mathbb{R}^d} f(y)p(x,s;y,t)\,dy - f(x) = \int_s^t \int_{\mathbb{R}^d} (\mathcal{A}f(y))p(x,s;y;r)\,dy\,dr. \tag{3.55}$$

Formally differentiating both sides with respect to t, we obtain the equation

$$\int_{\mathbb{R}^d} f(y)p_t(x,s;y,t)\,dy = \int_{\mathbb{R}^d} (\mathcal{A}f(y))p(x,s;y;t)\,dy. \tag{3.56}$$

Now, integrate by parts on the right hand side:

$$\begin{aligned}
\int_{\mathbb{R}^d} (\mathcal{A}f(y))p(x,s;y;t)\,dy &= \int_{\mathbb{R}^d} (\frac{1}{2}\sum_{ij} a_{ij}(x) f_{y_i y_j} + b(x)\cdot\nabla_y f)p(x,s;y;t)\,dy \\
= \int_{\mathbb{R}^d} f(y) &\left(\frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial_{y_i}\partial_{y_j}}(a_{ij}(x)p(x,s;y;t)) - \nabla_y\cdot(b(y)p(x,s;y;t))\right) dy \\
= \int_{\mathbb{R}^d} f(y) &\left(\mathcal{A}_y^* p(x,s;y,t)\right)\,dy.
\end{aligned} \tag{3.57}$$

Here $\mathcal{A}_y^*$ is the **adjoint operator** defined by

$$\mathcal{A}_y^* g(y) := \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial_{y_i}\partial_{y_j}}(a_{ij}(y)g(y)) - \nabla_y\cdot(b(y)g(y)). \tag{3.58}$$

In the integration by parts step, the boundary terms vanish since f has compact support. Therefore, $p(x,s;y,t)$ should satisfy

$$\int_{\mathbb{R}^d} f(y)\left(p_t(x,s;y,t) - \mathcal{A}_y^* p(x,s;y,t)\right)\,dy = 0. \tag{3.59}$$

Since $f(y)$ is chosen arbitrarily, and since we assume p to be sufficiently smooth, this implies that for each fixed x and s, the function $u(y,t) = p(x,s;y,t)$ satisfies $u_t = \mathcal{A}_y^* u$. That is,

$$\frac{\partial}{\partial t}p(x,s;y,t) = \mathcal{A}_y^* p(x,s;y,t). \tag{3.60}$$

As $t \searrow s$, $p(x,s;y,t)$ as a function of y converges to a delta distribution centered at $y = x$. This equation (3.60) is often called the **Kolmogorov forward equation** for the transition density $p(x,s;y,t)$. The term "forward" is applied since it describes the forward evolution of the probability density for X_t.

For fixed y and t, the function $u(x,s) = p(x,s;y,t)$ satisfies a different equation. To derive this equation, suppose that f is again smooth and compactly supported. We have already shown that the solution to the terminal value problem

$$w_s + \mathcal{A}_x w = 0, \quad s < t, \;\; x \in \mathbb{R}^d \tag{3.61}$$

with terminal data $w(x,t) = f(x)$ has the representation

$$w(x,s) = E\left[f(X_t) \mid X_s = x\right] = \int_{\mathbb{R}^d} f(y)p(x,s;y,t)\,dy \tag{3.62}$$

Formally differentiating the integral expression with respect to s and x and using (3.61) we find that

$$\int_{\mathbb{R}^d} f(y)(p_s(x,s;y,t)\,dy + \mathcal{A}_x p(x,s;y,t))\,dy = 0 \tag{3.63}$$

Since f was arbitrarily chosen this implies that for each y, $p_s(x,s;y,t) + \mathcal{A}_x p(x,s;y,t) = 0$. Since x and s were also arbitrarily chosen, this suggests that for each y and t fixed,

$$\frac{\partial}{\partial s} p(x,s;y,t) + \mathcal{A}_x p(x,s;y,t) = 0. \tag{3.64}$$

Since the coefficients defining the process X_t are independent of t, the transition density is a function of $t-s$:

$$p(x,s;y,t) = \rho(x,y,t-s) \tag{3.65}$$

for some function $\rho(x,y,r)$. Then (3.64) shows that for fixed y, $\rho(x,y,t)$ satisfies

$$\frac{\partial}{\partial t}\rho(x,y,t) = \mathcal{A}_x \rho(x,y,t) \tag{3.66}$$

This equation is often called the **Kolmogorov backward equation**.

4 Second order elliptic equations

4.1 Sobolev spaces

Weak derivatives

A weak derivative is a natural extension of the derivative to a non-differentiable function. In order to motivate this notion, let $u \in C^1(\mathbb{R}^n)$ and ϕ be a smooth compactly supported test function. Then we have:

$$\int_{\mathbb{R}^n} u \frac{\partial \phi}{\partial x_i} = -\int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i}\phi dx.$$

Note that the left side makes sense whether u is differentiable or not – all we need is that $u(x)$ is in $L^1_{loc}(\mathbb{R}^n)$, that is, u is integrable over compact sets. This motivates the following definition.

Definition 4.1 *Let $u, v \in L^1_{loc}(\mathbb{R}^n)$, then $v = \partial u/\partial x_j$ is the weak derivative of u with respect to x_j if for any $\phi \in C_c^\infty(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} u \frac{\partial \phi}{\partial x_i} dx = -\int_{\mathbb{R}^n} v\phi dx.$$

Example 1. Let $u(x) = |x|$, then for any $\phi \in C_c^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} |x|\phi'(x)dx = \int_{-\infty}^0 (-x)\phi'(x)dx + \int_0^\infty x\phi(x)dx = \int_{-\infty}^0 \phi(x)dx - \int_0^\infty \phi(x)dx = \int_{\mathbb{R}} \mathrm{sgn}(x)\phi(x)dx.$$

Therefore, the weak derivative $u'(x) = \mathrm{sgn} x$.

Example 2. Let $u(x) = \text{sgn}(x)$, we claim that the weak derivative of $u(x)$ does not exist. Indeed, for any test function $\phi \in C_c^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} u\phi' dx = \int_0^\infty \phi'(x)dx - \int_{-\infty}^0 \phi'(x)dx = -2\phi(0).$$

Let us now assume that there exists some function $v \in L^1_{loc}(\mathbb{R})$ such that for any function ϕ as above we have

$$\int_{\mathbb{R}} v(x)\phi(x)dx = 2\phi(0). \tag{4.1}$$

Choose a sequence $\phi_m(x)$ such that $\phi_m(0) = 1$, $\phi_m(x) = 0$ for $|x| \geq 1/m$ and $0 \leq \phi_m(x) \leq 1$ for all $x \in [-1/m, 1/m]$. Then we have

$$\left| \int_{\mathbb{R}} v(x)\phi_m(x)dx \right| \leq \int_{-1/m}^{1/m} |v(x)|dx \to 0,$$

as $m \to +\infty$, since $v \in L^1_{lot}(\mathbb{R})$. This contradicts (4.1) and shows that the weak derivative $u'(x)$ does not exist.

The definition of the higher order weak derivatives is a natural extension of the above. If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_m$, then $v = D^\alpha u$ in a domain U if $v \in L^1_{loc}(U)$, and for any function $\phi \in C_c^\infty(U)$ we have

$$\int_U uD^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx.$$

Definition 4.2 *The Sobolev space $W^{k,p}(U)$ consists of all functions $u \in L^1_{loc}(U)$ such that for each multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(U)$.*

When $p = 2$ we use a special notation $H^k(U) = W^{k,2}(U)$. The norm in $W^{k,p}(U)$ is defined as

$$\|u\|_{W^{k,p}(U)} = \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|u\|_{W^{k,\infty}(U)} = \sum_{|\alpha| \leq k} \sup_U |D^\alpha u|, \quad p = \infty.$$

Sobolev-type inequalities

Sobolev spaces are defined in terms of weak derivatives, which brings about a natural question of how "nice in the usual sense" are functions in the Sobolev space $W^{k,p}(U)$ for some fixed k and p. Are they continuous? Differentiable in there usual sense? This is very useful to know since norms in Sobolev spaces are in terms of integrals and are usually much easier to establish for solutions of PDEs than point-wise estimates that are required to prove continuity of differentiability of solutions.

Let us look at the following example: take $U = \{|x| \leq 1\}$ be the unit disk in two dimensions, and set $f(x) = x/\sqrt{x^2+y^2}$. This function is discontinuous, and has weak derivatives

$$\frac{\partial f(x,y)}{\partial x} = \frac{y^2}{(x^2+y^2)^{3/2}}, \quad \frac{\partial f(x,y)}{\partial y} = -\frac{xy}{(x^2+y^2)^{3/2}}.$$

Note that

$$\int_U \left|\frac{\partial f}{\partial x}\right| dxdy \leq \int_0^1 \frac{1}{r} r dr = 1,$$

and similarly for $\partial f/\partial y$. Therefore, f lies in the Sobolev space $W^{1,1}(U)$. On the other hand, we have

$$\int_U \left|\frac{\partial f}{\partial y}\right|^2 dxdy = \int_0^1 \int_0^{2\pi} \frac{r^4 \cos^2\phi \sin^2\phi}{r^6} r d\phi dr = +\infty,$$

hence f does no lie in the Sobolev space $H^1(U)$. This indicates that "maybe" Sobolev functions in $H^1(U)$ are "better" than those in $W^{1,1}(U)$. Sobolev inequalities provide a way to quantify that.

Gagliardo-Nirenberg inequality

Let us ask the following question; can we bound $\|u\|_{L^q(\mathbb{R}^n)}$ in terms of $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ with some p and q? The answer is obviously not since $u \equiv 1$ has an infinite L^q norm for any $1 \leq q < \infty$ but $\nabla u \equiv 0$. Let us restrict the question to functions $u \in C_c^\infty(\mathbb{R}^n)$ and ask whether it is true that

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}, \text{ for all } u \in C_c^\infty(\mathbb{R}^n). \tag{4.2}$$

The constant C should not depend on the function u – hence, in particular, it would not depend on the support of u. This is very suspicious if we think of some sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ that approximates the function $u \equiv 1$ as $m \to +\infty$. In order to see if we have a chance, consider a family of functions $u_\lambda(x) = u(\lambda x)$, with $\lambda > 0$, and see how (4.2) holds up as we vary the constant λ:

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(\lambda x)|^q dx\right)^{1/q} = \lambda^{-n/q}\|u\|_{L^q(\mathbb{R}^n)},$$

and

$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \lambda^p |\nabla u(\lambda x)|^p dx\right)^{1/p} = \lambda^{1-n/p}\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

If (4.2) holds for all $\lambda > 0$ we should have then

$$\lambda^{-n/q}\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-n/p}\|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{4.3}$$

for all $\lambda > 0$ and all $u \in C_c^\infty(\mathbb{R}^n)$. If we fix u in (4.3), we should have

$$\lambda^{n/p-n/q-1} \leq \frac{C\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^q(\mathbb{R}^n)}}.$$

This is only possible if

$$\frac{n}{p} - \frac{n}{q} = 1. \tag{4.4}$$

This tells us two things: first, given p, we can only hope to prove (4.2) for

$$q = \frac{np}{n-p}, \tag{4.5}$$

and that the range of p should be restricted to $1 \le p < n$.

Theorem 4.3 *(Gagliardo-Nirenberg inequality) Assume that $1 \le p < n$, and let $q = np/(n-p)$. There exists a constant $C > 0$ so that*

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)}, \text{ for all } u \in C_c^\infty(\mathbb{R}^n). \tag{4.6}$$

We will not prove this theorem here.

For bounded domains we have the following version.

Theorem 4.4 *Let $U \subset \mathbb{R}^n$ be a bounded domain with a C^1-boundary ∂U. Assume that $1 \le p < n$, $q = np/(n-p)$, and $u \in W^{1,p}(U)$. Then $u \in L^q(U)$ and*

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{1,p}(U)}. \tag{4.7}$$

The constant C depends only on p, n and U.

Morrey's inequality

In order to understand how being in some Sobolev space and the continuity of a function are related, recall the Hölder norm a function:

$$\|u\|_{C^{0,\alpha}(U)} = \sup_{x\in U}|u(x)| + \sup_{x,y\in U}\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Let us see when the inequality

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{4.8}$$

can hold. Once again, we fix $u \in C_c^\infty(\mathbb{R}^n)$, and consider the rescaled functions $u_\lambda(x) = u(\lambda x)$. The $W^{1,p}$-norm of the rescaled function is

$$\|u_\lambda\|^p_{W^{1,p}(\mathbb{R}^n)} = \|u_\lambda\|^p_{L^p} + \|\nabla u_\lambda\|^p_{L^p} = \lambda^{-n}\|u\|^p_{L^p} + \lambda^{p-n}\|\nabla u\|^p_{L^p}.$$

How does the Hölder norm scale with λ? We have

$$\|u_\lambda\|_{C^{0,\alpha}(U)} = \sup_x|u(\lambda x)| + \sup_{x,y}\frac{|u(\lambda x)-u(\lambda y)|}{|x-y|^\alpha} = \|u\|_{C(\mathbb{R}^n)} + \lambda^\alpha F_\alpha(u).$$

Here we have set

$$F_\alpha(u) = \sup_{x,y}\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Therefore, for (4.8) to hold we should have, at least, that

$$\|u\|_{C(\mathbb{R}^n)} + \lambda^\alpha F_\alpha(u) \le C(\lambda^{-n/p}\|u\|_{L^p} + \lambda^{1-n/p}\|\nabla u\|_{L^p}). \tag{4.9}$$

If $p \le n$ then fixing u and letting $\lambda \to +\infty$ in (4.9) we would reach a contradiction. Hence, we have to take $n < p \le +\infty$, and then we can take $\alpha = 1 - n/p$. This scaling analysis is confirmed by the following theorem.

