Winter 2010. MATH227. Solutions of Homework 3

Problem 1.

Applying the Ito formula to B_t^k we get:

$$B_t^k = \int_0^t k B_s^{k-1} dB_s + \frac{1}{2}\int_0^t k(k-1)B_s^{k-2} ds.$$

We take the expectation of both sides and note that B_t^{k-1} is square integrable and that we may apply Fubini's Theorem to $\mathbb{E}[\int_0^t k(k-1)B_s^{k-2}ds]$. Thus, we get $\beta_k(t) = \frac{k(k-1)}{2}\int_0^t \beta_{k-2}(s)ds$. Hence, $\mathbb{E}[B_s] = 0$ for all s implies that $\beta_k(t) = 0$ for all odd k and all t. For even $k = 2n$ we claim: $\beta_k(t) = \frac{k!\, t^n}{n!\, 2^n}$. Indeed, this is true for $k = 0$. Moreover, if it is true for $k = 2n$, then

$$\beta_{k+2}(t) = \frac{(k+2)(k+1)}{2}\int_0^t \frac{k!\, s^n}{n!\, 2^n} ds = \frac{k!\, t^{n+1}}{(n+1)!\, 2^{n+1}},$$

so the claim follows by induction.

Problem 2.

For $v(t,\omega) = 1$ we have $X_t^2 = B_t^2$. But for $t > 0$ it holds $\mathbb{E}[B_t^2] = t \neq 0 = \mathbb{E}[B_0]$. Since for any martingale Y it holds $\mathbb{E}[Y_t] = \mathbb{E}[Y_0]$, the process X_t^2, $t \geq 0$ is not a martingale.

On the other hand, applying Ito's formula we get:

$$dM_t = 2X_t dX_t + v(t)^2 dt - v(t)^2 dt = 2X_t v(t) dB_t.$$

By the properties of the stochastic integral M_t is a martingale provided that $X_t v(t)$ is in L^2 for any fixed $t \geq 0$. But the Ito isometry shows:

$$\mathbb{E}[(X_t v(t))^2] \leq M^2 \mathbb{E}[X_t^2] = M^2 \mathbb{E}\Big[\int_0^t v(s)^2 ds\Big] \leq M^4 t$$

where M is the constant in the statement of the problem. Hence, $X_t v(t)$ is in L^2 and M_t is indeed a martingale.

Problem 3.

Let $X_t = (1-t)\int_0^t \frac{dB_s}{1-s}$. Obviously, it suffices to prove $X_t \to 1$ almost surely. To this end, note that X_t is a martingale for $t \in [0,1)$, since its increments are mean zero normal random variables. Hence, we can apply Doob's inequality for continuous martingales to get:

$$\mathbb{P}(A_{n,k}) \equiv \mathbb{P}\Big(\sup_{t \in [1-2^{-n+1}, 1-2^{-n}]} |X_t| \geq \frac{1}{k}\Big) \leq k^2 \mathbb{E}[X_{1-2^{-n}}^2].$$

Moreover, the Ito isometry yields:

$$\mathbb{E}[X_{1-2^{-n}}^2] = 2^{-2n}\int_0^{1-2^{-n}} \frac{1}{(1-s)^2} ds = 2^{-2n}(2^n - 1).$$

Thus, $\sum_{n=1}^{\infty}\mathbb{P}(A_{n,k})<\infty$ for all k. Hence, for any fixed k there is a set Ω_k such that it holds $\sup_{t\in[1-2^{-n+1},1-2^{-n}]}|X_t(\omega)|<\frac{1}{k}$ for n large enough for all $\omega\in\Omega_k$ and $\mathbb{P}(\Omega_k)=1$. Moreover, for $\omega\in\Omega_k$ we have $\limsup_{t\uparrow 1}|X_t|\le\frac{1}{k}$, so for $\omega\in\bigcap_{k=1}^{\infty}\Omega_k$ it holds $\limsup_{t\uparrow 1}|X_t|=0$. But $\mathbb{P}(\bigcap_{k=1}^{\infty}\Omega_k)=1$, so $|X_t|\to 0$ almost surely.

For general a and b we can take $a(1-t)+bt+X_t$.

Problem 4.

Fix a $T>0$ and consider $v(t,x)=u(T-t,x)$ for $t\in[0,T]$, $x\in\mathbb{R}$. Then v satisfies $-v_t=\frac{\beta^2x^2}{2}v_{xx}-\alpha x v_x$. Now, fix an $x\in\mathbb{R}$ and consider the process $Z_t=xe^{\beta B_t-\beta^2t/2-\alpha t}$. By Ito's formula it solves:

$$dZ_t=Z_t\beta dB_t-Z_t\alpha dt.$$

Applying Ito's formula to $v(t,Z_t)$ we get:

$$dv(t,Z_t)=v_t(t,Z_t)dt+v_x(t,Z_t)(Z_t\beta dB_t-Z_t\alpha dt)+\frac{1}{2}v_{xx}(t,Z_t)Z_t^2\beta^2dt=v_x(t,Z_t)Z_t\beta dB_t.$$

Provided that $v_x(t,Z_t)Z_t$ is integrable with respect to B_t on $[0,T]$, we obtain that $v(t,Z_t)$ is a martingale on $[0,T]$. Hence,

$$u(t,x)=v(T-t,x)=\mathbb{E}[v(T,Z_t)|Z_{T-t}=x]=\mathbb{E}[f(Z_T)|Z_{T-t}=x]=\mathbb{E}[f(Z_t)].$$

The last equation follows from the fact that Z_T/Z_{T-t} has the same distribution as Z_t/Z_0.

