Homework 2

1. Consider the heat equation in the whole space $\mathbb{R}^n$:

$$u_t = \Delta u, \quad t > 0, \quad x \in \mathbb{R}^n,$$

with the initial data $u(0,x) = u_0(x) \in C_c^\infty(\mathbb{R}^n)$. Show that solution is given by

$$u(t,x) = \int_{\mathbb{R}^n} G(t,x,y) u_0(x,y) dy,$$

with

$$G(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/(4t)}.$$

Use this to show that

$$|u(t,x)| \leq \frac{C}{t^{n/2}} \|u_0\|_{L^1}.$$

2. Show that there exists a constant $C > 0$ so that for all functions $u \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ we have

$$\|u\|_{L^2}^{1+2/n} \leq C \|u\|_{L^1}^{2/n} \|\nabla u\|_{L^2}.$$

Use the scaling arguments to explain the exponents above. Hint: start with the identity

$$\int_{\mathbb{R}^n} |u(x)|^2 dx = \int |\hat{u}(\xi)|^2 d\xi,$$

split the integral in the right side into the regions $\{|\xi| \leq R\}$ and $\{|\xi| \geq R\}$, and show that for any $R > 0$ we have

$$\int |\hat{u}(\xi)|^2 d\xi \leq C_d R^d \|u\|_{L^1}^n + \frac{C}{R^2} \|\nabla u\|_{L^2}^2.$$

Finally, optimize over R.

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The rest of the homework deals with solutions of the parabolic problem

$$u_t = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right),$$

with the initial data $u(0,x) = u_0(x)$, also in the whole space $\mathbb{R}^n$. We assume that the matrix $a_{ij}(x)$ is positive definite and uniformly bounded in x: there exists $K > 0$ so that $|a_{ij}(x)| \leq K$, and

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \frac{1}{K} |\xi|^2,$$

for all $\xi \in \mathbb{R}^n$.

3. Show that the function $u(t,x)$ satisfies

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 dx = -\int |\nabla u|^2 dx.$$

4. Show that

$$I(t) = \int u(t,x)dx = \int u_0(x)dx.$$

5. Apply the inequality in Problem 2 and the identity in Problem 3, as well as the result of Problem 4 to show that

$$\|u(t)\|_{L^2} \leq \frac{C}{t^{n/4}}\|u_0\|_{L^1}.$$

Hint: this takes more than 5 minutes.

6. (i) Think of the operator S_t that maps u_0 to $u(t)$, as a mapping from L^1 to L^2. You have just shown in Problem 5 that $\|S_t\|_{L^1\to L^2} \leq C/t^{n/4}$. What is the adjoint operator S_t^*? Hint: it is S_t again, as a mapping L^2 to L^∞.

(ii) Show that $S_t = S_{t/2} \circ S_{t/2}$.

(iii) Use (ii) to show that

$$\|u(t)\|_{L^\infty} \leq \frac{C}{t^{n/4}}\|u_0\|_{L^2}.$$

(iv) Show that

$$\|u(t)\|_{L^\infty} \leq \frac{C}{t^{n/2}}\|u_0\|_{L^2},$$

as for the heat equation!