Theorem 4.5 *(Morrey's inequality) Assume that $n < p \le +\infty$. There exists a constant $C > 0$ that depends only on p and n so that*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{4.10}$$

holds for all $u \in C^1(\mathbb{R}^n)$, with $\alpha = 1 - p/n$.

Rellich-Kondrashov compactness theorem

Gagliardo-Nirenberg inequality in a bounded domain implies that if $1 \le p < n$ and $q = np/(n-p)$ then $W^{1,p}(U)$ is a subset of $L^q(\mathbb{R}^n)$. Rellich-Kondrashov theorem shows that it is actually a compact subset of $L^q(\mathbb{R}^n)$, which is extremely important for the PDE theory.

Definition 4.6 *Let X and Y be Banach spaces. Then X is compactly embedded in Y, written as $X \subset\subset Y$ if*
(i) there exists a constant $C > 0$ so that $\|x\|_X \le C\|x\|_Y$ for all $x \in$.
(ii) any bounded sequence x_n in X has a subsequence x_{n_k} that converges in Y.

It is crucial, of course, that it is only required in (ii) that the subsequence converges in the space Y and not in X!

Example. Let $Y = l^2$ and X be the set of all sequences x_n such that

$$\sum_{n=1}^\infty n^2|x_n|^2 < +\infty,$$

equipped with the norm

$$\|x\|_1 = \left(\sum_{n=1}^\infty n^2|x_n|^2\right)^{1/2}.$$

A good exercise is to check that X is compactly embedded in Y. The reason is, roughly, that X "behaves as a finite-dimensional subspace of Y". This is because if $\|x\|_1 < 1$ then the entries x_n decay as $|x_n| \le 1/n^2$, meaning that "only the few first entries of x play a role".

Theorem 4.7 *(Rellich-Kondrashov) Let U be a bounded open domain in $\mathbb{R}^n$ with a C^1-boundary ∂U. Suppose that $1 \le p < n$, then $W^{1,p}(U)$ is compactly embedded into $L^q(U)$ for each $1 \le q < np/(n-p)$.*

This theorem is crucial for construction of solutions of PDEs.

Traces and $W_0^{1,p}(U)$ spaces

As we will be talking about PDEs with boundary conditions, we need to be able to say what it means that a function in $W^{1,p}(U)$ (about which we do not know that it is continuous) vanishes on the boundary ∂U. This is done with the help of what is known as the trace operator. The basic estimate that makes it work is the following lemma that shows that for smooth functions their restriction to the boundary is bounded by the $W^{1,p}$-norm inside the domain.

Lemma 4.8 *Let U be a bounded domain with a C^1-boundary ∂U. There exists a constant $C > 0$ so that for all $u \in C^1(\bar{U})$ we have*

$$\|u\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}. \tag{4.11}$$

We will not prove this lemma but in order to understand why that can be true, consider the situation when U is a two-dimensional domain with a smooth boundary that contains the interval $[-1, 1]$ on the x-axis. Let $\zeta(x_1, x_2)$ be a C_c^∞ function such that $\zeta(x, 0) = 1$ for $x \in [-1/2, 1/2]$ and ζ is supported inside U. Assume also, for simplicity of notation that $u(x_1, x_2) \geq 0$ and let $\Gamma = \{(x, 0) : -1 \leq x \leq 1\}$. Then we have

$$\begin{aligned}\|u\|^p_{L^p(\Gamma)} &= \int_\Gamma |u(x,0)|^p dx = \int_\Gamma \zeta(x,0)u^p(x,0)dx = \int_U \frac{\partial}{\partial x_2}(\zeta(x_1,x_2)u^p(x_1,x_2))dx_1dx_2 \\ &= \int_U \frac{\partial \zeta}{\partial x_2} u^p dx_1dx_2 + p\int_U \zeta u^{p-1}\frac{\partial u}{\partial x_2}dx_1dx_2 = I + II.\end{aligned}$$

For the first term we have simply

$$I \leq C\int_U |u|^p dx_1dx_2.$$

The second can be estimated using the inequality

$$|a^{p-1}b| \leq |a|^p + |b|^p,$$

giving

$$II \leq C\int_U |u|^p dx_1dx_2 + C\int_U |\nabla u|^p dx_1dx_2.$$

Together, we have

$$\|u\|^p_{L^p(\Gamma)} \leq C\|u\|^p_{L^p(U)} + C\|\nabla u\|^p_{L^p(U)} = C\|u\|^p_{W^{1,p}},$$

which is almost (4.11) (almost because Γ is only part of the boundary of U). The general proof of Lemma 4.8 proceeds very similarly by making a change of variable to straighten the boundary.

Lemma 4.8 means that we can define the restriction (or trace) operator $T : C^1(\bar{U}) \to L^p(\partial U)$ that is bounded as in (4.11). Since the set $C^1(\bar{U})$ is dense in $W^{1,p}(U)$ we can extend T to all functions in $W^{1,p}(U)$ by continuity, preserving the bound

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}. \tag{4.12}$$

Now, we can say what a zero boundary condition means for a function $u \in W^{1,p}(U)$.

Definition 4.9 *A function* $u \in W_0^{1,p}(U)$ *if* $Tu = 0$.

We have the following version of Gagliardo-Nirenberg inequality for bounded domains and functions in $W_0^{1,p}(U)$.

Theorem 4.10 *(Poincaré's inequality) Let* $U \subset \mathbb{R}^n$ *be a bounded domain with a* C^1*-boundary* ∂U. *Assume that* $1 \le p < n$, *and* $u \in W_0^{1,p}(U)$. *Then* $u \in L^q(U)$ *and*

$$\|u\|_{L^q(U)} \le C\|\nabla u\|_{L^p(U)}, \tag{4.13}$$

for all $1 \le q \le np/(n-p)$. *The constant* C *depends only on* p, q, n *and* U.

Another useful result, also known as the Poincaré inequality is as follows.

Theorem 4.11 *(Poincaré's inequality) Let* $U \subset \mathbb{R}^n$ *be a bounded domain with a* C^1*-boundary* ∂U. *There exists a constant* $C > 0$ *so that for all functions* $u \in H_0^1(U)$ *we have*

$$\|u\|_{L^2(U)} \le C\|\nabla u\|_{L^2(U)}. \tag{4.14}$$

Let us prove Theorem 4.11. Assume that this is false, then there exists a sequence of functions $u_k \in H_0^1(U)$ such that

$$\|u_k\|_{L^2(U)} \ge k\|\nabla u_k\|_{L^2(U)}.$$

Consider the renormalized functions

$$v_k = \frac{u_k}{\|u_k\|_{L^2(U)}},$$

then we have

$$\|v_k\|_{L^2(U)} = 1, \quad \|\nabla v_k\|_{L^2(U)} \le \frac{1}{k}. \tag{4.15}$$

The functions v_k are uniformly bounded in the Hilbert space $H_0^1(U)$, hence there exists a subsequence n_k such that v_{n_k} converges weakly to a limit $v \in H_0^1(U)$. Moreover, Rellich-Kondrashov theorem implies that v_{n_k} converges strongly to v in $L^2(U)$. It follows that $\|v\|_{L^2(U)} = 1$ and $\|\nabla v\|_{L^2(U)} = 0$. The last condition implies that $v = \text{const}$ a.e. in U. As v vanishes on ∂U, it is an easy exercise to see from Lemma 4.8 that then $v \equiv 0$ a.e., contradicting the fact that $\|v\|_{L^2(U)} = 1$.

4.2 Weak solutions of boundary value problems

The weak formulation for the Poisson equation

We now turn to purely PDE questions of existence and regularity of solutions to elliptic boundary value problems. Let us first consider an example of the Poisson equation

$$\begin{aligned} -\Delta u &= f \text{ in } U, \\ u &= 0 \text{ on } \partial U, \end{aligned} \tag{4.16}$$

in a bounded domain U, with a given function $f \in L^2(U)$. Multiplying (4.16) by a test function $\phi \in C_c^\infty(U)$ and integrating by parts gives

$$\int_U \nabla u \cdot \nabla \phi dx = \int_U f \phi dx. \tag{4.17}$$

The left side makes sense as long as u and ϕ are in $H_0^1(U)$, which motivates the following definition.

Definition 4.12 *We say that $u \in H_0^1(U)$ is a weak solution of (4.16) if for any function $v \in H_0^1(U)$ we have*

$$\int_U \nabla u \cdot \nabla v dx = \int_U f v dx. \tag{4.18}$$

In order to show that (4.16) has a weak solution, let us consider $H_0^1(U)$ equipped with the inner product

$$\langle u, v \rangle_1 = \int_U \nabla u \cdot \nabla v dx,$$

and the norm

$$\|u\|_1 = \left(\int_U |\nabla u|^2 dx \right)^{1/2}.$$

One can verify that $H_0^1(U)$ is still a Hilbert space under this new inner product. Note, in particular, that $\|u\|_1 = 0$ implies that $u \equiv 0$ in U since $u = 0$ on the boundary ∂U. The linear functional

$$A(v) = \int_U f v dx$$

is bounded on $H_)^1(U)$ with that norm since

$$|A(v)| \leq \|f\|_{L^2} \|v\|_{L^2} \leq C \|f\|_{L^2} \|\nabla v\|_{L^2} = C \|f\|_{L^2} \|v\|_1,$$

by the Poincaré inequality (4.14). It follows now from the Riesz representation theorem for bounded functionals on Hilbert spaces that there exists an element $w \in H_0^1(U)$ such that for any function $v \in H_0^1(U)$ we have

$$A(v) = \langle w, v \rangle.$$

Writing this more explicitly, gives

$$\int_U f v = \int_U \nabla w \cdot \nabla v dx, \text{ for any } v \in H_0^1(U),$$

which means exactly that w is a weak solution of the Poisson equation (4.16).

The weak formulation for general elliptic problems

We now construct weak solutions for general elliptic problems of the form

$$\begin{aligned} \mathcal{L}u &= f \text{ in } U, \\ u &= 0 \text{ on } \partial U, \end{aligned} \tag{4.19}$$

posed in a bounded domain $U \subset \mathbb{R}^n$. Here $\mathcal{L}$ is an operator of the form

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u, \tag{4.20}$$

or, in a more compact notation,

$$\mathcal{L}u = -\nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u + c(x)u.$$

The operator $\mathcal{L}$ in (4.20) is in its so-called divergence form. Sometimes we will also consider operators in the non-divergence form:

$$\mathcal{L}u = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u. \tag{4.21}$$

Both forms have their advantages: the divergence form is well suited for energy methods (integration by parts), while the non-divergence form is convenient for applications of the maximum principle. One can always pass from one form to another by modifying the drift $b(x)$. We will always assume that $\mathcal{L}$ is elliptic, that is, there exists a constant $c_0 > 0$ so that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi\xi_j \geq c_0|\xi|^2, \tag{4.22}$$

for all $\xi \in \mathbb{R}^n$, and that the coefficients are bounded:

$$|a_{ij}(x)|, |b_i(x)|, |c(x)| \leq K, \tag{4.23}$$

with some $K > 0$.

We will define the notion of a weak solution for functions in $H_0^1(U)$, hence we need to reformulate the problem so that it would make sense for $H_0^1(U)$ functions. In order to do that, lest us take (4.19) in the divergence form (4.20), multiply it by a smooth test function $v \in C_c^\infty$ and integrate over U:

$$\int_U \sum_{i,j=1}^{n} \left(a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + v(x)(b(x) \cdot \nabla u) + c(x)uv \right) dx = \int_U fv dx. \tag{4.24}$$

The left side of (4.24) makes sense if $u \in H_0^1(U)$, and, in addition, the condition $u \in H_0^1(U)$ automatically enforces the boundary condition $u = 0$ on ∂U. In the spirit of (4.24) let us define a bilinear form

$$B(u,v) = \int_U \sum_{i,j=1}^{n} \left(a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + v(x)(b(x) \cdot \nabla u) + c(x)uv \right) dx, \tag{4.25}$$

define for functions $u, v \in H_0^1(U)$.

Definition 4.13 *We say that a function $u \in H_0^1(U)$ solves the boundary value problem*

$$\begin{aligned} &-\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f \text{ in } U, \\ &u = 0 \text{ on } \partial U, \end{aligned} \tag{4.26}$$

if for any $v \in H_)^1(U)$ we have

$$B(u,v) = \int_U fv dx. \tag{4.27}$$

Here $B(u,v)$ is the bilinear form defined by (4.26).

The Lax-Milgram lemma

The weak formulation (4.27) is very well suited for an application of the Lax-Milgram lemma.