Problem 5.

Fix $M>0$, $x\in\mathbb{R}$, $t\ge 0$ and $T>t$. Define $Z_t=\exp\Big(-\int_0^s b(X_u)dB_u-\int_0^s b(X_u)^2du\Big)$, $s\in[0,T]$. Then by Girsanov's Theorem the measure $\mathbb{Q}$ on $(\Omega,\mathscr{F}_T)$ given by its Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}=Z_T$ is well-defined and equivalent to $\mathbb{P}$. Moreover, under $\mathbb{Q}$ the process X_s, $s\in[0,T]$ is a Brownian motion started at x. Thus, the random variable X_t is normally distributed under $\mathbb{Q}$ and, in particular, $\mathbb{Q}(X_t\ge M)>0$. By the equivalence of the two measures this means $\mathbb{P}(X_t\ge M)>0$.

Now, assume $b<\varepsilon_0$ for some $\varepsilon_0<0$ (at this point there was a typo in the problem set!). Then

$$X_t=x+\int_0^t b(X_s)ds+B_t\le x+\varepsilon_0t+B_t=x+t\Big(\varepsilon_0+\frac{B_t}{t}\Big).$$

By the law of large numbers for the Brownian motion (or more generally for Levy processes), $\frac{B_t}{t}$ converges to $\mathbb{E}[B_1]=0$ almost surely. Hence, $\varepsilon_0+\frac{B_t}{t}$ converges to ε_0 and $x+t(\varepsilon_0+\frac{B_t}{t})$ to $-\infty$ almost surely. Thus, X_t converges to $-\infty$ as well. This is not a contradiction to the first part of the problem, since $\mathbb{P}(X_t\ge M)$ is positive, but can tend to zero for $t\to\infty$ for all $M\in\mathbb{R}$ (which in fact must be the case due to the findings in the second part of the problem).

Problem 6.

(i) Define u to be the unique solution of the ODE $rxu'(x) + \frac{\alpha^2}{2}x^2u''(x) = -1$ with boundary values $u(a) = 0$, $u(b) = 0$. Applying Ito's formula to $u(X_t)$ we obtain:

$$u(X_{\tau(x)\wedge t}) - u(x) = \int_0^{\tau(x)\wedge t} rX_su'(X_s) + \frac{\alpha^2 X_s^2}{2}u''(X_s)ds + \int_0^{\tau(x)\wedge t} \alpha X_s u'(X_s)dB_s.$$

Since u' is bounded on $[a, b]$ and X_s is in L^2 (it is log-normally distributed!), the second term is a martingale. Thus, taking expectations and using the ODE above we get:

$$\mathbb{E}[u(X_{\tau(x)\wedge t})] - u(x) = -\mathbb{E}[\tau(x) \wedge t].$$

Taking the limit $t \to \infty$, using the Dominated Convergence Theorem on the left-hand side (u is bounded on $[a, b]$) and the Monotone Convergence Theorem on the right-hand side we obtain $u(x) = \mathbb{E}[\tau(x)]$. Thus, the ODE above is the desired equation.

(ii) Here we consider again the ODE $rxv'(x) + \frac{\alpha^2}{2}x^2v''(x) = -1$ on $[a, b]$, but this time with boundary values $v(a) = 0$, $v(b) = 1$. The same application of Ito's formula as in (i) yields here:

$$v(x) = \mathbb{E}[v(X_{\tau(x)})] = \mathbb{E}[v(b)1_{\{X_{\tau(x)}=b\}}] = \mathbb{P}(X_{\tau(x)} = b).$$

Hence, it remains to solve the ODE. Suppose $0 \leq a < b$ and $\alpha^2 \neq 2r$. Then the ODE can be rewritten as

$$\frac{v''(x)}{v'(x)} = -\frac{2r}{\alpha^2}x^{-1},$$

so $\log v'(x) = -\frac{2r}{\alpha^2}\log x$ and $v'(x) = x^{-\frac{2r}{\alpha^2}}$. It follows that the general solution of the ODE is given by $c_1 + c_2x^{1-\frac{2r}{\alpha^2}}$. To find the constants we need to solve

$$\begin{aligned} 0 &= c_1 + c_2a^{1-\frac{2r}{\alpha^2}}, \\ 1 &= c_1 + c_2b^{1-\frac{2r}{\alpha^2}}. \end{aligned}$$

