Homework 4

1. Let u_k be a sequence of viscosity solutions of the Hamilton-Jacobi equation

$$u_t^k + H(\nabla u^k, x) = 0.$$

Suppose that $u^k \to u$ uniformly. Show that u is also a viscosity solution of

$$u_t + H(\nabla u, x) = 0.$$

2. Suppose that $u_\epsilon(t, x)$ satisfies the parabolic equation

$$\frac{\partial u_\epsilon}{\partial t} + H(\nabla u_\epsilon, x) = \epsilon \Delta u^\epsilon,$$

and $u^\epsilon(t, x) \to u$ uniformly. Show that $u(t, x)$ is a viscosity solution to

$$u_t + H(\nabla u, x) = 0.$$

3. Let $U \subset \mathbb{R}^n$ be an open bounded set, and set $u(x) = \mathrm{dist}(x, \partial U)$, for $x \in U$. Show that $u(x)$ is a viscosity solution to

$$|\nabla u| = 1 \text{ in } U.$$

4. Let $u(x)$ be a solution of the boundary value problem

$$\begin{aligned} &-\nabla \cdot (a(x)\nabla u) = f \text{ in } U, \\ &u(x) = 0 \text{ on } \partial U, \end{aligned}$$

and $v(t, x)$ solve the time-dependent problem

$$\begin{aligned} &v_t - \nabla \cdot (a(x)\nabla u) = 0 \text{ in } U, \\ &v(t, x) = 0 \text{ on } \partial U, \\ &v(0, x) = f(x), \text{ at } t = 0. \end{aligned}$$

Here $U(x)$ is a smooth bounded domain, and $x \in L^1(U) \cap L^\infty(U)$, and the function $a(x)$ is smooth, positive and bounded.
(i) Show that

$$u(x) = \int_0^\infty v(t, x)dt. \tag{0.1}$$

(ii) Use the ideas of one of the previous homeworks to show that there exists a constant $C > 0$ that may depend on the domain U and the function $a(x)$ only, so that

$$|v(t, x)| \leq \frac{C}{t^{n/2}} \|f\|_{L^1}.$$

(iii) Use (0.1) (and a few other things) to show that for any $p > n/2$ there exists a constant C_p so that

$$|u(x)| \leq C\|f\|_{L^p}.$$