Theorem 4.14 *(Lax-Milgram lemma) Let H be a Hilbert space, and B be a bi-linear form on H. Assume that there exist two constant $c_{1,2} >)$ so that*

$$|B((u,v)| \le c_1 \|u\|_H \|v\|_H,$$

and

$$c_2 \|u\|_H^2 \le B(u,u).$$

Then for any bounded linear functional A on H there exists a unique $w \in H$ such that

$$B(w,v) = A(v), \quad \text{for all } v \in H.$$

Existence of the weak solutions

Let us now verify that the assumptions of the Lax-Milgram lemma hold for the bilinear form $B(u,v)$ defined in (4.25).

Lemma 4.15 *There exists $c > 0$ so that $|B(u,v)| \le c||u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$, for all $u, v \in H_0^1(U)$.*

Proof. We simply check

$$|B(u,v)| \le \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(U)} \int_U |\nabla u||\nabla v| dx + \sum_{i=1}^n \|b_i\|_{L^\infty(U)} \int_U |\nabla u||v| dx$$
$$+ \|c\|_{L^\infty(U)} \int_U |u||v| dx \le C \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)},$$

with some appropriate constant C. □

Lemma 4.16 *There exist two constants $c_{1,2} > 0$ so that*

$$c_1 \|u\|_{H_0^1(U)}^2 \le B(u,u) + c_2 \|u\|_{L^2(U)}^2. \tag{4.28}$$

Proof. Note that, because of the ellipticity condition, we have

$$\sum_{i.j=1`}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \ge c_0 |\nabla u|^2,$$

hence

$$B(u,u) \ge c_0 \int_U |\nabla u|^2 dx - \sum_{i=1}^n \|b_i\|_{L^\infty(U)} \int_U |\nabla u||u| dx - \|c\|_{L^\infty(U)} \int_U |u|^2 dx$$
$$\ge c_0 \int_U |\nabla u|^2 dx - nK \int_U |\nabla u||u| dx - K \int_U |u|^2 dx.$$

Using the inequality

$$ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2,$$

gives

$$B(u,u) \ \ge c_0 \int_U |\nabla u|^2 dx - nK\varepsilon \int_U |\nabla u|^2 dx - \frac{nK}{\varepsilon} \int_U |u|^2 dx - K \int_U |u|^2 dx.$$

Choosing $\varepsilon = c_0/2nK$ gives

$$B(u,u) + c_2 \int_U |u|^2 dx \ge \frac{c_0}{2} \int_U |\nabla u|^2 dx,$$

with $c_2 = K + nK/\varepsilon$. $\square$

Theorem 4.17 *There exists a number γ so that for each $\mu \ge \gamma$ and each $f \in L^2(U)$ there exists a weak solution $u \in H_0^1(U)$ of the boundary value problem*

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u + \mu u = f(x) \ x \in U, \tag{4.29}$$
$$u = 0 \ \text{on } \partial U.$$

In addition, this solution satisfies

$$\|u\|_{H_0^1(U)} \le C\|f\|_{L^2(U)}, \tag{4.30}$$

with a constant $C > 0$ that does not depend on the function $f \in L^2(U)$.

Proof. Take c_2 from Lemma 4.16 and consider $\mu \ge c_2$. Define the bilinear form

$$B_\mu(u,v) = B(u,v) + \mu(u,v).$$

Recall that

$$(u,v) = \int_U uv dx$$

is the L^2 inner product. Then $B_\mu(u,v)$ satisfies the assumptions of the Lax-Milgram lemma. Given $f \in L^2(U)$ define also the linear functional

$$A(v) = (f,v) = \int_U fv dx.$$

The Lax-Milgram lemma now implies that there exists a unique $u \in h_0^1(U)$ such that

$$B_\mu(u,v) = (f,v), \quad \text{for all } v \in H_0^1(U). \tag{4.31}$$

This means exactly that u is a weak solution of (4.29). In order to get the bound for u in (4.30) note that (4.31) taken with $v = u$ gives

$$\int_U fu dx = B_\mu(u,u) = B(u,u) + \mu\|u\|^2_{L^2(U)}.$$

Using Lemma 4.16 gives

$$\int_U fudx \geq c_1 \|u\|^2_{H^1_0(U)} - c_2 \|u\|^2_{L^2(U)} + \mu \|u\|^2_{L^2(U)} \geq c_1 \|u\|^2_{H^1_0(U)}.$$

The Cauchy-Schwartz inequality implies then

$$c_1 \|u\|^2_{H^1_0(U)} \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)}.$$

Finally, the Poincaré inequality $\|u\|_{L^2(U)} \leq C \|u\|_{H^1_0(U)}$ implies that (4.30) holds. $\square$

Note that if the first order term in the operator $\mathcal{L}$ vanishes: $b_j \equiv 0$, and the zero order term is non-negative $c(x) \geq 0$, then we can take $c_2 = 0$ in Lemma 4.16, hence Theorem 4.17 applies, in this situation, to all $\mu \geq 0$.

Compactness of the inverse

Let us now, once again, take c_2 as in Lemma 4.16 and define the operator

$$\tilde{\mathcal{L}} u = \mathcal{L} u + c_2 u.$$

Theorem 4.17 says that the inverse operator $\mathcal{K} = \tilde{\mathcal{L}}^{-1}$ is defined and acts on $L^2(U)$. Let us now re-write the equation

$$\mathcal{L} u = f, \quad u \in H^1_0(U), \tag{4.32}$$

as follows. First, (4.32) is equivalent to

$$\tilde{\mathcal{L}} u = c_2 u + f, \tag{4.33}$$

which, in turn, can be re-formulated as

$$u = \mathcal{K}(c_2 u + f), \tag{4.34}$$

or

$$\mathcal{K} u - \frac{1}{c_2} u = h, \tag{4.35}$$

where $h = (1/c_2)\mathcal{K} f$. The key observation in understanding when (4.35) (and hence (4.32)) has a solution, is a the following lemma.

Lemma 4.18 *The operator $\mathcal{K}: L^2(U) \to L^2(U)$ is a bounded linear compact operator.*

Proof. The fact that $\mathcal{K}$ is a bounded operator follows from estimate (4.30). Compactness of $\mathcal{K}$ also follows from that estimate if we recall the Rellich-Kondrashov theorem – (4.30) says that if $\|f\|_{L^2(U)} \leq C_0$, then $\|\mathcal{K} f\|_{H^1_0(U)} \leq C C_0$, and Rellich-Kondrashiov theorem says that the set $\{\|u\|_{H^1_0(U)}\| \leq C C_0\}$ is a compact subset of $L^2(U)$. this means that $\mathcal{K}$ maps a bounded subset of $L^2(U)$ into a compact set and is therefore a compact operator. $\square$

As a consequence, we know that there exist an at most countable set $\Sigma = \{\lambda_1, \ldots, \lambda_n, \ldots\}$ of complex eigenvalues of the operator $\mathcal{K}$, and the only possible accumulation point of the eigenvalues is $\lambda = 0$. Moreover, for any $\lambda \notin \Sigma$, the equation

$$\mathcal{K} f - \lambda f = u,$$

has a unique solution for all $\lambda \in \mathbb{C}$ and all $u \in L^2(U)$. Now, we have two possibilities (the Fredholm alternative): if $1/c_2 \in \Sigma$ then (4.35) may have no solution but there exists a non-zero solution of the homogenous problem

$$\mathcal{K}u = \frac{1}{c_2}u, \quad u \not\equiv 0.$$

This is equivalent to the fact that the problem

$$\mathcal{L}u = 0, \quad u \in H_0^1(U), \tag{4.36}$$

has non-trivial solution. The second possibility is that $1/c_2 \notin \Sigma$, and then (4.35) has a unique solution. An alternative way to formulate this result is as follows.

Theorem 4.19 *There exists an at most countable set $\Sigma \in \mathbb{R}$ so that the boundary value problem*

$$\mathcal{L}u - \lambda u = f, \tag{4.37}$$

with the boundary condition $u = 0$ on ∂U has a unique weak solution $u \in H_)^1(U)$ for all $f \in l^2(U)$. If the set Σ is infinite then $\lambda_k \to +\infty$ as $k \to +\infty$.

The only claim we still need to verify here is that if $\mathcal{L}$ has infinitely many real eigenvalues λ_k then $\lambda_k \to +\infty$ as $k \to +\infty$. First, we have $|\lambda_k| \to +\infty$ since $(\mathcal{L} + c_2 I)^{-1}$ is compact. Moreover, if we have

$$\mathcal{L}u = \lambda u, \quad u \in H_0^1(U),$$

we can normalize u so that $\|u\|_{L^2(U)} = 1$. Then, we have

$$\lambda = \lambda\|u\|_{L^2(U)}^2 = \int_U (\mathcal{L}u)udx = \int_U \sum_{ij} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx + \int_U \sum_{j=1}^n b_j(x)\frac{\partial u}{\partial x_j}udx + \int_U c(x)u^2dx$$
$$\geq c_0 \int_U |\nabla u|^2 dx - C\int_U u^2(x)dx - \frac{c_0}{2}\int_U |\nabla u|^2 dx - C\int_U u^2(x)dx.$$

We used above the estimate

$$\left|\int_U \sum_{j=1}^n b_j(x)\frac{\partial u}{\partial x_j}udx\right| \leq C\int_U |\nabla u||u|dx \leq \frac{C \cdot C}{2c_0}\int |u|^2 dx + \frac{C}{2C}c_0\int |\nabla u|^2 dx$$

We conclude that

$$\frac{c_0}{2}\int_U |\nabla u|^2 dx \leq \lambda + K,$$

with some constant $K > 0$ that depends only on the domain U. It follows that $\lambda > -K$, and we are done. □

Regularity of solutions of elliptic equations

The Poisson equation

Let us consider solutions of the Poisson equation

$$-\Delta u = f, \tag{4.38}$$

in the whole space $\mathbb{R}^n$. This equation says that "the Laplacian of u is as good as f". Laplacian is just a linear combination of some second derivatives but it turns out that (4.38) implies, actually, that all second derivatives of u are "as good as f". In order to see that, at least on the formal level, let us assume that u vanishes sufficiently fast at infinity, and multiply (4.38) by Δu:

$$\begin{aligned}\int_{\mathbb{R}^n} f^2 dx &= \int_{\mathbb{R}^n} (\Delta u)^2 dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_j^2} dx = -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i}\frac{\partial^3 u}{\partial x_j^2 \partial x_i} dx \\ &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 dx.\end{aligned}$$

Therefore, for instance, if $f \in L^2(\mathbb{R}^n)$, and $u \in H^1(\mathbb{R}^n)$, it is reasonable to expect that $u \in H^2(\mathbb{R}^n)$. The calculation above assumes that u is sufficiently rapidly decaying at infinity and is smooth enough to justify integration by parts, making it only formal but it nevertheless captures the spirit of the elliptic regularity theory.

Interior regularity

We consider an elliptic operator in the divergence form

$$\mathcal{L}u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

with the coefficients a_{ij}, b_i and c that are uniformly bounded, and a_{ij}, as always, assumed to be uniformly positive definite. In addition, we assume here that $a_{ij} \in C^1(U)$.

Theorem 4.20 *Assume that $u \in H^1(U)$ is a weak solution of*

$$\mathcal{L}u = f, \tag{4.39}$$

with a function $f \in L^2(U)$. Then for any open subset $V \subset\subset U$ we have the estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{4.40}$$

The constant C depends only on U, V and coefficients of $\mathcal{L}$.

Note that we do not impose any boundary condition on u – this result holds locally inside U so the boundary condition does not matter! On the other hand, we do not allow the case $V = U$ – u may behave very badly near the boundary ∂U unless we impose a reasonable boundary condition.

Maximum principles

Here, we will consider elliptic operators in non-divergence form

$$\mathcal{L}u = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j},$$

with no zero-order term. As always, we assume that the ellipticity condition holds for a_{ij}. We also assume that all coefficients a_{ij} and b_j are continuous.

Theorem 4.21 *(Weak maximum principle) Assume $u \in C^2(U) \cap C(\bar{U})$, then, if $\mathcal{L}u \le 0$ in U then u attains its maximum over U on the boundary ∂U.*

If $\mathcal{L}u \ge 0$ then looking at $v(x) = -u(x)$ we deduce that u attains its minimum over U on the boundary ∂U.

Proof. First, assume that we have a strict inequality

$$\mathcal{L}u < 0 \text{ in } U. \tag{4.41}$$

If $u(x)$ attains its maximum at an interior point $x_0 \in U$ then $\nabla u(x_0) = 0$, and the Hessian matrix

$$H_{ij}(x_0) = \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)$$

is non-negative: for any vector $\xi \in \mathbb{R}^n$ we have

$$\sum_{i,j=1}^{n} H_{ij}(x_0)\xi\xi_j \le 0.$$

The matrix $a_{ij}(x_0)$ is positive-definite, hence it can be decomposed as

$$a_{ij}(x) = \sum_{k=1}^{n} \lambda_k \xi_i^{(k)} \xi_j^{(k)}.$$

Here $\xi^{(k)}$ is the normalized eigenvector of the matrix a corresponding to the eigenvalue $\lambda_k > 0$. Then we have

$$\mathcal{L}u(x_0) = -\sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) = -\sum_{i,j=1}^{n} a_{ij}(x_0)H_{ij}(x_0) = -\sum_{i,j,k=1}^{n} \lambda_k \xi_i^{(k)} \xi_j^{(k)} H_{ij}(x_0)$$
$$= -\sum_{k=1}^{n} \lambda_k \sum_{i,j=1}^{n} H_{ij}(x_0)\xi_i^{(k)}\xi_j^{(k)} \ge 0,$$

which is a contradiction to (4.41).