The solution to this system is given by:

$$c_1 = \frac{-a^{1-\frac{2r}{\alpha^2}}}{b^{1-\frac{2r}{\alpha^2}} - a^{1-\frac{2r}{\alpha^2}}}, \quad c_2 = \frac{1}{b^{1-\frac{2r}{\alpha^2}} - a1 - \frac{2r}{\alpha^2}}.$$

Finally, we plug in the values of the constants into the general solution to obtain:

$$v(x) = \frac{x^{1-\frac{2r}{\alpha^2}} - a^{1-\frac{2r}{\alpha^2}}}{b^{1-\frac{2r}{\alpha^2}} - a^{1-\frac{2r}{\alpha^2}}}.$$

In the case $\alpha^2 = 2r$ the same procedure yields $v(x) = \frac{\log x - \log a}{\log b - \log a}$.

(iii) The same procedure as in (i) and (ii) shows that w is the unique solution of $rxw'(x) +$

$\frac{\alpha^2}{2}x^2w''(x) = -g(x)$, $x \in (a,b)$ with boundary values $w(a) = w(b) = 0$. The substitution $x = e^t$ yields the constant coefficient linear second-order ODE:

$$\frac{\alpha^2}{2}\frac{d^2w}{dt^2} + (r - \alpha^2/2)\frac{dw}{dt} = -g(e^t), \quad t \in (\log a, \log b).$$

Its solution can be readily expressed as an integral involving g.

Problem 7.

We can easily compute:

$$\begin{aligned} u_t(t,x) &= \frac{d}{dt}v(x^2/t) = -v'(x^2/t)\frac{x^2}{t^2}, \\ u_x(t,x) &= \frac{d}{dx}v(x^2/t) = v'(x^2/t)\frac{2x}{t}, \\ u_{xx}(t,x) &= v''(x^2/t)\frac{4x^2}{t^2} + v'(x^2/t)\frac{2}{t}. \end{aligned}$$

Hence, $u_t = u_{xx}$ is equivalent to $-v'(x^2/t)\frac{x^2}{t^2} = v''(x^2/t)\frac{4x^2}{t^2} + v'(x^2/t)\frac{2}{t}$. Multiplying this equation by x and setting $z = x^2/t$ we see that the equation is equivalent to

$$0 = 4zv''(z) + (2+z)v'(z)$$

for all $z > 0$. Next, we have

$$\frac{d}{ds}(e^{-s/4}s^{-\frac{1}{2}}) = -\frac{1}{4}e^{-s/4}s^{-\frac{1}{2}} - \frac{1}{2}e^{-s/4}s^{-\frac{3}{2}} = -\frac{s+2}{4s}e^{-s/4}s^{-\frac{1}{2}}.$$

Thus, $\int_0^t e^{-s/4}s^{-\frac{1}{2}}ds$ solves the above ODE. Moreover, the constant function 1 solves the ODE as well. Since we deal with a homogeneous linear ODE of order two, we may conclude that all solutions are linear combinations of the two solutions (the two solutions a linearly independent!). Finally,

$$\frac{d}{dx}v(x^2/t) = v'(x^2/t)\frac{2x}{t} = C_1e^{-s/4}s^{-\frac{1}{2}}\Big|_{s=x^2/t}\frac{2x}{t} = C_1e^{-\frac{x^2}{4t}}(x^2/t)^{-\frac{1}{2}}\frac{2x}{t} = \frac{2C_1}{\sqrt{t}}e^{-\frac{x^2}{4t}}.$$

Thus, by choosing $C_1 = \frac{1}{4\sqrt{\pi}}$ we recover the heat kernel.

Problem 8.

From page 56 of the lecture notes we have

$$u(x) = \mathbb{E}[g(X_\tau)] - \mathbb{E}\left[\int_0^\tau f(X_s)ds\right]$$

where $X_t = x + \sqrt{2}B_t$, B_t is an n-dimensional Brownian motion and $\tau = \tau_x$ is the exit time of X_t from $B(0,1)$. We easily obtain the bound

$$|u(x)| \le \|g\|_\infty + \|f\|_\infty \cdot \mathbb{E}\left[\int_0^\tau 1ds\right].$$

Thus, it suffices to show that $\sup_{x \in B(0,1)} \mathbb{E}[\tau_x] < \infty$. But by the same argument we know that $w(x) = \mathbb{E}[\tau_x]$ solves $\Delta w = -1$ on $B(0,1)$, $w = 0$ on $\partial B(0,1)$. By the maximum principle the solution to this BVP is unique. Moreover, $\frac{1}{2n}(1 - \|x\|_2^2)$ solves this BVP. Thus, $\mathbb{E}[\tau_x] = \frac{1}{2n}(1 - \|x\|_2^2)$ which is bounded by $\frac{1}{2n}$ on $B(0,1)$.