Homework 1

1. Use the fact that $s_{j+1}B_{s_{j+1}} - s_jB_{s_j} = s_j(B_{s_{j+1}} - B_{s_j}) + (s_{j+1} - s_j)B_{j+1}$ to show directly that
$$\int_0^t s\, dB_s(\omega) = tB_t - \int_0^t B_s\, ds \tag{0.1}$$
Would the same formula hold if B_t were actually differentiable and the first integral were given by a Riemann-Stieljes integral?

2. For $F(s,\omega) \in \mathcal{L}^4$, let
$$Z_t = e^{\int_0^t F^2(s,\omega)\, dB_s - \frac{1}{2}\int_0^t |F|^4\, ds} \tag{0.2}$$
Use Itô's formula to derive a stochastic differential equation (SDE) for Z_t. Write your answer in integral form.

3. Suppose that $u(x,t)$ is smooth and satisfies the terminal value problem
$$u_t + u_{xx} - \mu u_x + \lambda u = 0, \quad t < T, \ \ x \in \mathbb{R} \tag{0.3}$$
with terminal data $u(x,T) = \phi(x) \in C_0^\infty(\mathbb{R})$. Use Itô's fromula and the product rule to derive a stochastic representation for $u(x,t)$.

4. What stochastic differential equation is satisfied by $Y_t = e^{\lambda t}\sin(\alpha B_t)$? Under what conditions on λ and α is Y_t a martingale (with respect to the Brownian filtration)?

5. Suppose that $D \subset \mathbb{R}^d$ is a smooth bounded domain and that $u(x)$ and $v(x)$ both satisfy the equation
$$\frac{1}{2}\Delta u + b(x) \cdot \nabla u = 0, \quad x \in D \tag{0.4}$$
where $b(x)$ is a smooth, bounded vector field. Suppose that $u(x) = g(x)$ for $x \in \partial D$ and that $v(x) = f(x)$ for $x \in \partial D$, where f and g are smooth functions.
   (i) Derive a stochastic representation for $u(x)$ and $v(x)$ in terms of a stochastic process X_t^x.
   (ii) Suppose that for any set $\Gamma \subset \partial D$ having positive Lebesgue measure, the process X_t^x has a nonzero probability of hitting the boundary first at Γ. That is, if γ_D^x is the first hitting time of X_t^x to the boundary ∂D, then we suppose that $P(X_{\gamma_D}^x \in \Gamma) > 0$ for all $x \in D$. Use this assumption and your answer in part (i) to prove a strong comparison principle: if $f \geq g$ for all $x \in \partial D$ with $f > g$ somewhere on the boundary, then $v > u$ for all $x \in D$.

6. Define $\beta_k(t) = \mathbb{E}(B_t^k))$, where B_t is the standard Brownian motion. Use Ito's formula and induction to find an expression for β_k for all positive integers $k \geq 1$.

7. Let X_t be an Ito integral
$$X_t = \int_0^t v(s,\omega) dB_s,$$
with a bounded function v, that is, $|v(t,\omega)| \leq M$ for all $t \geq 0$ and all $\omega \in \Omega$. Then X_t is a martingale. Give an example of $v(t,\omega)$ such that X_t^2 is not a martingale. Show that if v is bounded then
$$M_t = X_t^2 - \int_0^t |v(s,\omega)|^2 ds$$
is a martingale.

8. Let

$$Y_t = t + (1-t)\int_0^t \frac{dB_s}{1-s}$$

and show that $\lim_{t\to 1} Y_t = 1$ almost surely. Hence, Y_t connects $Y_0 = 0$ and $Y_1 = 1$.

9. Show that the solution $u(t,x)$ of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\beta^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} - \alpha x \frac{\partial u}{\partial x}, \quad x \in \mathbb{R},$$

with the initial data $u(0,x) = f(x)$ may be written as

$$u(t,x) = \mathbb{E}\left\{ f(xe^{\beta B_t + (\alpha - \beta^2/2)t}) \right\}.$$

10. Let $b(x)$, $x \in \mathbb{R}$, be a smooth bounded function and define the process X_t by

$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x.$$

(i) Prove that for all $M > 0$, $x \in \mathbb{R}$ and $t > 0$ we have $P(X_t^x \geq M) > 0$. Hint: Girsanov's theorem is helpful here (look it up in the notes). (ii) Assume that $b(x) \leq \epsilon_0$ for all x with some $\epsilon_0 > 0$. Show that $X_t^x \to +\infty$ as $t \to +\infty$ almost surely. Why does this not contradict (i)?

11. Let (a,b) be a bounded interval, set

$$dX_t = rX_t + \alpha X_t dB_t, \quad X_0 = x \in (a,b).$$

(i) Let $\tau(x)$ be the exit time for X_t^x from (a,b). Find an equation for $u(x) = \mathbb{E}(\tau(x))$.
(ii) Compute $P(X_{\tau(x)} = b)$.
(iii) Let g be a bounded continuous function defined on (a,b). Find

$$w(x) = \mathbb{E}\left[\int_0^{\tau(x)} g(X_t)dt\right].$$

12. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n$ centered at $x = 0$. Use the probabilistic interpretation to show that there exists a constant $C > 0$ so that any solution of the Poisson equation $\Delta u = f$ in $B(0,1)$ with the boundary condition $u(x) = g(x)$ for x with $|x| = 1$ satisfies

$$|u(x)| \leq C(\max_{|x|=1}|g(x)| + \max_{|x|\leq 1}|f(x)|).$$