In the general case, if we only have $\mathcal{L}u \le 0$, set

$$v(x) = u(x) + \varepsilon e^{\lambda x_1},$$

then

$$\mathcal{L}v = \mathcal{L}u + \varepsilon\mathcal{L}(e^{\lambda x_1}) \leq \varepsilon\mathcal{L}(e^{\lambda x_1}) = \varepsilon[-\lambda^2 a_{11} + \lambda b_1]e^{\lambda x_1}.$$

As the matrix a_{ij} is positive-definite, $a_{11}(x) \geq c_0 > 0$ in U. Therefore, if we choose $\lambda > 0$ sufficiently large (independent of $\varepsilon > 0$), we will have $\mathcal{L}v < 0$ and thus $v(x)$ would attain its maximum on the boundary ∂U:

$$v(x) \leq M_\varepsilon = \max_{y\in\partial U} v(y), \quad \text{for all } x \in U.$$

However, for $\lambda > 0$ fixed we have

$$u(x) = v(x) - \varepsilon e^{\lambda x_1} \leq v(x) \leq M_\varepsilon,$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ we conclude that

$$u(x) \leq M = \max_{y\in\partial U} u(y),$$

and we are done. □

Let us now consider domains satisfying the interior ball condition: for every $x \in \partial U$ there exists a ball $B \subset U$ such that $x \in \partial B$ – this condition is automatically satisfied if ∂U is C^2.

Lemma 4.22 *(Hopf's Lemma) Assume that $u \in C^2(U) \cap C(\bar{U})$ and $\mathcal{L}u \leq 0$ in U. Suppose in addition, that U satisfies the interior ball condition and there exists a point $x_0 \in \partial U$ such that $u(x_0) > u(x)$ for all $x \in U$, then*

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Proof. Let us assume that the ball that satisfies the interior ball condition at x_0 is $B(y, r)$. We will choose $\lambda > 0$ so that the function

$$v(x) = e^{-\lambda|x-y|^2} - e^{-\lambda r^2}$$

would satisfy

$$\mathcal{L}v \leq 0. \tag{4.42}$$

Let us compute

$$\mathcal{L}v = e^{-\lambda|x-y|^2}\sum_{i,j=1}^{n} a_{ij}(-4\lambda^2(x_i - y_i)(x_j - y_j) + 2\lambda\delta_{ij}) - e^{-\lambda|x-y|^2}\sum_{j=1}^{n} b_j(-2\lambda(x_j - y_j))$$
$$\leq e^{-\lambda|x-y|^2}\Big[- c_0\lambda^2|x-y|^2 + 2\lambda\sum_{i=1}^{n} a_{ii} + 2\lambda|b||x-y|\Big].$$

If we choose now $\lambda > 0$ sufficiently large (depending on r), we have, in the annulus $D = \{r/2 < |x-y| < r\}$:

$$\mathcal{L}v \leq e^{-\lambda|x-y|^2}\Big[- c_0\lambda^2\frac{r^2}{4} + 2\lambda\sum_{i=1}^{n} a_{ii} + 2\lambda|b|r\Big] \leq 0.$$

Next, note that since $u(x_0) > u(x)$ for all $x \in U$, hence there exists $\varepsilon > 0$ so that in the smaller ball we have

$$u(x_0) \geq u(x) + \varepsilon v(x) \text{ for } x \in B(y, r/2).$$

We also have

$$u(x_0) \geq u(x) + \varepsilon v(x) \text{ for } x \in \partial B(y, r),$$

simply because $v = 0$ on $\partial B(y, r)$. Now, we are ready to apply the weak maximum principle in the annulus D to the function

$$p(x) = u(x) + \varepsilon v(x) - u(x_0).$$

This function satisfies $\mathcal{L}p(x) \leq 0$ in D, and $p(x) \leq 0$ on ∂R. Hence, $p(x) \leq 0$ in D by the weak maximum principle. As $p(x_0) = 0$, we have

$$\frac{\partial p}{\partial \nu}(x_0) \geq 0.$$

It follows that

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x_0) > 0,$$

and we are done. □

Theorem 4.23 *(Strong maximum principle) Assume that U is a connected open bounded domain, $u \in C^2(U) \cap \bar{U}$ and $\mathcal{L}u \leq 0$ in U. If u attains its maximum at an interiir point of U then $u \equiv const$ in U.*

Proof. Let $M = \max_{\bar{U}} u(x)$ and consider the set $C = \{x \in U : \ u(x) = M\}$. Assume that $u \not\equiv M$, then $V = \{x \in U : \ u(x) < M\}$ is a non-empty open set. Choose a point $y \in V$ such that $\mathrm{dist}(y, C) < \mathrm{dist}(y, \partial U)$. Let B be the largest ball centered at y that lies in V. Then there exists a point $z \in C$ such that $z \in \partial B$. Then V satisfies the interior ball condition at z, hence $\partial u / \partial \nu(z) > 0$. But this is a contradiction – u attains its interior maximum at z whence $\nabla u(z) = 0$. □

5 Hamilton-Jacobi equations

We will now study solutions of the initial value problem for the Hamilton-Jacobi equations of the form

$$\begin{aligned} &u_t + H(\nabla u, x) = 0 \\ &u(0, x) = u_0(x). \end{aligned} \tag{5.1}$$

In order to explain how such problems come about we need to recall some basic notions form the control theory.

5.1 Deterministic Optimal Control

Consider the following abstract optimization problem. Let $y(s) : [t, T] \to \mathbb{R}^d$ denote the **state of a system** at time $s \in [t, T]$. This vector function could represent many things like the position and orientation of an aircraft, the amount of capital available to a government, or the wealth of an individual investor. We'll suppose that $y(s)$ satisfies the ordinary differential equation

$$\begin{aligned} y'(s) &= f(y(s), \alpha(s)), \quad s \in [t, T] \\ y(t) &= x \in \mathbb{R}^d \end{aligned} \tag{5.2}$$

(If $d > 1$, this is a system of ODEs.) The function $f(y, \alpha) : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ models the system dynamics. We'll suppose that f is bounded and Lipschitz continuous. The function $\alpha(s) : [t, T] \to A$ is called a **control**. The control takes values in the set A, a compact subset of $\mathbb{R}^m$. The set of all possible controls or **admissible controls** will be denoted by $\mathcal{A}_{t,T}$:

$$\mathcal{A}_{t,T} = \{\alpha(s) : [t, T] \to A \mid \ \alpha(s) \text{ is measureable}\}. \tag{5.3}$$

When the dependence on t and T is clear from the context, we will simply use $\mathcal{A}$ instead. By choosing α we have some control over the course of the system $y(t)$. For example, in a mechanics application $\alpha(s)$ might represent a throttle control, which determines how much thrust comes from an engine. Or, in an economics application α might represent the rate at which economic resources are consumed.

So, we choose a control $\alpha(\cdot)$ and the system evolves according to (5.2); we'd like to control the system in an optimal way, in the following sense. Suppose that the function $g(x) : \mathbb{R}^d \to \mathbb{R}$ represents a **final payoff**; this is a reward which depends on the final state of the system at time T. Also, the function $r(x, \alpha) : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ represents a **running payoff** or **running cost**. If $r > 0$, this would represent additional path-dependent payoff; if $r < 0$, this would represent operational costs incurred before the final payoff at time T. Given the initial state of the system $y(t) = x$, the optimization problem is to find an optimal control $\alpha^*(\cdot)$ that maximizes net profit:

$$J_{x,t}(\alpha^*) = \max_{\alpha(\cdot) \in \mathcal{A}} J_{x,t}(\alpha) = \max_{\alpha(\cdot) \in \mathcal{A}} \left[\int_t^T r(y(s), \alpha(s))\, ds + g(y(T)) \right] \tag{5.4}$$

If $r < 0$, this may be thought of as the optimal balance between payoff and running costs. Although, we may be able to control the system (by choosing α) so that the final payoff $g(y(T))$ is large, it may be very expensive to arrive at this state. So, we want to find the optimal control that balances these competing factors.

Even if an optimal control does not exist, we may study the function

$$u(x, t) = \max_{\alpha(\cdot) \in \mathcal{A}} J_{x,t}(\alpha) = \max_{\alpha(\cdot) \in \mathcal{A}} \left[\int_t^T r(y(s), \alpha(s))\, ds + g(y(T)) \right] \tag{5.5}$$

This function is called the **value function** associated with the control problem. It depends on x and t through the initial conditions defining $y(s)$. There are many interesting mathematical questions related to this optimization problem. For example:

1. For given (x,t), is there an optimal control α^*?
2. If so, how can we compute it?
3. How does the value function u depend on x and t? Does it satisfy a PDE?

Some Examples

Here are two examples which fit into this abstract framework.

Example 1: Suppose you want to drive a boat from position x_0 at time t to a position x_f at time T. Let $x(s)$ denote the position of the boat, $v(s)$ denote the velocity, then a simple model for the boat dynamics might be

$$\begin{aligned} x'(s) &= v(s) \\ v'(s) &= \frac{\alpha(s)}{(m_1 + m(s))} - \beta(v(s)) \\ m'(s) &= -k|\alpha(s)| \end{aligned}$$

Here $m(s)$ is the mass of the boat's fuel, m_1 is the boat's dry weight, the function $\beta(v) \geq 0$ models drag as the boat moves through the water. The vector $\alpha(s)$ represents a throttle and direction control, and its magnitude is proportional to the rate of fuel consumption (k is a proportionality constant). The acceleration is also proportional to the throttle control parameter.

How should we steer the boat in order to minimize fuel consumption? In this setting, the system state $y(s)$ is the vector $y(s) = (x(s), v(s), m(s))$. One way to model this problem would be to find

$$\max_{\alpha} J_{x_0,t}(\alpha) = \max_{\alpha} \left[m(T) + p(x(T)) \right] \tag{5.6}$$

Here $p \leq 0$ might be a function satisfying $p(x) = 0$ if x is close to x_f and $p(x) \ll -1$ if x is far from x_f. So, although we don't need to land precisely at x_f, there is a big penalty for leaving the boat far from x_f. There is no "running cost" in this model. Notice that it is possible for $m(s)$ to become negative, which is non-physical. We could fix this modeling issue by modifying the equations appropriately or by applying an additional **state constraint** of the form $0 \leq m(s)$.

Example 2: Here is a variant of a classic example studied by F. P. Ramsey (see *The Economic Journal*, Vol. 38, No. 152, 1928). The problem is to determine how much of a nation's resources should saved and how much should be consumed. Let $c(s)$ denote the rate of capital consumption, let $p(s)$ denote the rate of capital production, and let $k(s)$ denote the amount of capital at time s. Then the rate change in capital is the difference between the rates of production and consumption:

$$k'(s) = p(s) - c(s). \tag{5.7}$$

Suppose that the production is related to capital and consumption as $p(s) = P(c(s), k(s))$. Suppose also that consumption is related to capital according to $c(s) = \alpha(s)C(k(s))$, where

$\alpha(s)$ is a control. Therefore,

$$k'(s) = P(\alpha(s)C(k(s)), k(s)) - \alpha(s)C(k(s)). \tag{5.8}$$

Given current level of capital $k(t) = k_0$, we'd like to choose a level of consumption (by choosing α) which maximizes the total utility; this goal might be modeled by the optimal control problem

$$\max_{\alpha} J_{k_0,t}(\alpha) = \max_{\alpha} \left[\int_t^T U(c(s))\, ds + U_T(k(T)) \right] = \max_{\alpha} \left[\int_t^T U(\alpha(s)C(k(s)))\, ds + U_T(k(T)) \right].$$

Here U is a utility function, and U_T models some payoff representing the utility of having left-over capital $k(T)$ at time T.

5.2 The Dynamic Programming Principle

Theorem 5.1 *Let $u(x,t)$ be the value function defined by (5.5). If $t < \tau \leq T$, then*

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s), \alpha(s))\, ds + u(y(\tau), \tau) \right]. \tag{5.9}$$

The relation (5.44) is called the **Dynamic Programming Principle**, and it is a fundamental tool in the analysis of optimal control problems. It says that if we know the value function at time $\tau > t$, we may determine the value function at time t by optimizing from time t to time τ and using $u(\cdot, \tau)$ as the payoff. Roughly speaking, this is reminiscent of the Markov property of a stochastic process, in the sense that if we know $u(x,\tau)$ we can determing $u(\cdot, t)$ for $t < \tau$ without any other information about the control problem beyond time τ (ie. times $s \in [\tau, T]$). (More precisely, it means that $u(x,t)$ satisfies what is called a semi-group property.)

