Homework 3

1. Define $\beta_k(t) = \mathbb{E}(B_t^k))$, where B_t is the standard Brownian motion. Use Ito's formula and induction to find an expression for β_k for all positive integers $k \geq 1$.

2. Let X_t be an Ito integral
$$X_t = \int_0^t v(s,\omega)dB_s,$$
with a bounded function v, that is, $|v(t,\omega)| \leq M$ for all $t \geq 0$ and all $\omega \in \Omega$. Then X_t is a martingale. Give an example of $v(t,\omega)$ such that X_t^2 is not a martingale. Show that if v is bounded then
$$M_t = X_t^2 - \int_0^t |v(s,\omega)|^2 ds$$
is a martingale.

3. (i) Let
$$Y_t = t + (1-t)\int_0^t \frac{dB_s}{1-s}$$
and show that $\lim_{t\to 1} Y_t = 1$ almost surely. Hence, Y_t connects $Y_0 = 0$ and $Y_1 = 1$.
(ii) Generalize (i) to construct a process of the same kind that connects $Y_0 = a$ and $Y_1 = b$ for arbitrary $a, b \in \mathbb{R}$.

4. Show that the solution $u(t,x)$ of the initial value problem
$$\frac{\partial u}{\partial t} = \frac{\beta^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} - \alpha x \frac{\partial u}{\partial x}, \quad x \in \mathbb{R},$$
with the initial data $u(0,x) = f(x)$ may be written as
$$u(t,x) = \mathbb{E}\left\{ f(xe^{\beta B_t + (\alpha - \beta^2/2)t}) \right\}.$$

5. Let $b(x)$, $x \in \mathbb{R}$, be a smooth bounded function and define the process X_t by
$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x.$$
(i) Prove that for all $M > 0$, $x \in \mathbb{R}$ and $t > 0$ we have $P(X_t^x \geq M) > 0$. Hint: Girsanov's theorem is helpful here (look it up in the notes). (ii) Assume that $b(x) \leq \epsilon_0$ for all x with some $\epsilon_0 > 0$. Show that $X_t^x \to +\infty$ as $t \to +\infty$ almost surely. Why does this not contradict (i)?

6. Let (a,b) be a bounded interval, set
$$dX_t = rX_t + \alpha X_t dB_t, \quad X_0 = x \in (a,b).$$
(i) Let $\tau(x)$ be the exit time for X_t^x from (a,b). Find an equation for $u(x) = \mathbb{E}(\tau(x))$.
(ii) Compute $P(X_{\tau(x)} = b)$.
(iii) Let g be a bounded continuous function defined on (a,b). Find
$$w(x) = \mathbb{E}\left[\int_0^{\tau(x)} g(X_t)dt\right].$$

7. Let $u(t,x) = v(x^2/t)$. Show that $u_t = u_{xx}$ if and only if

$$4zv''(z) + (2+z)v'(z) = 0 \qquad (*)$$

for all $z > 0$. Show that the general solution of (*) is

$$v(z) = c_1 \int_0^z \frac{e^{-s/4}}{\sqrt{s}} ds + c_2.$$

Finally, differentiate $v(x^2/t)$ with respect to x and choose c_1 so as to get the heat kernel in one dimension.

8. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n$ centered at $x = 0$. Use the probabilistic interpretation to show that there exists a constant $C > 0$ so that any solution of the Poisson equation $\Delta u = f$ in $B(0,1)$ with the boundary condition $u(x) = g(x)$ for x with $|x| = 1$ satisfies

$$|u(x)| \leq C(\max_{|x|=1} |g(x)| + \max_{|x|\leq 1} |f(x)|).$$