Proof of Theorem 5.1: At the heart of this proof of the Dynamic Programming Principle is the observation that any admissible control $\alpha \in \mathcal{A}_{t,T}$ is the combination of a control in $\mathcal{A}_{t,\tau}$ with a control in $\mathcal{A}_{\tau,T}$. We will express this relationship as

$$\mathcal{A}_{t,T} = \mathcal{A}_{t,\tau} \oplus \mathcal{A}_{\tau,T}. \tag{5.10}$$

This notation $\oplus$ means that if $\alpha_t(s) \in \mathcal{A}_{t,\tau}$ and $\alpha_\tau(s) \in \mathcal{A}_{\tau,T}$, then the control defined by splicing α_t and α_τ according to

$$\alpha(s) = (\alpha_t \oplus \alpha_\tau)(s) := \begin{cases} \alpha_t(s), & s \in [t, \tau] \\ \alpha_\tau(s), & s \in [\tau, T] \end{cases} \tag{5.11}$$

is an admissible control in $\mathcal{A}_{t,T}$. On the other hand, if we have $\alpha \in \mathcal{A}_{t,T}$, then by restricting the domain of α to $[t,\tau]$ we obtain an admissible control in $\mathcal{A}_{t,\tau}$. Similarly, by restricting the domain of α to $[\tau, T]$ we obtain an admissible control in $\mathcal{A}_{\tau,T}$.

The function u is defined as

$$\begin{aligned} u(x,t) &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^T r(y(s), \alpha(s))\, ds + g(y(T)) \right] \\ &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s), \alpha(s))\, ds + \int_\tau^T r(y(s), \alpha(s))\, ds + g(y(T)) \right]. \end{aligned}$$

Notice that the first integral on the right depends only on y and α up to time τ, while the last two terms depend on the values of y and α after time τ. Since a control $\alpha \in \mathcal{A}_{t,T}$ may be decomposed as $\alpha = \alpha_1 \oplus \alpha_2$ with $\alpha_1 \in \mathcal{A}_{t,\tau}$ and $\alpha_2 \in \mathcal{A}_{\tau,T}$, we may maximize over each component in the decomposition:

$$\begin{aligned} u(x,t) &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + \int_\tau^T r(y(s),\alpha(s))\,ds + g(y(T)) \right] \\ &= \max_{\alpha_1\in\mathcal{A}_{t,\tau},\alpha_2\in\mathcal{A}_{\tau,T},\alpha=\alpha_1\oplus\alpha_2} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + \int_\tau^T r(y(s),\alpha(s))\,ds + g(y(T)) \right]. \end{aligned}$$

On the right hand side, the system state $y(t)$ is determined by (5.2) with $\alpha = \alpha_1 \oplus \alpha_2 \in \mathcal{A}_{t,T}$. Therefore, we may decompose the system state as $y(s) = y_1 \oplus y_2$ where $y_1(s) : [t,\tau] \to \mathbb{R}^d$ and $y_2(s) : [\tau, T] \to \mathbb{R}^d$ are defined by

$$\begin{aligned} y_1'(s) &= f(y_1(s),\alpha_1(s)), \quad s \in [t,\tau] \\ y_1(t) &= x \end{aligned}$$

and

$$\begin{aligned} y_2'(s) &= f(y_2(s),\alpha_2(s)), \quad s \in [\tau,T] \\ y_2(\tau) &= y_1(\tau) = y(\tau). \end{aligned}$$

Here we use $\oplus$ to denote the splicing or gluing of y_1 and y_2 to create $y(t) : [t,T] \to \mathbb{R}^d$. Therefore,

$$u(x,t) = \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \max_{\alpha_2\in\mathcal{A}_{\tau,T}, y_2(\tau)=y_1(\tau)} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + \int_\tau^T r(y_2(s),\alpha_2(s))\,ds + g(y_2(T)) \right],$$

where the initial point for $y_2(\tau)$ is $y_2(\tau) = y_1(\tau)$. Observe that y_1 depends only on x and α_1, not on y_2 or α_2. Since the first integral depends only on α_1 and y_1, this may be rearranged as

$$\begin{aligned} u(x,t) &= \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \max_{\alpha_2\in\mathcal{A}_{\tau,T}, y_2(\tau)=y_1(\tau)} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + \int_\tau^T r(y_2(s),\alpha_2(s))\,ds + g(y_2(T)) \right] \\ &= \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + \max_{\alpha_2\in\mathcal{A}_{\tau,T}, y_2(\tau)=y_1(\tau)} \left(\int_\tau^T r(y_2(s),\alpha_2(s))\,ds + g(y_2(T)) \right) \right] \\ &= \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + u(y_1(\tau),\tau) \right] \qquad \text{(using the definition of } u\text{)} \\ &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + u(y(\tau),\tau) \right] \end{aligned} \tag{5.12}$$

This completes the proof. □

Notice that in this proof we have not assumed that an optimal control exists.

5.3 The Hamilton-Jacobi-Bellman Equation

How does the value function depend on x and t? Is it continuous in (x,t)? Is it differentiable? Does it satisfy a PDE? Unfortunately, the value function may not be differentiable, even in simple examples! Here is one interesting example of this fact. Suppose that $f(x, \alpha) = \alpha$, $g \equiv 0$, and $r(x, \alpha)$ is defined by

$$r(x, \alpha) = -\mathbb{I}_D(x) = \begin{cases} -1, & x \in D \\ 0, & x \in \mathbb{R}^d \setminus D \end{cases} \tag{5.13}$$

where $D \subset \mathbb{R}^d$ is some bounded set. Suppose that the set of admissible controls is defined by (5.49) with $A = \{|\alpha| \leq 1\}$. In this case, $y'(s) = \alpha(s)$ and $|y'(s)| \leq 1$. Therefore, the value function may be written as

$$u(x, t) = \max_{y:[t,T]\to\mathbb{R}^d, |y'|\leq 1,\ y(t)=x} \left[\int_t^T -\mathbb{I}_D(y(s))\, ds \right]. \tag{5.14}$$

Clearly $u(x, t) \leq 0$, and the optimum is obtained by paths that spend the least amount of time in the set D. If $x \in \mathbb{R}^d \setminus D$, then $u(x, t) = 0$, because we could take $y(s) = x$ for all $s \in [t, T]$. In this case, the system state doesn't change, so the integral is zero, which is clearly optimal. On the other hand, if $x \in D$ then the optimal control moves $y(s)$ to $\mathbb{R}^d \setminus D$ as quickly as possible and then stays outside D. Since $|y'(s)| \leq 1$, this implies that the value function is given explicitly by

$$u(x, t) = -\min\left((T - t), dist(x, \mathbb{R} \setminus D)\right) \tag{5.15}$$

where

$$dist(x, \mathbb{R} \setminus D) = \inf_{y \in \mathbb{R}\setminus D} |x - y|, \tag{5.16}$$

is the Euclidean distance from x to the outside of D. Albeit continuous, this function may not be differentiable! For example, suppose that $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ is the unit disk. In this case,

$$u(x, t) = \begin{cases} |x| - 1, & |x| \leq 1 \\ 0, & |x| \geq 1 \end{cases} \tag{5.17}$$

for $t \leq T - 1$. Thus $u(x, t)$ is not differentiable at the origin $x = (x_1, x_2) = (0, 0)$ for $t < T - 1$.

So, in general, the value function may not be differentiable. However, one can still derive a PDE satisfied by the value function. If the value function is differentiable, this equation is satisfied in the classical sense. At points where the value function is not differentiable, one can show that the value function (assuming it is at least continuous) satisfies the PDE in a weaker sense. This weaker notion of "solution" is called a "viscosity solution" of the PDE. For the moment, we will formally compute as if the value function were actually differentiable. We will come back to the weak solutions later.

For now, let us use the Dynamic Programming Principle to formally derive an equation solved by the value function $u(x, t)$. The Dynamic Programming Principle does not require differentiability of the value function; however, in our computations we assume that the value function is continuous and differentiable with respect to both x and t. The Dynamic Programming Principle tells us that

$$u(x, t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s), \alpha(s))\, ds + u(y(\tau), \tau) \right]. \tag{5.18}$$

To formally derive a PDE for u, we let $h \in (0, T-t)$ and set $\tau = t + h < T$, then

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^{t+h} r(y(s),\alpha(s))\,ds + u(y(t+h),t+h) \right]. \tag{5.19}$$

We'll assume that nearly optimal controls are approximately constant for $s \in [t, t+h]$.

First, consider the term $u(y(t+h), t+h)$. From the chain rule and our assumption that u is continuously differentiable in x and t, we conclude that

$$u(y(t+h),t+h) = u(y(t),t) + hy'(t)\cdot\nabla u(y(t),t) + hu_t(y(t),t) + o(h). \tag{5.20}$$

Recall that a function $\phi(h)$ is said to be $o(h)$ ("little oh of h") if $\lim_{h\to 0}(\phi(h)/h) = 0$. So, (5.20) says that $u(y(t+h), t+h)$ is equal to a linear function of h plus something that is $o(h)$ (i.e. higher order than h, but not necessarily $O(h^2)$). Therefore,

$$\begin{aligned} &u(y(t+h),t+h) = u(y(t),t) + hf(y(t),\alpha(t))\cdot\nabla u(y(t),t) + hu_t(y(t),t) + o(h) \\ &= u(x,t) + hf(x,\alpha(t))\cdot\nabla u(x,t) + hu_t(x,t) + o(h). \end{aligned} \tag{5.21}$$

Now, plug this into (5.19):

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^{t+h} r(y(s),\alpha(s))\,ds + u(x,t) + hf(x,\alpha(t))\cdot\nabla u(x,t) + hu_t(x,t) + o(h) \right]. \tag{5.22}$$

The term $u(x,t)$ may be pulled out of the maximum, so that it cancels with the left hand side:

$$0 = hu_t(x,t) + o(h) + \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^{t+h} r(y(s),\alpha(s))\,ds + hf(x,\alpha(t))\cdot\nabla u(x,t) \right]. \tag{5.23}$$

Now, divide by h and let $h \to 0$.

$$0 = u_t(x,t) + \frac{o(h)}{h} + \max_{\alpha(\cdot)\in\mathcal{A}} \left[\frac{1}{h}\int_t^{t+h} r(y(s),\alpha(s))\,ds + f(x,\alpha(t))\cdot\nabla u(x,t) \right]. \tag{5.24}$$

If $\alpha(s)$ is continuous at t, then as $h \to 0$,

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} r(y(s),\alpha(s))\,ds = r(y(t),\alpha(t)) = r(x,\alpha(t)) \tag{5.25}$$

So, if the nearly optimal controls are approximately constant for $s \in [t, t+h]$, then by letting $h \to 0$ in (5.24) we conclude that

$$u_t(x,t) + \max_{a\in A}\left[r(x,a) + f(x,a)\cdot\nabla u(x,t)\right] = 0, \quad x \in \mathbb{R}^d,\ t < T. \tag{5.26}$$

This equation is called the **Hamilton-Jacobi-Bellman equation**. The function $u(x,t)$ also satisfies the terminal condition

$$u(x,T) = g(x). \tag{5.27}$$

Notice that the HJB equation if a first-order, fully nonlinear equation, having the form

$$u_t + H(\nabla u, x) = 0$$

where the function H is defined by

$$H(p, x) = \max_{a \in A} \left[r(x, a) + f(x, a) \cdot p \right], \quad p \in \mathbb{R}^d, \tag{5.28}$$

and is sometimes called the **Hamiltonian**.

In addition to telling us how the value function depends on x and t, this PDE suggests what the optimal control should be. Suppose $u(x, t)$ is differentiable and solves the PDE in the classical sense. Then

$$u_t + H(\nabla u, x) = 0, \tag{5.29}$$

where $H(p, x)$ is defined by (5.28). Then the optimal control is computed by finding $(y^*(s), \alpha^*(s))$ which satisfies

$$H(\nabla u(y^*(s), s), y^*(s)) = r(y^*(s), \alpha^*(s)) + f(y^*(s), a^*(s)) \cdot \nabla u(y^*(s), s), \tag{5.30}$$

and

$$\begin{aligned} &\frac{d}{dt} y^*(s) = f(y^*(s), \alpha^*(s)), \quad s > t \\ &y^*(t) = x. \end{aligned} \tag{5.31}$$

Examples

Example 1: In this example we want to maximize utility from consumption of our resources over time interval $[t, T]$. If $c(s)$ is the rate of consumption, then we model the utility gained from this consumption as

$$U = \int_t^T e^{-\mu s} \psi(c(s))\, ds. \tag{5.32}$$

For example, we might choose ψ to be an increasing, concave function of c like $\psi(c) = c^\nu$ for some power $\nu \in (0, 1)$. The factor $e^{-\mu s}$ is a discount factor. Let us suppose that the rate of consumption is $c(s) = \alpha(s) y(s)$, where y is our total wealth and α is a control. So, α is approximately the proportion of total wealth consumed in a unit of time. Instead of consuming resources, we might invest them in such a way that our total wealth satisfies the ode

$$y'(s) = qy(s) - c(s) = qy(s) - \alpha(s) y(s) \tag{5.33}$$

Without consumption, our wealth would grow exponentially at rate $q > 0$. So, investing some of our wealth might allow us to consume more in the long run. The control $\alpha(s)$ should satisfy some realistic constraints, for example $\alpha \in A = [0, \bar{a}]$. Thus, if x denotes the current wealth $y(t) = x$, the control problem is to find

$$u(x, t) = \max_{\alpha \in A} \left[\int_t^T e^{-\mu s} (\alpha(s) y(s))^\nu \, ds + g(y(T)) \right] \tag{5.34}$$

The function g models some utility of leaving an amount of unused wealth $y(T)$ at the final time. In this example, we can actually compute $y(s)$ explicitly:

$$y(s) = y(t)e^{q(t-s)-\int_t^s \alpha(\tau)\,d\tau} \tag{5.35}$$

To determine the associated HJB equation, notice that $f(x,a) = (q-a)x$ and $r(x,a,s) = e^{-\mu s}(ax)^\nu$. Therefore, we expect the value function to satisfy

$$u(x,t) + \max_{a\in A}\left[e^{-\mu s}(ax)^\nu + (q-a)x\cdot\nabla u\right] = 0, \tag{5.36}$$

perhaps in a weak sense (viscosity solutions) rather than a classical sense.

Example 2: Here's an example from engineering. Suppose that $r \equiv 0$, $f(x,a) = -v(x) + a$, and $A = \{|a| \le \mu_0\}$. In this case, the HJB equation is

$$u_t(x,t) + \max_{|a|\le\mu_0}\left[-v(x)\cdot\nabla u(x,t) + a\cdot\nabla u(x,t)\right] = 0, \quad x\in\mathbb{R}^d,\ t<T. \tag{5.37}$$

It is easy to see that the optimal a is $a = \mu_0(\nabla u)/|\nabla u|$, so that the equation becomes

$$u_t - v(x)\cdot\nabla u + \mu_0|\nabla u| = 0 \tag{5.38}$$

The function $G(x,t) = u(x,T-t)$ satisfies

$$G_t + v(x)\cdot\nabla G = \mu_0|\nabla G|. \tag{5.39}$$

In the combustion community, this equation is called the "G-equation" and is used in computational models of premixed turbulent combustion. The level set $\{G=0\}$ is considered to be the flame surface. The parameter μ_0 corresponds to the laminar flame speed (i.e. the flame speed without the turbulent velocity field v).

Infinite Time Horizon

So far we have considered a deterministic control problem with **finite time horizon**. This means that the optimization involves a finite time interval and may involve a terminal payoff. One might also consider an optimization posed on an infinite time interval. Suppose that $y:[t,\infty)\to\mathbb{R}^d$ satisfies

$$\begin{aligned} y'(s) &= f(y(s),\alpha(s)), \quad s\in[t,\infty) \\ y(t) &= x\in\mathbb{R}^d. \end{aligned} \tag{5.40}$$

Now the domain for the control is also $[t,\infty)$. We'll use $\mathcal{A} = \mathcal{A}_{t,\infty}$ for the set of admissible controls. For $x\in\mathbb{R}^d$, define the value function

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} J_{x,t}(\alpha) = \max_{\alpha(\cdot)\in\mathcal{A}}\left[\int_t^\infty e^{-\lambda s} r(y(s),\alpha(s))\,ds\right]. \tag{5.41}$$

The exponential term in the integral is a discount factor; without it, the integral might be infinite. Notice that there is no terminal payoff, only running payoff. This optimal control problem is said to involve an **infinite time horizon**. Notice also that the value functions depends on t in a trivial way. Since r and f do not depend on t, we may change variables to see that

$$u(x,t) = e^{-\lambda t} u(x,0). \tag{5.42}$$

So, to find $u(x,t)$ it suffices to compute

$$u(x) = \max_{\alpha(\cdot)\in\mathcal{A}} J_x(\alpha) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_0^\infty e^{-\lambda s} r(y(s),\alpha(s))\, ds \right] \tag{5.43}$$

where $\mathcal{A} = \mathcal{A}_{0,\infty}$.

Theorem 5.2 (Dynamic Programming Principle) *Let $u(x)$ be the value function defined by (5.43). For any $x \in \mathbb{R}^d$ and $h > 0$,*

$$u(x) = \max_{\alpha(\cdot)\in\mathcal{A}_{0,h}} \left[\int_0^h e^{-\lambda s} r(y(s),\alpha(s))\, ds + e^{-\lambda h} u(y(h)) \right] \tag{5.44}$$

Proof: Exercise.

Using the Dynamic Programming Principle, one can formally derive the HJB equation for the infinite horizon control problem. The equation is:

$$-\lambda u + \max_{a\in A} \left[r(x,a) + f(x,\alpha)\cdot\nabla u \right] = 0 \tag{5.45}$$

which has the form

$$-\lambda u + H(\nabla u, x) = 0 \tag{5.46}$$

with the Hamiltonian $H(p,x)$ defined by

$$H(p,x) = \max_{a\in A} \left[r(x,a) + f(x,a)\cdot p \right] \tag{5.47}$$

Exercise: Check these computations.

5.4 Brief Introduction to Stochastic Optimal Control

Thus far, we have considered deterministic optimal control in which the dynamic behaviour of the system state is deterministic. In a stochastic optimal control problem, the system state $y(s)$ is a stochastic process. Consequently, the controls also will be stochastic, since we may want to steer the system in a manner that depends on the system's stochastic trajectory. To this end, we now suppose that the **system state** $Y_s(\omega) : [t,T]\times\Omega \to \mathbb{R}^d$ now satisfies the stochastic differential equation (or system of equations)

$$\begin{aligned} dY_s &= f(Y_s,\alpha_s,s)ds + \sigma(Y_t,\alpha_s,s)dB_s, \quad s \ge t \\ Y_t &= x, \quad a.s. \end{aligned} \tag{5.48}$$

where B_s is a n-dimensional Brownian motion defined on probability space $(\Omega,\mathcal{F},\{\mathcal{F}_s\}_{s\ge t},P)$, and σ is a $d\times n$ matrix.

We control the system state through the **control process** $\alpha_s(\omega) : [t,T] \times \Omega \to \mathbb{R}^m$ which is adapted to the filtration $\{\mathcal{F}_s\}_{s\geq t}$. The set of admissible controls is now

$$\mathcal{A}_{t,T} = \{\alpha_s(\omega) : [t,T] \times \Omega \to A \mid \ \alpha_s \ \text{ is adapted to the filtration } \ \{\mathcal{F}_s\}_{s\geq t}\}\,. \tag{5.49}$$

The assumption that the controls are adapted means that we cannot look into the future; the control can only be chosen on the basis of information known up to the present time. Supposing that σ and f satisfy the usual bounds and continuity conditions, the stochastic process $Y_s(\omega)$ is uniquely determined by the initial condition $Y_t = x$ and the control process $\alpha_s(\omega)$.

Given a time $T > t$, the abstract stochastic optimal control problem is to maximize

$$\max_{\alpha \in \mathcal{A}_{t,T}} J_{x,t}(\alpha(\cdot)) = \max_{\alpha \in \mathcal{A}_{t,T}} E\left[\int_t^T r(Y_s, \alpha_s, s)\, ds + g(Y_T) \mid Y_t = x\right] \tag{5.50}$$

As before, the function $r(y, \alpha, s)$ represents a running payoff (or running cost, if $r < 0$), and g represents a terminal payoff (or terminal cost, if $g < 0$). Since the system state is a stochastic process, the net payoff is a random variable, and our goal is to maximize the expected payoff. Even if an optimal control process does not exist, we may define the value function to be

$$u(x,t) = \max_{\alpha \in \mathcal{A}_{t,T}} E\left[\int_t^T r(Y_s, \alpha_s, s)\, ds + g(Y_T) |\ Y_t = x\right] \tag{5.51}$$

Notice that the value function is *not* random.

Variations

There are variations on this theme. For example, we might add the possibility of a payoff based on a stopping criteria. In this case, we want to maximize:

$$\max_{\alpha \in \mathcal{A}_{t,T}} E\left[\int_t^{\gamma \wedge T} r(Y_s, \alpha_s, s)\, ds + g(Y_T)\mathbb{I}_{\{\gamma \geq T\}} + h(Y_\gamma)\mathbb{I}_{\{\gamma < T\}} \mid Y_t = x\right] \tag{5.52}$$

Here, the random variable $\gamma(\omega)$ is a stopping time. The function h represents some payoff that is incurred if $\gamma < T$ (or, this may represent a penalty if $h < 0$).

In (5.51) the time horizon is finite. One could also pose an optimal control problem on an infinite time horizon. For example, one might consider maximizing

$$\max_{\alpha \in \mathcal{A}} J_{x,t}(\alpha(\cdot)) = \max_{\alpha \in \mathcal{A}} E\left[\int_t^\gamma e^{-\lambda s} r(Y_s, \alpha_s)\, ds + e^{-\lambda \gamma} h(Y_\gamma)\right] \tag{5.53}$$

where γ is a stopping time.

5.5 Dynamic Programming Principle for Stochastic Control

For the stochastic control problem there is a Dynamic Programming Principle that is analogous to the DPP for deterministic control. Using the Markov Property of the stochastic process Y_t, one can easily prove the following:

Theorem 5.3 *Let $u(x,t)$ be the value function defined by (5.51). If $t < \tau \le T$, then*

$$u(x,t) = \max_{\alpha \in \mathcal{A}_{t,\tau}} E\left[\int_t^\tau r(Y_s, \alpha_s, s)\, ds + u(Y_\tau, \tau) \mid Y_t = x\right] \tag{5.54}$$

Proof: Exercise. The idea is the same as in the case of deterministic control. Split the integral into two pieces, one over $[t,\tau]$ and the other over $[\tau, T]$. Then condition on $\mathcal{F}_\tau$ and use the Markov property, so that the second integral and the payoff may be expressed in terms of $u(Y_\tau, \tau)$. $\square$

5.6 HJB equation

Using the Dynamic Programming Principle, one can formally derive a PDE for the value function $u(x,t)$. As in the case of deterministic optimal control, one must assume that the value function is sufficiently smooth. Because the dynamics are stochastic, we want to apply Itô's formula in the way that we used the chain rule to derive the HJB equation for deterministic control. Thus, this formal computation requires that the value function by twice differentiable.

From Itô's formula we see that

$$\begin{aligned} u(Y_\tau, \tau) - u(x,t) &= = \int_t^\tau \left[u_t(Y_s, s) + f(Y_s, \alpha_s, s) \cdot \nabla u(Y_s, s)\right] ds \\ &\quad + \int_t^\tau \frac{1}{2} \sum_k \sum_{i,j} u_{x_i x_j}(Y_s, s) \sigma^{jk}(Y_s, \alpha_s, s) \sigma^{ik}(Y_s, \alpha_s, s)\, ds \\ &\quad + \int_t^\tau (\nabla u(Y_s, s))^T \sigma(Y_s, \alpha_s, s)\, dB_s \\ &= \int_t^\tau u_t(Y_s, s) + \mathcal{L}^\alpha u(Y_s, s)\, ds + \int_t^\tau (\nabla u(Y_s, s))^T \sigma(Y_s, \alpha_s, s)\, dB_s \end{aligned} \tag{5.55}$$

where $\mathcal{L}$ is the second order differential operator

$$\begin{aligned} \mathcal{L}^\alpha u &= f(y, \alpha, s) \cdot \nabla u(y,s) + \frac{1}{2} \sum_k \sum_{i,j} u_{y_i y_j}(y,s) \sigma^{jk}(y, \alpha_s, s) \sigma^{ik}(y, \alpha_s, s) \\ &= f(y, \alpha, s) \cdot \nabla u(y,s) + \frac{1}{2} \mathrm{tr}(D^2 u(y,s) \sigma(y, \alpha, s) \sigma^T(y, \alpha, s)) \end{aligned} \tag{5.56}$$

and $D^2 u$ is the matrix of second partial derivatives. Now we plug this into the DPP relation (5.54) and use the fact the martingale term in (5.55) has zero mean. We obtain:

$$\begin{aligned} 0 &= \max_{\alpha \in \mathcal{A}_{t,\tau}} E\left[\int_t^\tau r(Y_s, \alpha_s, s)\, ds + u(Y_\tau, \tau) - u(x,t) \mid\ Y_t = x\right] \\ &= \max_{\alpha \in \mathcal{A}_{t,\tau}} E\left[\int_t^\tau r(Y_s, \alpha_s, s)\, ds + \int_t^\tau u_t(Y_s, s) + \mathcal{L}^\alpha u(Y_s, s)\, ds \mid\ Y_t = x\right] \end{aligned} \tag{5.57}$$

Finally, let $\tau = t + h$, divide by h and let $h \to 0$, as in the deterministic case. We formally obtain the HJB equation

$$u_t(x,t) + \max_{a \in A} \left[r(x, a, t) + \mathcal{L}^a u(x,t)\right] = 0. \tag{5.58}$$

This may be written as

$$u_t(x,t) + \max_{a\in A}\left[r(x,a,t) + \frac{1}{2}\mathrm{tr}(D^2u(x,t)\sigma(x,a,t)\sigma^T(x,a,t)) + f(x,a,t)\cdot\nabla u(x,t)\right] = 0 \tag{5.59}$$

which is, in general, a fully-nonlinear, *second order* equation of the form

$$u_t + H(D^2u, Du, x, t) = 0 \tag{5.60}$$

Notice that the equation is deterministic. The set of possible control values $A \subset \mathbb{R}^m$ is a subset of Euclidean space, and the maximum in the HJB equation (5.68) is over this deterministic set, not over the set $\mathcal{A}$.

HJB for the infinite horizon problem

Deriving the HJB for the infinite horizon problem is similar. Suppose $r = r(y,a)$ and $f = f(y,a)$. Suppose that u is the value function defined by

$$u(x) = \max_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-\lambda s} r(Y_s,\alpha_s)\,ds \mid\ Y_0 = x\right] \tag{5.61}$$

Then the Dynamic Programming Principle shows that for any $\tau > 0$

$$u(x) = \max_{\alpha\in\mathcal{A}} E\left[\int_0^\tau e^{-\lambda s} r(Y_s,\alpha_s)\,ds + e^{-\lambda\tau}u(Y_\tau) \mid\ Y_0 = x\right] \tag{5.62}$$

Using Itô's formula as before, we formally derive the second order equation equation

$$-\lambda u(x) + \max_{a\in A}\left[r(x,a) + \mathcal{L}^a u(x)\right] = 0 \tag{5.63}$$

5.7 Examples

Example 1: In this example, we consider the problem of portfolio optimization. We already considered the problem of maximizing utility from consumption of resources. In that model, we assumed that in the absense of consumption the individual's total wealth grows exponentially according to a deterministic rate (interest rate). Now we suppose that the individual may invest money in various asset classes and or consume resources. For simplicity we suppose that the individual may invest in either a stock (risky) or a bond (risk free growth). Suppose that the stock and bond satsfiy the equations

$$\begin{aligned} db_s &= rb_s\,ds \\ dP_s &= \mu P_s\,ds + \sigma P_s\,dB_s \end{aligned} \tag{5.64}$$

Here B_s is a standard Brownian motion. The individual may decide how much money to consume, how much money to invest in stock, and how much money to invest in bonds. Let c_s denote the consumption rate. Let π_s denote the proportion of investments to put in stocks. Now, if Y_s is the individual's wealth at time s, Y_s satisfies the equation

$$\begin{aligned} dY_s &= (1-\pi_s)rY_s\,ds + \pi_s\mu Y_s\,ds + \pi_s\sigma Y_s\,dB_s - c_s\,ds \\ &= \left[((1-\pi_s)r + \pi_s\mu)Y_s - c_s\right]\,ds + \pi_s\sigma Y_s\,dB_s \end{aligned} \tag{5.65}$$

Lets suppose that $c_s = \eta_s Y_s$ with $\eta \geq 0$, so that the consumption rate cannot be negative. Then the control has two components: $\alpha_s = (\eta_s, \pi_s)$. Now our goal is to maximize the discounted utility

$$J_x(\alpha(\cdot)) := \int_0^\infty e^{-\lambda s} U(c_s)\, ds = \int_0^\infty e^{-\lambda s} U(\eta_s Y_s)\, ds \tag{5.66}$$

where $U(c)$ is a utility function, typically increasing and concave. So the value function is

$$u(x) = \max_{\alpha \in \mathcal{A}} \int_0^\infty e^{-\lambda s} U(\eta_s Y_s)\, ds \tag{5.67}$$

Notice that u depends on π_s through the definition of Y_s.

The HJB equation has the form

$$-\lambda u(x) + \max_{\eta,\, \pi} \left[U(\eta y) + ((1-\pi)r + \pi\mu - \eta) x u_x(x) + \frac{\pi^2\sigma^2}{2} x^2 u_{xx}(x) \right] = 0. \tag{5.68}$$

This example is based on the paper by Robert Merton, *Optimal Consumption and Portfolio Rules in a Continuous-Time Model*, J. Econ. Theory, **3**, 1971 pp. 373-413. The model can be solved explicitly. See lecture notes by M. Soner for details.

Example 2: This example comes from the interesting paper by Leung, Sircar, and Zariphopoulou cited above. An investor want to optimize her portfolio, investing in either stock or bond, with no consumption. The problem is that the company in which she invests may default. If this occurs, she must sell her stock at the market price and put her money in the bond (she can't re-invest in another stock for the time interval under consideration). The stock price is modeled as:

$$dP_s = \mu P_s ds + \sigma P_s dB_s^1 \tag{5.69}$$

where B_s^1 is a Brownian motion. The value of the firms assets is modeled as

$$dV_s = \nu V_s ds + \eta V_s (\rho dB_s^1 + \rho' dB_s^2) \tag{5.70}$$

where B_s^2 is an independent Brownian motion, and $\rho \in (-1, 1)$, $\rho' = \sqrt{1-\rho^2}$. Default occurs if the firm's price P_s drops below a boundary $\tilde{D}_s = De^{-\beta(T-s)}$, $s \in [0, T]$, $D > 0$. Choosing to invest a ratio π_s in the stock (as in the preceding example), the investor's wealth is Y_s where

$$dY_s = (1-\pi_s) r Y_s\, ds + \pi_s \mu Y_s\, ds + \pi_s \sigma Y_s\, dB_s \tag{5.71}$$

and r is the interest rate for the bond. The initial condition is $Y_t = y$. We define the default time, a stopping time, by

$$\gamma_t = \inf\{\tau \geq t \mid V_\tau \leq \tilde{D}_\tau\} \tag{5.72}$$

If $\gamma_t > T$, no default occurs over the time interval in question.

Therefore, the investor wants to optimize

$$u(x, y, t) = \max_{\pi_s(\cdot)} E\left[U(Y_T)\mathbb{I}_{\{\gamma_t > T\}} + U(Y_{\gamma_t} e^{r(T-\gamma_t)})\mathbb{I}_{\{\gamma_t \leq T\}} \mid V_t = x, Y_t = y \right] \tag{5.73}$$

Here $U(y)$ is a utility function ($U(y) = -e^{-hy}$ is used in the paper, $h > 0$). In this case, the HJB equation is

$$u_t + \mathcal{L}u + rxu_x + \max_\pi \left[\frac{1}{2}\sigma^2\pi^2 u_{xx} + \pi(\rho\sigma\eta u_{xy} + (\mu - r)u_x) \right] = 0 \tag{5.74}$$

where $\mathcal{L}$ is the operator

$$\mathcal{L}u = \frac{\eta^2 y^2}{2} u_{yy} + \nu y u_y. \tag{5.75}$$

The domain for u is $\{(t, y, x) \mid t \in [0, T], x \in \mathbb{R},\ y \in [\tilde{D}(t), \infty)\}$, and the boundary condition is

$$u(x, y, T) = U(x), \qquad u(x, De^{-\beta(T-t)}, t) = U(xe^{r(T-t)}). \tag{5.76}$$

There are other examples in this paper illustrating techniques for valuation of credit derivatives.

5.8 The basic theory of Hamilton-Jacobi equations

The Euler-Lagrange equations

We now describe the approach to the Hamilton-Jacobi equations in terms of calculus of variations rather than optimal control.

Let $L(q, x)$ be a smooth function, $q, x \in \mathbb{R}^n$ called the Lagrangian. Fix two points $x, y \in \mathbb{R}^n$ and consider the class of admissible functions

$$\mathcal{A} = \{w \in C([0, t]; \mathbb{R}^n) :\ w(0) = y,\ w(t) = x\},$$

that is $w(t)$ are smooth paths that start at y at time zero, and end at x at time t. Define the functional

$$I(w) = \int_0^t L(\dot{w}(s), w(s))ds.$$

The basic problem of the calculus of variations is to find the optimal curve $w(t)$:

$$\text{find } I^* = \min_{w \in \mathcal{A}} I(w),$$

and, if possible, the optimal path $z(s) \in \mathcal{A}$ such that $I(z) = I^*$. Let us first assume that such $z(s)$ exists and deduce some of its properties.

Theorem 5.4 *(Euler-Lagrange equations) The function $z(s)$ satisfies the Euler-Lagrange equations*

$$-\frac{d}{ds}[\nabla_q L(\dot{z}(s), z(s))] + \nabla_x L(\dot{z}(s), z(s)) = 0, \quad 0 \le s \le t. \tag{5.77}$$

Proof. Let $v(t)$ be a smooth function such that $v(0) = v(t) = 0$ and consider $w_\tau(s) = z(s) + \tau v(s)$. Set also $r(\tau) = I(w_\tau)$. As $z(s)$ minimizes $I(w)$ over $\mathcal{A}$ and $w_\tau \in \mathcal{A}$ for all τ, we have $r'(0) = 0$. Let us now compute $r'(\tau)$:

$$r(\tau) = \int_0^t L(\dot{z}(s) + \tau\dot{v}(s), z(s) + \tau v(s))ds,$$

so

$$r'(\tau) = \int_0^t [\nabla_q L \cdot \dot{v}(s) + \nabla_x L \cdot v(s)] ds = \int_0^t [-\frac{d}{ds}\nabla_q L + \nabla_x L] \cdot v(s) ds.$$

We integrated by parts in the second equality above, and used the fact that the boundary terms vanish since $v(0) = v(t) = 0$. Since $r'(0) = 0$ for all $v(s)$ as above, we should have

$$-\frac{d}{ds}\nabla_q L(\dot{z}(s), z(s)) + \nabla_x L(\dot{z}(s), z(s)) = 0,$$

which is (5.77). □

The above computation shows that if $z(s)$ is a minimizer then it has to satisfy the Euler-Lagrange equation (5.77). However, of course, it is possible that $z(s)$ is a critical point of $I(w)$ but not its minimum – in that case $z(s)$ also satisfies the Euler-Lagrange equations.

The Hamilton equations

There is a nice connection between the Euler-Lagrange equations and the Hamilton equations of classical mechanics. We assume that the equation

$$p = \nabla_q L(q, x) \tag{5.78}$$

can be solved uniquely as an equation for q, as a smooth function of p and x. If that is the case, we can define the Hamiltonian

$$H(p, x) = p \cdot q(p, x) - L(q(p, x), x), \tag{5.79}$$

with the function $q(p, x)$ defined implicitly by (5.78).

Let us now assume that $z(s)$ is the solution of the Euler-Lagrange equations, and set

$$p(s) = \nabla_q L(\dot{z}(s), z(s)), \tag{5.80}$$

that is,

$$\dot{z}(s) = q(p(s), z(s)). \tag{5.81}$$

Differentiating (5.79) in p_j gives

$$\frac{\partial H(p(s), z(s))}{\partial p_j} = q_j(p(s), z(s)) + \sum_{i=1}^{n} p_i(s)\frac{\partial q_i}{\partial p_j} - \sum_{i=1}^{m} \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial p_j} = q_j.$$

We used (5.80) in the last step. Using this in (5.81) gives

$$\dot{z}_j(s) = \frac{\partial H(p(s), z(s))}{\partial p_j}. \tag{5.82}$$

The Euler-Lagrange equations say that

$$\dot{p}_j(s) = \frac{\partial L}{\partial x_j}. \tag{5.83}$$

Differentiating (5.79) in x gives:

$$\frac{\partial H}{\partial x_j} = \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial x_j} - \frac{\partial L}{\partial x_j} - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial x_j} = -\frac{\partial L}{\partial x_j}.$$

Now, putting this together with (5.82)-(5.83) gives the Hamiltonian system

$$\dot{z}(s) = \nabla_p H(p(s), z(s)), \quad \dot{p}(s) = -\nabla_z H(p(s), z(s)). \tag{5.84}$$

The Legendre transform

Let us now assume that the Lagrangian does not depend on the variable x: $L = L(q)$. Then the Hamiltonian $H(p)$ is

$$H(p) = p \cdot q(p) - L(q(p)), \tag{5.85}$$

with q being the solution of

$$p = \nabla_q L(q). \tag{5.86}$$

In order to ensure that the function $q(p)$ is well-defined let us assume that the function $L(q)$ is convex and

$$\lim_{|q|\to+\infty} \frac{L(q)}{|q|} = +\infty. \tag{5.87}$$

Let us now fix p and consider the function $r(q) = p \cdot q - L(q)$. This function is convex and $r(q) \to -\infty$ as $|q| \to +\infty$. Therefore, $r(q)$ attains a unique maximum at the point where $p = \nabla L(q)$, which is exactly the same equation as (5.86). Therefore, we may reformulate (5.85) as

$$H(p) = \sup_q (p \cdot q - L(q)). \tag{5.88}$$

The function $H(p)$ defined by (5.88) is called the Legendre transform of $L(q)$, denoted as $H(p) = L^*(q)$.

Theorem 5.5 *Assume that the function $L(q)$ is convex and (5.87) holds, then $H(p)$ is also convex, and*

$$\lim_{|p|\to+\infty} \frac{H(p)}{|p|} = +\infty. \tag{5.89}$$

Moreover, $L(q)$ is the Legendre transform of the function H.

Proof. The function $s(p; q) = p \cdot q - L(q)$ is an affine function of p for each q fixed. Therefore, $H(p)$ is a supremum of a family of affine functions – hence, it is convex. Indeed, for any $\lambda \in (0, 1)$ we have

$$\begin{aligned}
&H(\lambda p_1 + (1-\lambda)p_2) = \sup_q (\lambda p_1 + (1-\lambda)p_2 \cdot q) - L(q) \\
&= \sup_q [\lambda p_1 \cdot q - \lambda L(q) + (1-\lambda)p_2 \cdot q - (1-\lambda)L(q)] \\
&\leq \sup_q [\lambda p_1 \cdot q - \lambda L(q)] + \sup_q [(1-\lambda)p_2 \cdot q - (1-\lambda)L(q)] = \lambda H(p_1) + (1-\lambda)H(p_2),
\end{aligned}$$

hence $H(p)$ is convex.

In order to see that (5.89) holds, fix $\lambda > 0$ and take $\bar{q} = \lambda p/|p|$ in the definition of $H(p)$, then $|\bar{q}| \leq \lambda$, hence

$$H(p) \geq p \cdot \bar{q} - L(\bar{q}) = \lambda|p| - L(\bar{q}) \geq \lambda|p| - \sup_{|q|\leq\lambda} L(q).$$

It follows that

$$\lim_{|p|\to+\infty} \frac{H(p)}{|p|} \geq \lambda,$$

for each $\lambda > 0$, thus (5.89) holds.

In order to show that $L(q)$ is actually the Legendre transform of $H(p)$, note that, for all p and q we have

$$H(p) \geq p \cdot q - L(q),$$

whence

$$L(q) \geq p \cdot q - H(p).$$

It follows that $L(q) \geq H^*(q)$. But we also have

$$H^*(q) = \sup_p [p \cdot q - H(p)] = \sup_p [p \cdot q - \sup_y [p \cdot y - L(y)]] = \sup_p \inf_y [p \cdot (q - y) + L(y)]. \quad (5.90)$$

As the function $L(q)$ is convex, for each q there exists $s(q)$ such that the graph of $L(y)$ lies above the corresponding hyperplane:

$$L(y) \geq L(q) + s \cdot (y - q).$$

Let us take $p = s(q)$ in (5.90):

$$H^*(q) \geq \inf_y [s \cdot (q - y) + L(y)] \geq L(q). \quad (5.91)$$

We conclude that $H^*(p) = L(q)$. □

The Hopf-Lax formula

We now relate the variational problem that looked at to the Hamilton-Jacobi equations. Consider the initial value problem

$$u_t + H(\nabla u) = 0, \quad t > 0, \; x \in \mathbb{R}^n, \quad (5.92)$$

with the initial data $u(0, x) = g(x)$. The initial data $g(x)$ is globally Lipschitz continuous:

$$\mathrm{Lip}(g) = \sup_{x,y \in \mathbb{R}^n} \frac{|g(x) - g(y)|}{|x - y|} < +\infty. \quad (5.93)$$

We assume that $H(p)$ is convex and satisfies the growth condition (5.89). Let us define

$$u(t, x) = \inf \left[\int_0^t L(\dot{w}(s)) ds + g(y) : \; w(0) = y, \; w(t) = x \right], \quad (5.94)$$

with the infimum taken over all C^1 functions $w(t)$ that satisfy the constraint $w(t) = x$. Here $L(q)$ is the Legendre transform of the function $H(p)$. We will show that expression (5.94) gives a solution of the Hamilton-Jacobi equation (5.92).

Theorem 5.6 *(Hopf-Lax formula) The function $u(t, x)$ defined by (5.94) can be written as*

$$u(t, x) = \min_{y \in \mathbb{R}^n} \left[tL\left(\frac{x - y}{t} \right) + g(y) \right]. \quad (5.95)$$

Proof. First, for any $y \in \mathbb{R}^n$ we may take a "test path"

$$w(s) = y + \frac{s}{t}(x - y),$$

leading to

$$u(t,x) \le \int_0^t L(\frac{x-y}{t})ds + g(y) = tL\left(\frac{x-y}{t}\right) + g(y).$$

As a consequence, we have

$$u(t,x) \le \inf_{y \in \mathbb{R}^n} \left[tL\left(\frac{x-y}{t}\right) + g(y) \right].$$

On the other hand, Jensen's inequality implies that for any test path $w(s)$ we have

$$\frac{1}{t}\int_0^t L(\dot{w}(s))ds \ge L\left(\frac{1}{t}\int_0^t \dot{w}(s)ds\right).$$

Therefore,

$$\int_0^t L(\dot{w}(s))ds \ge tL\left(\frac{x-y}{t}\right),$$

where $y = w(0)$, and thus

$$u(t,x) \ge \inf_{y \in \mathbb{R}^n} \left[tL\left(\frac{x-y}{t}\right) + g(y) \right].$$

Thus, we have shown that

$$u(t,x) = \inf_{y \in \mathbb{R}^n} \left[tL\left(\frac{x-y}{t}\right) + g(y) \right].$$

The fact that the infimum in the right side is actually achieved follow from the fact that for each t and x fixed the function

$$r(y) = tL\left(\frac{x-y}{t}\right) + g(y)$$

tends to $+\infty$ as $|y| \to +\infty$. This is because $L(y)$ is super-linear at infinity, and g is globally Lipschitz. $\square$

A formal computation of the Hamilton-Jacobi equation

Let us now show why we expect the function given by the Hopf-Lax formula to satisfy the hamilton-Jacobi equation, assuming that it is as smooth as needed. For simplicity, assume that $x \in \mathbb{R}$. Let z be such that

$$u(t,x) = tL(\frac{x-z}{t}) + g(z).$$

Then z is determined by the condition

$$g'(z) = L'(\frac{x-z}{t}), \tag{5.96}$$

hence we have

$$u_t = L(\frac{x-z}{t}) - L'(\frac{x-z}{t})z_t - \frac{(x-z)}{t}L'(\frac{x-z}{t}) + g'(z)z_t = L(\frac{x-z}{t}) - \frac{(x-z)}{t}L'(\frac{x-z}{t}).$$

Moreover,

$$u_x = L'((x-z)/t), \tag{5.97}$$

hence the above can be written as

$$u_t = L(\frac{x-z}{t}) - u_x\frac{(x-z)}{t}. \tag{5.98}$$

On the other hand, in the definition of $H(p)$ we have

$$H(p) = \sup_{y\in\mathbb{R}}(py - L(y)) = pq - L(q),$$

with q determined by the relation $p = L'(q)$. Therefore,

$$H(u_x) = u_x q - L(q),$$

with q such that $u_x = L'(q)$. But (5.97) implies that then $q = (x-z)/t$, and (5.98) is nothing but the Hamilton-Jacobi equation

$$u_t + H(u_x) = 0.$$

The rigorous derivation of the Hamilton-Jacobi equation

Let us now verify that the Hopf-Lax formula is Lipschitz continuous.

Lemma 5.7 *Let $u(t,x)$ be defined by (5.95). Then the function $u(t,x)$ is Lipschitz continuous in x for $t \geq 0$ and $x \in \mathbb{R}^n$, and $u(t,x) \to g(x)$ as $t \to 0$.*

Proof. Take $x_1, x_2 \in \mathbb{R}^n$, and choose y so that

$$u(t,x_1) = tL(\frac{x_1-y}{t}) + g(y),$$

then, choosing $z = x_2 - x_1 + y$ below, gives

$$\begin{aligned} u(t,x_1) - u(t,x_2) &= \min_{z\in\mathbb{R}^n}\left[tL\left(\frac{x_2-z}{t}\right) + g(z)\right] - tL(\frac{x_1-y}{t}) - g(y) \\ &\leq g(x_2 - x_1 + y) - g(y) \leq Lip(g)|x_1 - x_2|. \end{aligned}$$

Switching the roles of x_1 and x_2 gives Lipschitz continuity in x:

$$|u(t,x_1) - u(t,x_2)| \leq Lip(g)|x_1 - x_2|.$$

In order to verify the initial condition, note that choosing $y = x$ gives

$$u(t,x) \le tL(0) + g(x), \tag{5.99}$$

but we also have

$$\begin{aligned} u(t,x) &= \min_y \left[tL\left(\frac{x-y}{t}\right) + g(y)\right] \ge \min_y \left[tL\left(\frac{x-y}{t}\right) + g(x) - Lip(g)|x-y|\right] \\ &= g(x) + \min_z [tL(z) - Lip(g)t|z|] = g(x) + t\min_z [L(z) - Lip(g)|z|]. \end{aligned}$$

once again, as $L(z)$ grows super-linearly at infinity, we have

$$\min_z [L(z) - Lip(g)|z|] > -\infty,$$

hence

$$u(t,x) \ge g(x) - Ct. \tag{5.100}$$

We conclude that $u(t,x) \to g(x)$ as $t \to 0$. $\Box$

In order to show that $u(t,x)$ is Lipschitz continuous in time, we need the following lemma (which is essentially a version of the dynamic programming principle).

Lemma 5.8 *For each $x \in \mathbb{R}^n$, and $0 \le s < t$ we have*

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left[(t-s)L\left(\frac{x-y}{t-s}\right) + u(s,y)\right]. \tag{5.101}$$

Proof. Choose z so that

$$u(s,y) = sL(\frac{y-z}{s}) + g(z).$$

Let us write

$$\frac{x-z}{t} = (1-\frac{s}{t})\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}.$$

As L is convex, it follows that

$$\begin{aligned} u(t,x) &\le tL(\frac{x-z}{t}) + g(z) \le t(1-\frac{s}{t})L(\frac{x-y}{t-s}) + sL(\frac{y-z}{s}) + g(z) \\ &= (t-s)L(\frac{x-y}{t-s}) + sL(\frac{y-z}{s}) + g(z) = (t-s)L(\frac{x-y}{t-s}) + u(s,y), \end{aligned}$$

and thus

$$u(t,x) \le \inf_{y\in\mathbb{R}^n} \left[(t-s)L(\frac{x-y}{t-s}) + u(s,y)\right].$$

As the function $u(s,y)$ is actually continuous in y (this follows from Lemma 5.7), and $|u(s,y)|$ grows not faster than linearly at infinity (that follows from (5.99)-(5.100)), the infimum in the right side is actually attained:

$$u(t,x) \le \min_{y\in\mathbb{R}^n} \left[(t-s)L(\frac{x-y}{t-s}) + u(s,y)\right].$$

In order to show the opposite inequality, choose z so that

$$u(t,x) = tL(\frac{x-z}{t}) + g(z),$$

and set

$$y = \frac{s}{t}x + (1-\frac{s}{t})z.$$

Then, we have

$$\frac{x-y}{t-s} = \frac{x-z}{t} = \frac{y-z}{s},$$

hence

$$(t-s)L(\frac{x-y}{t-s}) + u(s,y) \le (t-s)L(\frac{x-z}{t}) + sL(\frac{y-z}{s}) + g(z) = tL(\frac{x-z}{t}) + g(z) = u(t,x).$$

This proves (5.101). □

Lemma 5.9 *The function $u(t,x)$ defined by (5.95) is Lipschitz continuous in t for $t \ge 0$ and $x \in \mathbb{R}^n$.*

Proof. Combing the ideas in the proof of Lemma 5.7 (see (5.99)-(5.100)) with the result of Lemma 5.8 gives

$$u(s,x) - C(t-s) \le u(t,x) \le u(s,x) + C(t-s),$$

and we are done. □

Since the function $u(t,x)$ is Lipschitz in t and x, it is differentiable almost everywhere.

Theorem 5.10 *The function $u(t,x)$ defined by (5.95) is Lipschitz continuous in t and x, differentiable almost everywhere and solves the initial value problem*

$$u_t + H(\nabla u) = 0, \quad t > 0, \; x \in \mathbb{R}^n, \tag{5.102}$$

with $u(0,x) = g(x)$.

Proof. It remains only to verify that at the points (t,x) where both u_t and ∇u exist, the Hamilton-Jacobi equation (5.102) is satisfied. Fix $q \in \mathbb{R}^n$, $h > 0$, then we have, according to Lemma 5.8:

$$u(x+hq, t+h) = \min_{y \in \mathbb{R}^n} \left[hL(\frac{x+hq-y}{h}) + u(t,y) \right] \le hL(q) + u(t,x).$$

It follows that

$$u_t(t,x) + q \cdot \nabla u(t,x) \le L(q),$$

for all $q \in \mathbb{R}^n$. Therefore, we have

$$u_t(t,x) + H(\nabla u(t,x)) = u_t(t,x) + \min_{q \in \mathbb{R}^n}(q \cdot \nabla u(t,x) - L(q)) \le 0. \tag{5.103}$$

Next, we show the opposite inequality. Choose z so that

$$u(t,x) = tL(\frac{x-z}{t}) + g(z).$$

Given $h > 0$, set

$$y = \frac{t-h}{t}x + (1 - \frac{t-h}{t})z = x - h\frac{(x-z)}{t}, \tag{5.104}$$

so that

$$\frac{x-z}{t} = \frac{y-z}{t-h}.$$

We have

$$u(t,x) - u(t-h,y) \geq tL(\frac{x-z}{t}) + g(z) - \left[(t-h)L(\frac{y-z}{t-h}) + g(z)\right] = hL(\frac{x-z}{t}).$$

Keeping in mind expression (5.104), and letting $h \to 0$ gives

$$u_t(t,x) + \frac{1}{t}(x-z)\cdot\nabla u(t,x) \geq L(\frac{x-z}{t}).$$

It follows that

$$u_t(t,x) + H(\nabla u(t,x)) = u_t(t,x) + \min_{q\in\mathbb{R}^n}(q\cdot\nabla u(t,x) - L(q)) \geq 0,$$

which, together with (5.103) finishes the proof. □